Mathematical Models, Analysis and Simulation
Part I, Fall 2009

August 21, 2009

Homework 6, Two-point Boundary Value Problems with FEM, Max. Score: 7.0, Deadline Sun, Nov. 22

In this homework you study two point boundary value problems, using a matlab program `adfem`. The program is in an ascii-file available on the WWW page of the course. Save it and extract the individual files by using an editor of your choice. There is a readme file with useful information. A copy of the m-files can be found in `/info/mmas/part1/studentsm-files` and on the course's home page.

The program solves the two point boundary value problem

$$\begin{aligned} -(du')' + cu' + ru &= f \text{ in } (a,b), \\ u(a) = ... \quad &\text{or} \quad u'(a) + ku(a) = ... \\ u(b) = ... \quad &\text{or} \quad u'(b) + ku(b) = ... \end{aligned} \tag{1}$$

with piecewise linear finite elements. The program starts by the command "`adfem`" in matlab. `Adfem` solves (1) adaptively by error control in energy norm, L_2 norm or pointwise. The error control is based on *a posteriori* error estimates of the form

$$\|u - U\| \leq C\|R(U)h^k\|. \tag{2}$$

The constant C is in the case of L_2 and L_∞ norms determined by solving a certain dual problem. The function $R(U)$ measures the residual.

(2.0) 1a. Consider the boundary value problem

$$(a(x)u')' = f(x), x \in (0,1), u(0) = u'(1) = 0, \tag{3}$$

with natural (Neumann) boundary condition at $x = 1$ and essential (Dirichlet) condition at $x = 0$. Determine the stiffness matrix and the load vector in the case $f(x) = 1, a(x) = 1 + x$ for a FEM for (3) with piecewise linear elements and a uniform mesh. In particular, study by numerical experiments how the natural boundary condition is approximated by the FEM.

(2.0) 1b. Solve problem 1a with the program `adfem` and compare the cases

$$a(x) = 1,\ f(x) = 1/x^{3/2},\ (i), f(x) = 1,\ (ii) \tag{4}$$

and explain why error control in the energy norm for case (i) does not work. Test other norms. Do they work? Try large tolerance first.

(1.0) 2a. *A tent problem.*

Consider an elastic membrane spanned over a circular ring. Investigate if it is possible to support the membrane, which is loaded by its own weight, by using a thin supporting pole in the center of the membrane.

Using polar coordinates with r the distance to the center and assuming radial symmetry, this problem may be formulated as the following boundary value problem:

$$-(ru')' = rf(r), r \in I = (0, 1), u(0) = 1, u(1) = 0, \tag{5}$$

where $f(r) = -1$, and $u' = u'(r) = \frac{du}{dr}$. This is because the transversal displacement u of an elastic membrane may be modeled by $-\Delta u = f$ in two dimensions and in polar coordinates with radial symmetry $\Delta u = \frac{1}{r}(ru')'$. The support of the membrane in the center by means of a thin pole is modeled by the boundary condition $u(0) = 1$.

The basic question is now if the boundary value problem (5) has a reasonable solution $u(r)$ or not. If yes, you can go ahead and start the production of the membrane with supporting pole and the design should work in practice. If not, you may want to reformulate (5) into something more reasonable and accordingly redesign the support of the membrane at $r = 0$. To investigate the nature of the solutions of (5) you may use the program `adfem` and see what you get (the teacher might get problems with the system manager if you do not *stop* the computation, with "control c", after a (short) while). You may also try to solve (5) analytically and see what you get.

In case you find the proposed model (5) not reasonable, propose a modified better model and a corresponding improved design of the support of the membrane.

(2.0) 2b. We know that boundary value problems which are minimizers of positive definite problems have a unique solution. What about (5), are the requirements satisfied? At the heart of the matter is the boundary condition $u(0) = 1$. Can we really specify the value of $u(0)$ in the model? This question is related to the possibility to find functions u_n, satisfying the boundary conditions $u_n(0) = 1, u_n(1) = 0$, such that the internal elastic energy of the membrane tends to zero, i.e

$$\int_0^1 r(u_n'(r))^2 dr \to 0, \quad \text{as } n \to \infty. \tag{6}$$

Try to find an analytic or a computational answer whether such a sequence can be found, and the properties of its limit (if any).