Lecture 4: Dynamical systems, differential equations

(as the lecture was given, not as it was planned :-))

Phase Space analysis

The differential initial value problem is $d\mathbf{u}/dt = \mathbf{f}(\mathbf{u})$, $\mathbf{u}(0) = \mathbf{u}0$, with $\mathbf{u}(t)$ a real s-vector.
We "plot" the solution in R^s x $[0,T)$ and look at the projection on R^s. As time runs, the point traces out a curve, the trajectory. Different initial values give a family of curves, which portrays the properties of the solution set, hence called *the phase portrait*.
We think of the differential equation as giving, for each point, the tangent of the trajectory; plotting many such arrows (Matlab: `quiver`) also produces a phase portrait. The length of the arrow is the "velocity", $\|d\mathbf{u}/dt\|_2$. Going back to lect 1 and the rigid rotation,

$$dx/dt = -\omega y, dy/dt = \omega x$$

$$\frac{d\mathbf{u}}{d(\omega t)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{u}$$

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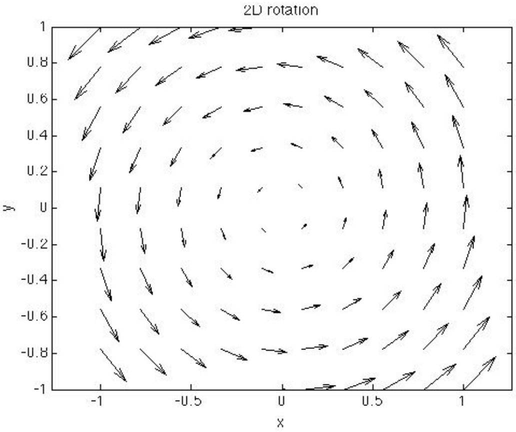
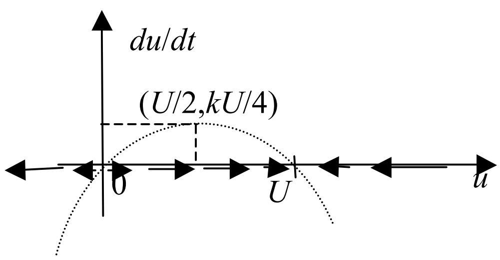
x = linspace(-1,1,10);
[xx,yy]=meshgrid(x,x);
dxdt = -yy;
dydt =  xx;
quiver(xx,yy,dxdt,dydt,'k');
axis ([-1 1 -1 1])
axis equal
```



The idea of "flow" immediately makes one think of where the flow can go, i.e., which part of the phase space is accessible for initial values in some domain, say D.
Folklore theorem: Let D be nice enough to have a boundary Γ which possesses an outward normal $\mathbf{n}$ everywhere (except at isolated points). Then, if $\mathbf{f}(\mathbf{u}).\mathbf{n} \leq 0$ on Γ, a solution starting in D can never leave.
Folklore proof: The conclusion is obvious for strict inequality $\mathbf{f}(\mathbf{u}).\mathbf{n} < 0$. It is possible to show that it holds also for $\leq$, but one has to restrict Γ not to have cusps (corners with 0 interior angle).

Example –
The "Logistic" growth equation $\dot{u} = a(u)u, a = k(1 - u/U)$ models growth, impeded by "crowding". If u grows to U, the growth stops. k is the net growth rate for small population. One can write an analytical solution, but an inspection of the phase space – the real line– reveals its main features. The dotted parabola is du/dt vs. u, as read off from the differential equation.



For negative u, du/dt < 0, and $u(t)$ decreases monotonically and ever faster;
In $0 < u < U$, u grows, fastest for $u = U/2$, slows down close to U. For $u > U$, u decreases towards U. The Folklore theorem

guarantees that u stays in $D = [0,U]$ forever if started there. The points 0 and U are critical points u^* (see Lect. 4). 0 is ***unstable***: $u(0)>0$ makes u tend to U, and $u(0)<0$ makes u decrease forever. U is *asymptotically stable,* indeed, any positive $u(0)$ gives $\lim_{t\to\infty} u(t) = U$.

Linearization gives $J(u) = k(1-2u/U)$, for the perturbation $w(t)$ from u^* linear, constant coefficient equations:

$$0: J = -k, w = w_0 e^{-kt} \to 0$$

$$U: J = +k, w = w_0 e^{+kt} \to \infty$$

which confirms and details our conclusions from study of the phase line.

The second example is the classical predator-prey model named for Vito Volterra and James Lotka, developed in WW1. Consider a population of u_1 foxes (predators) and u_2 rabbits (prey). The number of encounters per unit time will be proportional to the product $u_1 u_2$, by reasoning similar to the mass action law of chemical kinetics. In the absence of prey, the fox population declines, and in the absence of foxes, the rabbit population grows rapidly. So, with the a_i and b_i positive,

