Lecture 4: Dynamical systems, differential equations

Strang: Ch. 1.5, Oscillations, Ch 2.2;

The name of the game is to derive properties of sets of solutions to a differential (or difference) equation. Often the set is generated by an initial value problem with different initial conditions and/or different "parameter" values. The standard form is a system of s first order equations,

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u},t), \mathbf{u}(0) = \mathbf{u}_0, u = (u_1(t), u_2(t), \ldots, u_s(t))^T, \mathbf{f} = (f_1(\mathbf{u},t), \ldots, f_s(\mathbf{u},t))^T$$

u is the state-vector. Higher order equations are easily rewritten as systems by introducing new variables for the derivatives. The solution is visualized either in R^s x [0,T] or as a parametrized curve in R^s: the phase space (Strang: "Shows where but not when"). Often we write the system in *autonomous* form (no explicit dependence on t), no restriction, we can always add the equation $du_{s+1}/dt = 1$ and use u_{s+1} for *t.*

Existence and uniqueness

Def. f is Lipschitz continuous in D if, for **x** and **y** in D, there is an L such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|$$

Thm. Let **f** be Lipschitz continuous in an open domain D containing $\mathbf{u}_0$. Then, the initial value problem has a unique, continuously differentiable solution on $[0,T)$ for some $T > 0$.

Qualitative theory

"where but not when …"

Growth, decay and oscillations; periodic solutions, critical points, phase portraits for 2D linear systems, Lyapunov stability

Non-linear

- Limit cycles
- Poincare-Bendixson
- Center Manifold theorem

Critical points

If $\mathbf{f}(\mathbf{u}^*) = 0$ we call $\mathbf{u}^*$ a critical point. Any critical point is a constant solution to the differential equation. One studies the behavior of solutions which make small excursions in the neighborhood of u* by *linearization*:

Say $\mathbf{u}^* + \mathbf{v}(t)$ is a solution, then

$$\frac{d}{dt}(\mathbf{u}^* + \mathbf{v}) = \frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{u}^* + \mathbf{v}) = \underbrace{\mathbf{f}(\mathbf{u}^*)}_{0} + \underbrace{\mathbf{J}(\mathbf{u}^*)}_{\substack{\text{constant} \\ \text{matrix } \mathbf{A}}}\mathbf{v} + O(\|v\|^2), J_{ij} = \frac{\partial f_i(\mathbf{u})}{\partial u_j}$$

and unless **A** vanishes (too much …) the dynamics is determined by **A**. It is possible to make a precise description of the different types of phase portraits possible for a 2 x 2 system. The key is the ability to write the analytical solution in terms of the eigen values and vectors (and a little more in the non-diagonalizable case).

For example,

Definition: Lyapunov stability: A solution **u** is L.-stable for $t > 0$ if, whenever another solution **v** has $\|\mathbf{u}(0)-\mathbf{v}(0)\| < \varepsilon$, then **v** eventually never strays more than $E(\varepsilon)$ from **u** where $E(\varepsilon)$ is continuous at 0.

Asymptotic stability means that $\lim_{t\to\infty} \|\mathbf{u}(t) - \mathbf{v}(t)\| = 0$ whenever ε is small enough.

Example: The mass-spring-damper model
The spring is Hookean with spring constant *K*, the dashpot gives force proportional to velocity, damping coefficient *D*, and the device is acted upon by a time-harmonic force $F\cos\omega t$

$$ma = f : m\frac{dv}{dt} = -Kx - Dv + F\cos\omega t;$$

$$\text{Kinematics, relation between velocity } v \text{ and position } x : \frac{dv}{dt} = v$$

This model has five parameters (m,K,D,F,ω), but by choosing suitable scaling factors (equivalent to using different units) we can bring the number of parameters down to two:
Set $x = Lx'$ and $t = T\,t'$ with x' and t' non-dimensional length and time

$$\frac{mL}{T^2}\frac{d^2x'}{dt'^2} + KLx' + D\frac{L}{T}\frac{dx'}{dt'} = F\cos\omega Tt';$$

$$\frac{d^2x'}{dt'^2} + \frac{KT^2}{m}x' + \frac{DT}{m}\frac{dx'}{dt'} = \frac{FT^2}{mL}\cos\omega Tt';$$

