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2D1266, Mathematical models, Analysis and Simulation, part I
Saturday January 12th 2002, 8-13

Closed book examination 5 hours. A sum of credits of 50, homework included, is certainly enough to pass. The results will be announced no later than January 7th.

P1. Given the following $n \times n$-matrix A with elements

$$a_{ii} = a, \quad i = 1, \dots, n \qquad a_{ij} = 1, i \neq j$$

- **a) (2)** What is $rank(A)$ when i) $a = 1$, ii) $a \neq 1$? Motivate your answer.
- **b) (3)** Which are the eigenvalues of A?
- **c) (2)** For which values of a is A positive definite?

P2. For the solution of the normal equations $A^T Ax = A^T b$, the following iterative method can be used:

$$x_{k+1} = x_k - hA^T(Ax_k - b), \quad h > 0$$

The matrix A has dimension $m \times n, \quad m > n$.

- **a) (2)** Let $P = A^T A$ have the eigenvalues $\lambda_i, \quad i = 1, 2, \dots n$. Show that they are nonnegative.
- **b) (2)** Let the solution of the normal equations be x^*. Let $e_k = x^* - x_k$. Show that $e_{k+1} = B(h)e_k$, where $B(h) = I - hP$.
- **c) (1)** Assume that a lower bound $a > 0$ and an upper bound b are known for λ_i. Give upper and lower bounds for the eigenvalues of B.
- **d) (2)** Prove that the iteration converges for $h = 1/b$.
- **e) (3)** With the information given in c) give the best value of h, i.e. the value for which the convergence is fastest.

P3. Given the vibration equation in mechanics

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F \sin \omega t \quad (\star)$$

where all parameters m, c, k, F and ω are positive quantities.

- **a) (3)** When there is no driving function, i.e. when the right hand side is zero, which are the critical points of the differential equation? Are they stable? Sketch the phase portraits in the neigbourhood of the critical points in the three cases i) $c^2 - 4km > 0$ (2 real roots λ_1, λ_2), ii) $c^2 - 4km = 0$ (1 real double root) and iii) $c^2 - 4km < 0$ (2 complex roots).

b) (3) Show that the general solution of ($\star$) can be written

$$u(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + Asin(\omega t) + Bcos(\omega t)$$

What is λ_1 and λ_2? Show that the first two terms approach zero when t increases in all three cases i), ii) and iii) given above.
Hence the stationary solution will consist of the last two terms.

b) (2) Show that the amplitude C of the stationary solution is

$$C = \frac{F}{\sqrt{(m\omega^2 - k)^2 + (c\omega)^2}}$$

Sketch the curves $C(\omega)$ on an ω-interval $[0, \Omega]$ for a small value of c and a large value of c.

P4. Given the boundary value problem

$$-\frac{d^2u}{dx^2} + u = f(x), \quad u(0) = 0, \quad u(1) = 0$$

a) (3) In the weak form (variational form) formulation: Find $u \in V$ such that

$$a(u, v) = L(v), \qquad for \quad all \quad v \in V$$

specify the bilinear form $a(u, v)$, the functional $L(v)$ and the function space V.

b) (3) In the Galerkin method formulation of the problem: Find $u_h \in V_h$ such that

$$a(u_h, v) = L(v), \qquad for \quad all \quad v \in V_h$$

describe the function space V_h in case V_h consists of "roof" functions defined on an equidistant grid on $[0, 1]$. If the stepsize h is $h = 1/3$, how does the ansatzfunction look like? What is the dimension of the matrix A assiciated with $a(u_h, v)$?

P5. Consider the conservation law $u_t + (Q(u))_x = 0$, where $Q(u) = u(1 - u)$.

a) (2) Formulate the upstream method for the initial value problem $u(x, 0) = f(x)$, $-\infty < x < \infty, \quad f(x) > 0$.

b) (3) Write the equation for the characteristics and show that i) u is constant along a characteristic, ii) the characteristics are straight lines.

c) (2) At $t = 0$, the concentration has a discontinuity at $x = 0$:

$$f(x) = 1/4, \quad x \leq 0, \qquad f(x) = 3/4, \quad x > 0$$

Find the shock speed and plot the solution at $t = 1$.

d) (2) Suppose that $Q(u) = u(1 - u - u_x/u)$. You may assume that $u > 0$. Sketch how this would change the plot of c).

P6. Given a least squares problem with linear constraints

$$\min_{Cx=c} \frac{1}{2} \|Ax - a\|_2^2$$

where A is $m \times n$, $m > n$, $rank(A) = n$ and C is $p \times n$, $p < n$, $rank(C) = p$. The x-value for which the minimum is taken is denoted by $\hat{x}$.

a) (3) Show that $\hat{x}$ satisfies the following linear system of equations

$$\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T a \\ c \end{pmatrix}$$

where λ is the Lagrange multiplier vector connected to this constrained problem.

b) (3) The matrix in the linear system in a) is obviously symmetric. Is it also positive definite? Give a proof or give a counter example.

c) (4) If the linear system in a) is solved we can write the solution as

$$\hat{x} = Pa + Qc$$

Give the explicit form of the two matrices P and Q as functions of A and C.