$$\begin{aligned} \frac{du_1}{dt} &= u_1(b_1 u_2 - a_1) \\ \frac{du_2}{dt} &= u_2(a_2 - b_2 u_1) \end{aligned} \qquad (1)$$

First we note that the solution stays positive: Let D be the positive quadrant, then on its boundary $u_1 = 0$ $\mathbf{n} = (-1,0)$ and $\mathbf{n}.d\mathbf{u}/dt = 0$; on $u_2 = 0$, $\mathbf{n} = (0,-1)$ and $\mathbf{n}.d\mathbf{u}/dt$ vanishes there, too.
Critical points:

$$\frac{du_1}{dt} = 0: u_1 = 0 \cup u_2 = a_1 / b_1$$

$$\frac{du_2}{dt} = 0: u_2 = 0 \cup u_1 = a_2 / b_2$$

so 1) $\mathbf{u}^* = (0,0)$ and 2) $\mathbf{u}^* = (a_2/b_2, a_1/b_1)$. The Jacobian is

$$\mathbf{J}(\mathbf{u}) = \begin{pmatrix} b_1 u_2 - a_1 & b_1 u_1 \\ -b_2 u_2 & a_2 - b_2 u_1 \end{pmatrix};$$

$$\mathbf{J}(\mathbf{0}) = \begin{pmatrix} -a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \mathbf{J}(\mathbf{u}_2^*) = \begin{pmatrix} 0 & \frac{b_1 a_2}{b_2} \\ -\frac{b_2 a_1}{b_1} & 0 \end{pmatrix}$$

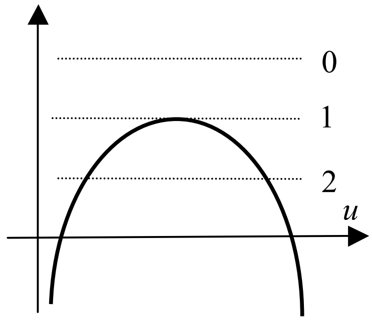
For perturbations $\mathbf{w}$ from $\mathbf{u}^*$,

$$0: \mathbf{w}(t) = e^{\begin{pmatrix} -a_1 & 0 \\ 0 & a_2 \end{pmatrix} t} \mathbf{w}(0) = \begin{pmatrix} e^{-a_1 t} u_1(0) \\ e^{a_2 t} u_2(0) \end{pmatrix}$$

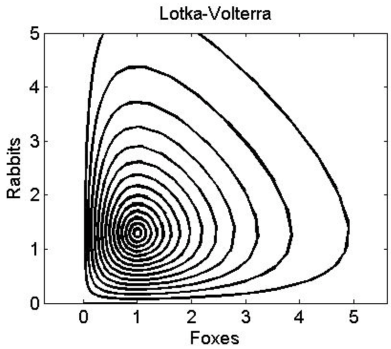
is *unstable*, a saddle point: u_1 (the foxes) disappear and u_2 grows. The positive point $\mathbf{u}_2^*$: the eigenvalues of $\mathbf{J}(\mathbf{u}^*)$ are $\pm i\sqrt{a_1 a_2}$, and the perturbation make harmonic oscillations around the critical point. The *local* phase portrait is a set of ellipses,

neutrally stable. This does not prove that the positive point is stable for the full system, but at least that the phase portrait is close to the elliptic.
But the neutral stability is true; It happens that all solutions are periodic; the system admits an integration:

$$\frac{du_1}{du_2} = \frac{u_1(b_1u_2 - a_1)}{u_2(a_2 - b_2u_1)}, \frac{a_2 - b_2u_1}{u_1} du_1 = \frac{b_1u_2 - a_1}{u_2} du_2 :$$
$$a_2 \ln|u_1| - b_2u_1 = b_1 \ln|u_2| - a_1u_2 + C$$



The LHS and RHS are both convex functions (*f*'' < 0 for $u > 0$) of u_1 and u_2, viz. Thus, for a given value of u_1 (and C) the equation to solve for u_2 has zero, one (tangency) or two solutions.
That is exactly a recipe for drawing a convex closed curve.
And here is the result, $a_1 = 1.3$, the others all 1. In particular, the linearization showed the period of small oscillations to be $\frac{2\pi}{\sqrt{a_1a_2}}$.



How to compute the period of a *large* oscillation, now we know the solution is periodic?
The formula is simple, for instance,

$$T = \oint dt = \oint \frac{du_1}{u_1(b_1u_2 - a_1)}$$

but the problem is to distinguish when a whole revolution has been traversed. Note that the discrete values produced by the numerical scheme will not be exactly periodic.
Let us instead record the angle $\phi(t)$ swept by the radius vector from $\mathbf{u}^*$

$$\frac{d\phi}{dt} = \frac{(\dot{u}_1, \dot{u}_2, 0) \times (u_1 - u_1^*, u_2 - u_2^*, 0)}{(u_1 - u_1^*)^2 + (u_2 - u_2^*)^2} = \frac{\dot{u}_1 \cdot (u_2 - u_2^*) - \dot{u}_2 \cdot (u_1 - u_1^*)}{(u_1 - u_1^*)^2 + (u_2 - u_2^*)^2}$$
$$= \frac{b_1u_1(u_2 - u_2^*)^2 + b_2u_2(u_1 - u_1^*)^2}{(u_1 - u_1^*)^2 + (u_2 - u_2^*)^2}$$

so, bounded above and below. Adjoin the ϕ-differential equations to the model and change to ϕ as independent variable:

$$\frac{du_1}{d\phi} = \frac{dt}{d\phi} \cdot \dot{u}_1$$
$$\frac{du_2}{d\phi} = \frac{dt}{d\phi} \cdot \dot{u}_2$$
$$\frac{dt}{d\phi} = \frac{(u_1 - u_1^*)^2 + (u_2 - u_2^*)^2}{b_2u_2(u_1 - u_1^*)^2 + b_1u_1(u_2 - u_2^*)^2}$$

(du_1/dt etc. from eqn. 1) and run from 0 to 2π.