Several choices: We take the time-scale from the harmonic forcing, $T = 1/\omega$
and the length scale, too, $L = FT^2/m = F/(\omega^2 m)$, and skip the primes on variables:

$$\frac{d^2x}{dt^2} + ax + b\frac{dx}{dt} = \cos t;\ a = \frac{K}{m\omega^2}, b = \frac{D}{m\omega}$$

where now the spring constant *a* and the damping coefficient *b* are non-dimensional and properly scaled.
The state vector is $\mathbf{u} = (x,v)^T$, the first order system is

$$\frac{d\mathbf{u}}{dt} + \mathbf{Au} = \mathbf{f}, \mathbf{A} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \cos t \\ 0 \end{pmatrix}$$

See Strang pp xx for the discussion on the solutions and the different phase portraits for 2D linear systems.
Note: The matrix exponential is easily computed when **A** is diagonalizable, but the critically damped **A** has only one eigenvector.

On energy conservation
The work done against the spring force is stored as elastic energy in the spring and can be converted into other forms subsequently. The work done against the damping force is another kind: it is a dissipative process which converts mechanical energy into heat. The system above must be augmented by a container for the thermal energy to be closed in the thermodynamical sense. Assume that the heat produced in the damper heats a mass with thermal capacity *C* (J/K), and temperature *T*, then, in the absence of driving force,

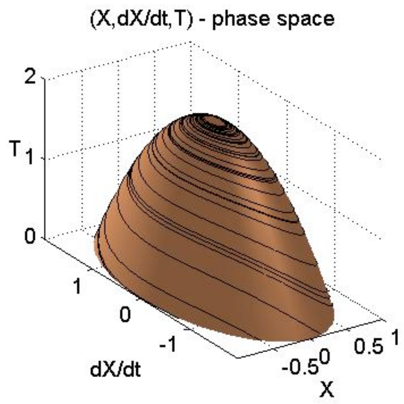
$$\frac{d^2x}{dt^2} + ax + b\frac{dx}{dt} = 0;$$

$$C\frac{dT}{dt} = b\frac{dx}{dt}\cdot\frac{dx}{dt} \geq 0 \quad \text{(power = force x velocity, non-negative!)}$$

The augmented system admits an integral:

$$E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{a}{2}(x)^2 + CT$$

because $dE/dt = 0$ (multiply the force equation by dx/dt, add to the T-equation, and check that this = dE/dt). Note that it does not depend explicitly on b, but the trajectories do, of course. The trajectories in this 3D phase space are on paraboloids (different for different initial data) with the T-axis as axis, and elliptical cross section. If $b > 0$ a trajectory moves towards its apex, where T is largest: all kinetic energy converts to heat, eventually. For the underdamped case, the motion is a spiral; the over- and critically damped cases gives more monotonic curves. If $b = 0$ the motion stays on a plane $T = T(0)$ and is periodic.



Ellipses and hyperbolas

Consider a quadratic form without cross terms, $Ax^2 + By^2 = C$ where A is positive. Divide by $|C|$ to produce

$$\frac{x^2}{|C|/A} + \text{sgn}(B)\frac{y^2}{|C|/|B|} = \text{sgn}(C)$$

$$\left(\frac{x}{a}\right)^2 + \text{sgn}(B)\left(\frac{y}{b}\right)^2 = \text{sgn}(C), a = \sqrt{|C|/A}, b = \sqrt{|C|/|B|}$$

Four cases:

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       sgn(B) |       +       |       -       |
sgn(C)    ---+---------------+---------------+
          +  | Ellipse       | Hyperbola(x)  |
          ---+---------------+---------------+
          -  | Empty         | Hyperbola(y)  |
          ---+---------------+---------------+
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Lanchester's law of squares

As a simple, deterministic model of a battle, suppose that $R(t)$ red and $G(t)$ green units begin fighting at t=0, and that each unit destroys r or g (the *fighting effectiveness*) enemy units in one unit of time, so that

$$\frac{dR}{dt} = -gG, \frac{dG}{dt} = -rR, G(0) = G_0, R(0) = R_0 \qquad (1)$$

The fight stops when one army has been obliterated. We leave aside any discussion of the realism of the model.

