Chapter 1: Differential-Algebraic Equations

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Mathematical Models, Analysis and Simulation, Part I

The Problem

Consider the differential equation

$$A(t,x)x' + g(t,x) = 0$$

or, more generally, $F(t,x,x') = 0$. Here, all involved functions $x: I \to \mathbb{R}^n$ etc are vector-valued functions!

The matrix-valued function $A(t,x)$ is assumed to be singular for all values of their arguments.

Applications:

- Electrical circuits (see Strang, sections 2.4, 2.6, pp. 179–181, handout for homework)
- Constraint mechanical multibody systems
- Singular perturbed problems (covered in next lecture)
- (Semi-) Discretization of multiphysics systems (covered in later lectures)

Transient Analysis of Electrical Networks: MNA

- A (linear) network consists of (linear) resistors, capacitors, and inductors, as well as voltage and current sources.
- All elements are characterized by voltage-current characteristics ($i \equiv y$ in Strang):
  - Resistor: $i = Gv$
  - Capacitor: $i = Cdv/dt$
  - Inductor: $v = Ldi/dt$
  - Voltage source: $v = E$
  - Current source: $i = I$
- The topology is described using *incidence matrices*.
- Kirchhoff's laws: $Ai = 0$, $v = A^T x$ where x denotes the node potentials.

$$\begin{pmatrix} A_C C A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} x \\ y_L \\ y_V \end{pmatrix} + \begin{pmatrix} A_R G A_R^T & A_L & A_V \\ -A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y_L \\ y_V \end{pmatrix} = \begin{pmatrix} -A_I I \\ 0 \\ E \end{pmatrix}$$

Constraint Mechanical Multibody Systems

In case of holonomic constraints, Lagrangian calculus of the first kind gives:

$$M(p)p'' = f(p,p') - R(p)^T\lambda,$$
$$0 = r(p)$$

where $R(p) = (\partial/\partial p)r(p)$.

p – (generalized) positions

Reformulation as a first order system: $x = (p, v, \lambda)$, $v = p'$

$$A(t,x) = \begin{pmatrix} I & 0 & 0 \\ 0 & M(p) & 0 \\ 0 & 0 & 0 \end{pmatrix}, g(t,x) = \begin{pmatrix} -v \\ -f(p,v) + R(p)^T\lambda \\ r(p) \end{pmatrix}$$

The Mathematical Pendulum

A pendulum is fixed at origo in the (x,y)-plane. Langrangian calculus of the first kind provides the mathematical model,

$$mx'' = -\lambda x,$$
$$my'' = -mg - \lambda y,$$
$$0 = x^2 + y^2 - l^2.$$

The first two equations are differential ones while the last is an algebraic equation.

Note: Often, the algebraic relations are "hidden" in the system and not that easy to identify.

A more general example of this type is given in Strang, p. 180.

Singular Perturbed Odes

Consider

$$y' = f(y,z),$$
$$\varepsilon z' = g(y,z)$$

Assume ε to be a small paramter. Often, the solution of the system obtained by formally setting $\varepsilon = 0$ is a good approximation to the original one. See next lecture.

Partial Differential Equations

- Navier-Stokes equation
  - Momentum equation: $\frac{\partial}{\partial t}u - \nu\Delta u + u \cdot \nabla u + \nabla p = 0$
  - Incompressibility condition: $\operatorname{div} u = 0$
- Heat equation (in more detail later)
  - Conservation law: $\frac{\partial}{\partial t}T + \operatorname{div} \Phi = 0$
  - Material law: $\Phi = -k\nabla T$

Multiphysics problems lead to mixed systems, so-called *partial differential-algebraic equation*.

This is a subject of active research.

Semiexplicit Systems

$$\boxed{\begin{aligned} y' &= f(y,z), \\ 0 &= g(y,z) \end{aligned}}$$

is a *semiexplicit dae* with $x = (y,z)$.

- The second equation respresents a *constraint* on the solution $x = (y,z)$.
- It would be natural to consider the (restricted) dynamics on the manifold

$$\mathscr{S} = \{(y,z)|g(y,z) = 0\}.$$

- Hence, differential equations on manifolds are the "natural" framework.

Here, we make an elementary analysis, only!

Note: Compare the relation between Lagrangian calculus of first and second kinds!

Linear Constant Coefficient Daes

$$\boxed{Ax'(t) + Bx(t) = q(t)}$$

with A, B being square $n \times n$-matrices. A dae is obtained if A is singular.

Definition: (A, B) forms a regular matrix pencil iff there exists a $\lambda \in \mathbb{C}$ such that $\lambda A + B$ is nonsingular. Otherwise, the matrix pencil is called singular.

Note: If (A, B) is a singular pencil, the homogeneous initial value problem

$$Ax' + Bx = 0, \quad x(0) = 0$$

has infinitely many solutions!

The Index of a Dae

Theorem: For any regular matrix pencil (A, B), there exist nonsingular matrices E, F such that

$$EAF = \begin{pmatrix} I & \\ & J \end{pmatrix}, \quad EBF = \begin{pmatrix} W & \\ & I \end{pmatrix},$$

where J is a nilpotent Jordan block matrix,

$$J^{\mu-1} \neq 0, J^{\mu} = 0 \text{ for some } \mu \in \mathbb{N}.$$

By definition $\mu = 0$ means that the second block row is missing, i.e., A is nonsingular.