Which side wins? – has an easy answer

How long does it take? – hard work

We first note that both R and G decrease monotonically until the end.

Solution

1) the hard way:

$$\mathbf{u} = \begin{pmatrix} R \\ G \end{pmatrix}, d\mathbf{u}/dt = -\begin{pmatrix} 0 & g \\ r & 0 \end{pmatrix}\mathbf{u},$$

$$\lambda_{1,2} = \pm\rho = \pm\sqrt{rg}, \mathbf{S} = \begin{pmatrix} \sqrt{g} & \sqrt{g} \\ \sqrt{r} & -\sqrt{r} \end{pmatrix}, \mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{1/g} & \sqrt{1/r} \\ \sqrt{1/g} & -\sqrt{1/r} \end{pmatrix}$$

$$\mathbf{u}(t) = \mathbf{S}\begin{pmatrix} e^{-\rho t} & 0 \\ 0 & e^{+\rho t} \end{pmatrix}\mathbf{S}^{-1}\mathbf{u}(0) = \frac{1}{2}\mathbf{S}\begin{pmatrix} e^{-\rho t}(\sqrt{1/g}R_0 + \sqrt{1/r}G_0) \\ e^{+\rho t}(\sqrt{1/g}R_0 - \sqrt{1/r}G_0) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-\rho t}(R_0 + \sqrt{g/r}G_0) + e^{+\rho t}(R_0 - \sqrt{g/r}G_0) \\ e^{-\rho t}(\sqrt{r/g}R_0 + G_0) - e^{+\rho t}(\sqrt{r/g}R_0 - G_0) \end{pmatrix}$$

$$= \begin{pmatrix} R_0 \cosh\rho t - \sqrt{g/r}G_0 \sinh\rho t \\ G_0 \cosh\rho t - \sqrt{r/g}R_0 \sinh\rho t \end{pmatrix}$$

G wins if R becomes zero. That happens, if ever, when $\tanh\rho t = \frac{R_0}{G_0}\sqrt{\frac{r}{g}}$. But only when $\frac{R_0}{G_0}\sqrt{\frac{r}{g}} < 1$, or $rR_0^2 < gG_0^2$

which is Lanchester's law of squares:

Numbers count as much as fighting effectiveness squared.

2) the easy way: Multiply the R equation by rR and the G equation by gG and *subtract*:

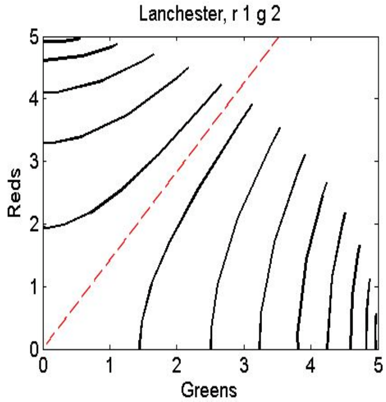
$$rR\frac{dR}{dt} - gG\frac{dG}{dt} = -rgGR + grRG = 0$$

$$so: rR^2 - gG^2 = const. = rR_0{}^2 - gG_0{}^2$$

Since both R and G decrease, it follows that if the *const.* > 0, R must prevail, since R = 0 would render the expression negative. Another way of arriving at the result is to disregard time t and consider the (R,G)-plane. From the R and G equations immediately follows that

$$\frac{dR}{dG} = \frac{-gG}{-rR}, rRdR - gGdG = 0$$

The traces of the time histories $R(t)$ and $G(t)$ on the (R,G)-plane are hyperbolas with asymptotes forming angle $\phi = \arctan\sqrt{\frac{r}{g}}$ with the axes.



Balls on springs: the Lagrange way

The equations of motion can be easily derived from the Lagrange function – the difference between the kinetic energy T and the potential energy W, when such exists for the acting external forces. This immediately implies that the *sum* of the kinetic and potential energies is a constant of motion.