μ is called the *index* of the matrix pencil (A, B) and of the dae.

$\begin{pmatrix} I & \\ & J \end{pmatrix}, \begin{pmatrix} W & \\ & I \end{pmatrix}$ is called the *Kronecker canonical form* of the pencil (A, B).

The Index of a Dae (cont)

Use the change of variables

$$\begin{pmatrix} y \\ z \end{pmatrix} = F^{-1}x$$

and scale the dae by E:

$$EAF(F^{-1}x)' + EBF(F^{-1}x) = Eq(t)$$

$$\begin{pmatrix} I & \\ & J \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}' + \begin{pmatrix} W & \\ & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p(t) \\ r(t) \end{pmatrix}$$

or, equivalently:

$$y' + Wy = p(t),$$
$$Jz' + z = r(t).$$

The first equation is a usual (explicit) ode.

Why Is The Index Important?

$\mu = 0$ The second row is missing. The dae is in fact an ode.

$\mu = 1$ Hence, $J = 0$ such that the second row reads

$$z = r.$$

Nothing special.

Note: While y is obtained by *integrating*, z is determined by a pure algebraic realtion. Moreover, the initial value for z cannot be chosen freely, it is fixed by the right hand side r

Higher Index Problems: $\mu = 2$

Here, $J^1 \neq 0$ while $J^2 = 0$. Multiply the second row by J:

$$J^2 z' + Jz = Jr \Longrightarrow (Jz)' = (Jr)' \Longrightarrow$$
$$z = r - Jz' = r - (Jr)'$$

- Some components of z are given by algebraic relations, *others by differentiated components of the right hand side*.
- Our “integration” problem contains a *differentiation problem*!

What is bad with differentiation problems?

- Consider $x(t) = q'(t)$
- Add a small perturbation, $x_\varepsilon(t) = (q + \varepsilon\cos(\omega t))'$
- It holds $\|q - (q + \varepsilon\cos(\omega\cdot)\| = |\varepsilon|$
- Nevertheless, $\|x - x_\varepsilon\| = |\omega\varepsilon|$ may become arbiritrarily large!
- *Differentiating is an ill-posed problem.*

Notes

- For $\mu \geq 3$, it holds

$$z = \sum_{j=0}^{\mu-1} (-1)^j (J^j r)^{(j)}$$

- Relations of the type $(Jz)' = (Jr)'$ are called *hidden constraints.*
- Initial value problems become solvable for *consistent initial values*, only,

$$x(0) = F\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = F\begin{pmatrix} y_0 \\ z(0) \end{pmatrix}$$

  This includes the hidden constraints!

Index Reductions

Start with a semiexplicit index-1 system,

$$y' = -B_{11}y - B_{12}z + p,$$
$$0 = -B_{21}y - B_{22}z + r.$$

This dae has index 1 if and only if B_{22} is nonsingular.

Then:

$$z = B_{22}^{-1}(-B_{21}y + r),$$
$$y' = (-B_{11} + B_{12}B_{22}^{-1}B_{21})y + p - B_{12}B_{22}^{-1}r.$$

- This system is an index-1 system as before.
- The differential equation is completely decoupled such that the ode theory applies.
- Can one even avoid the assignment for z?

Index Redutions (cont)

Differentiate the constraint in the original dae:

$$0 = -B_{21}y' - B_{22}z' + r'$$

Then, the system reads:

$$\begin{pmatrix} I & 0 \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p \\ r' \end{pmatrix}.$$

This is an index-0 dae (an ode)

Conclusion: By differentiation of the algebraic constraint, the index can be reduced by one.

- The index-reduced system is *not* equivalent to the original one!
- The new system has more degrees of freedom (initial values for z).
- How to do that in more implicitely given systems?
- Do there exist better index reduction methods? Yes and No!

The Differentiation Index

Definition: For a general dae $F(x', x, t) = 0$, the *index* along a solution trajectory $x(t)$ is the minimum number of differentiations of the system which would be required to solve for x' uniquely in terms of x and t.

- If there does not exist such a value, the index is undefined.
- For many applications, the index is known or can easily be checked. (Introductory examples!)
- Often, structural considerations help, but this may be misleading.
- *The differentiation index may underestimate the sensitivity with respect to perturbations.*

Examples

1. Consider

$$x_1' = x_3,$$
$$0 = x_2(1 - x_2),$$
$$0 = x_1 x_2 + x_3(1 - x_2) - t.$$

The system has two (continuous) solutions, one with $x_2 \equiv 0$ and one with $x_2 \equiv 1$.

- If $x_2 \equiv 0$, the system has differentiation index 1.
- If $x_2 \equiv 1$, the system has differentiation index 2.

2. Consider the system

$$x_1' = x_3,$$
$$x_2' = 0,$$
$$0 = x_1 x_2 + x_3(1 - x_2) - t.$$

Now, the index depends on the initial conditions. If $x_2(0) = 0$, the index is 1, and if $x_2(0) = 1$, the index equals 2.