Lagranges' equations of motion for a system described by s degrees of freedom x_i

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, i = 1,2,\ldots,s$$

The balls & springs example:

$$T = 1/2 m_1 \dot{x}_1^2 + 1/2 m_2 \dot{x}_2^2$$

$$W = -m_1 g x_1 - m_2 g x_2 + 1/2 K_1 (x_1 - l_1)^2 + 1/2 K_2 (x_2 - x_1 - l_2)^2$$

$$L = T - W : \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) + \frac{\partial W}{\partial x_i} = 0$$

$$\frac{d}{dt}(m_1 \dot{x}_1) = m_1 \ddot{x}_1 = m_1 g - K_1 (x_1 - l_1) + K_2 (x_2 - x_1 - l_2)$$

$$m_2 \ddot{x}_2 = m_2 g - K_2 (x_2 - x_1 - l_2)$$

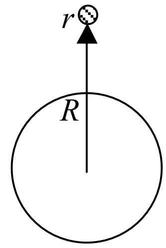
$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{\text{Mass matrix}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}}_{\text{Stiffness matrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m_1 g + K_1 l_1 - K_2 l_2 \\ m_2 g + K_2 l_2 \end{pmatrix}$$

The Lagrange formalism avoids the introduction of internal forces which have subsequently to be eliminated, and deals elegantly with systems described by any coordinates (angles and distances and …). It needs extension to handle dissipative elements, like joints with friction. The equations of motion for the double pendulum come out easily, anyway.

Simplification & scaling

The projectile problem (Lin & Segel):

A projectile fired vertically from surface of planet, under gravity



$$m\frac{d^2r}{dt^2} = \frac{-GmM}{(R+r)^2}; GM/R^2 = g:$$

$$\frac{d^2r}{dt^2} = \frac{-gR^2}{(R+r)^2}, r(0) = 0, \frac{dr}{dt} = V$$

Three parameters: R (earth radius), g (gravitational acceleration on surface), and V, initial velocity, and two scales: length and time, to choose.

Look for a non-dimensional combination

$$\pi = g^x R^y V^z$$

and the conditions are

length units: x + y + z = 0

time units: 2x + z = 0, so z = -2x, y = -(x+z) = x

and

$$\pi = (gR/V^2)^x$$

This is the single parameter we expect to wind up with in our non-dimensional formulation of the model.

Choose length scale L, $r = Lr'$ and time scale T, $t = T t'$, and then skip the primes:

$$\frac{L}{T^2}\frac{d^2r'}{dt'^2} = \frac{-gR^2}{(R+Lr')^2}, r'(0) = 0, \frac{L}{T}\frac{dr'}{dt'} = V$$

$$\frac{d^2r}{dt^2} = -\frac{gT^2}{L}\frac{1}{(1+\frac{L}{R}\cdot r)^2}, r(0) = 0, \frac{dr}{dt} = \frac{VT}{L}$$

First attempt: Unsuccessful

An obvious choice is

$L = R$ – compare projectile altitude with earth radius, and

$T = R/V$ - time to travel R

$$\frac{d^2r}{dt^2} = -\alpha\frac{1}{(1+r)^2}, r(0) = 0, \frac{dr}{dt}(0) = 1, \alpha = \frac{gR}{V^2}$$

Consequences:

Small V gives α large, and no interesting limit of the dynamics.

With α small (0), the dynamics reduces to $dr/dt = V$ – a particle not influenced by gravity.

We conclude that this scaling is uninformative and does not help in simplifying the model.

Next attempt: Successful:

Choose L and T to make $gT^2/L = 1$, and $VT/L = 1$:

$$T = V/g, L = V^2/g$$

This makes $L/2$ the distance traveled by a particle accelerating at g from rest, and T is the time for retardation by g from V to 0.

$$\frac{d^2 r}{dt^2} = -\frac{1}{(1+\beta r)^2}, r(0) = 0, \frac{dr}{dt}(0) = 1, \beta = \frac{1}{\alpha} = \frac{V^2}{gR}$$

Consequences:
$\beta = 0$ gives $d^2r/dt^2 = -1$, $r(t) = t-t^2/2$, and the max. height reached is 1/2.
large β i.e. large V means gravity doesn't much retard the movement: $r(t) = t$.
We may try to find an approximation for small but non-zero β. Assuming that r is a regular function of β, we make the power series ansatz $r = \sum \beta^j r_j(t)$

$$(1+\beta(\sum \beta^j r_j)(\beta \sum \beta^j r_j + 2)) \cdot \sum \beta^j r_j'' = -1:$$

$$\beta^0 : r_0'' = -1 : r_0 = t - t^2/2$$

$$\beta^1 : r_1'' + 2r_0 r_0'' = 0 : r_1'' = 2t - t^2, r_1 = t^3/3 - t^4/12$$

etc.

A population model

It is reported in DN Sep. 13 that the Russian nativity is 1.4 children per woman. Unless the boy/girl distribution is very skewed, in the absence of immigration, the Russians face severe population decline. How severe?
We assume that as many boys as girls are born, then it is enough to count the women. Let *X* be the number of infant women, and *Y* be the number of fertile women, and *Z* the number of babushkas.

Consider the population at time t_n, where a timestep $t_{n+1} - t_n$ is chosen to be a sizable fraction of the age-spans of the three groups.
In a timestep,

- A net fraction *a* of the infants mature. Mortality is neglected.
- The fertiles give birth to *bY* female infants, and a fraction *c* becomes old (or die);
- A fraction *d* of the old die.

Thus,

$$x_{n+1} = (1-a)x_n + by_n; y_{n+1} = ax_n + (1-c)y_n; z_{n+1} = cy_n + (1-d)z_n$$

$$\mathbf{u}_{n+1} = \begin{pmatrix} 1-a & b & 0 \\ a & 1-c & 0 \\ 0 & c & 1-d \end{pmatrix} \mathbf{u}_n, \begin{matrix} 0 < a < 1 \\ 0 < b < \infty \\ 0 < c < 1 \end{matrix}$$

The qualitative question is: How large must the parameter *b* be for the population to be stable, and what is the equilibrium age-distribution? Note: $b > 1$ is OK
Remembering the Stochastic matrix discussion, or rather the Perron-Frobenius theorem, we see that:
The eigenvalue of **A** with largest real part is positive and has a positive eigenvector which is a stable equilibrium. **A** has one eigenvalue $1-d$, eigenvector $(0,0,1)^T$ which cannot dominate. The product of the eigenvalues of the upper left 2x2 part is $(1-a)(1-c) > 0$ so *all* eigenvalues are positive, and also the eigenvectors.

There remains only to find the critical *b*-value for which there is an eigenvalue 1:

$$(1-a-1)(1-c-1) - ab = 0, \because b = c$$

- the *a*-coefficient is determined only by the timestep chosen and the length of infancy and as such does not influence the population dynamics.
- The *c*-coefficient can include untimely death, hence can be controlled by health measures, but only to a certain degree. There remains to control *b*!
- … Nativity must exactly match the loss of fertile individuals to *age* and untimely death.

How do the numbers come out? Assume that fertility is *F* yrs. With a time step of *k* yrs we get $c = k/F$. So the equilibrium *b* is k/F; over the whole fertility period $F/k\, b = 1$, and including boys a nativity of 2 would be required, which we knew, of course, without doing the eigenvalues …
How quickly does the population decay with nativity 1.4? Assuming infancy is 20 yrs, and fertility 30, and a timestep of 10,

$$\mathbf{A} = \begin{pmatrix} 0.5 & 0.7/3 & 0 \\ 0.5 & 2/3 & 0 \\ x & x & x \end{pmatrix}, \lambda_{\max} = 0.93$$

a loss of 7% in 10 yrs, or 18.3% in 30 yrs.

Note: With the equilibrium b, the age profile predicted by the model is completely flat: (X, Y) is proportional to (2,3) – matching the assumed timespans of infancy and fertility. This is another result of our neglect of untimely mortality.