Lecture 5: Phase portraits and bifurcation

The phase portraits for linear, constant coefficient 2D systems

$$d\mathbf{u}/dt = \mathbf{A}\mathbf{u}$$

can be completely characterized, and we proceed to make a map of the different types. The eigenvalues of **A** are λ_1 and λ_2, and the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.The plots are shown in (x_1,x_2)-coordinates. For diagonalizable systems with real eigenvalues this means

$$\mathbf{u}(t) = x_1(t)\mathbf{v}_1 + x_2(t)\mathbf{v}_2$$

For complex eigenvalues, $\lambda = \mu$ +- $i\omega$, $\mathbf{v}_2 = conj(\mathbf{v}_1)$ and we choose $\mathbf{w}_+ = \mathrm{Re}(\mathbf{v})$ and $\mathbf{w}_- = \mathrm{Im}(\mathbf{v})$ as basis vectors. Then $\mathbf{A}\mathbf{w}_{+,-} = (\mu \pm \omega)\mathbf{w}_{+,-}$ and $\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

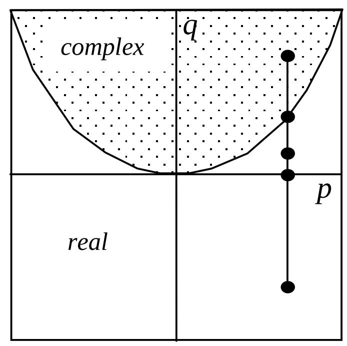
The characteristic equation is

$$0 = \det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = \lambda^2 - p\lambda + q,\ p = tr\mathbf{A},\ q = \det(\mathbf{A})$$

$D = p^2/4 - q$

Real: $D > 0$; Complex: $D < 0$; Double: $D = 0$.

The (p,q) – parametrization discriminates all the cases, except the two variants of $D=0$, diagonalizable and defective (=non-diagonalizable).



1. D > 0

i) Negative: $\lambda_1 < \lambda_2 < 0$

$$x_1 = x_1^0 e^{\lambda_1 t}, x_2 = x_2^0 e^{\lambda_2 t}, both \to 0$$

$$x_1 / x_2 \to 0 : \text{final approach to 0 along } x_1 = 0$$

A stable *node.*

ii) Both signs: $\lambda_1 < 0 < \lambda_2$

$$x_1 = x_1^0 e^{\lambda_1 t} \to 0, x_2 = x_2^0 e^{\lambda_2 t} \to \infty$$

$$x_1 / x_2 \to 0 : \text{final divergence to } \infty \text{ along } x_1 = 0$$

A *saddle*, unstable, except for points with $x_2 = 0$

iii) Positive: $0 < \lambda_1 < \lambda_2$

$$x_1 = x_1^0 e^{\lambda_1 t}, x_2 = x_2^0 e^{\lambda_2 t}, both \to \infty$$

$$|x_2| = C|x_1|^{\lambda_2 / \lambda_1}$$

Same as i), with direction on trajectories reversed, and 1 and 2 switched, an *unstable* node.

Note: If the matrix is singular, $q = 0$, the origin is not the only critical point. Rather, all multiples of the zero eigenvector **v** are critical, so the dynamics takes place along lines at an angle to **v** – a solution stays on a line.

Exercise:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$. What angle does a trajectory make with the u_1-axis?

Example:

`http://www-math.mit.edu/daimp/LinPhasePorMatrix.html`

lets you play with the entries of the matrix, shows eigenvalues and phase portrait, and names it.
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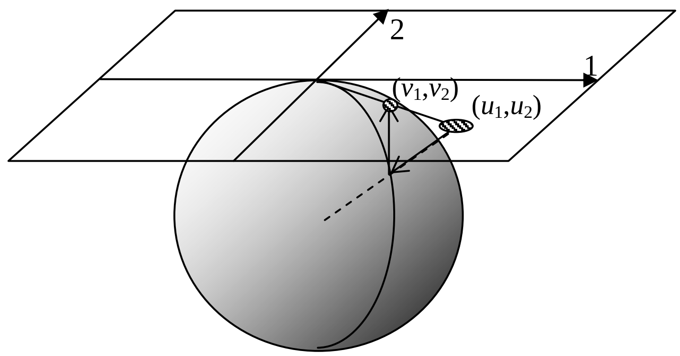
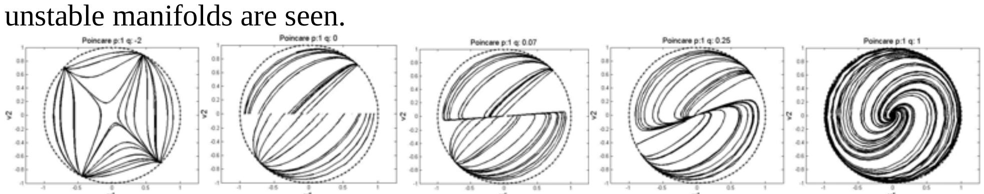


The first complete map of this kind was probably made by H.Poincaré about 100 years ago.
Poincarés (half)sphere, mapped onto the unit disk: $\mathbf{v} = \mathbf{u} / \sqrt{1+u_1^2+u_2^2}$ shows the *whole* phase space.
The Poincaré plots below show a saddle, a degenerate (q=0) node, a proper node, a "one-tangent" degenerate node, and a spiral (focus). The plots were made by integration forward *and* backward in time, so both stable and unstable manifolds are seen.



The portraits correspond to the points marked on the (p,q)-map above, with "bifurcations" - changes to the type of portrait - at q = 0 and 0.25.

Bifurcations
Example
A point mass m on a rod of length l rotates around the z-axis with angular velocity ω. Gravitational acceleration is g in the negative z-direction. No friction or other damping. We derive the dynamics by Lagrange's equations, and in keeping with Lagrange's tradition there is no drawing:

Gravitational + "centrifugal" potential energy:

$$W = mgz + 1/2m(l\sin\theta)^2\omega^2 = -mgl\cos\theta + 1/2m\omega^2 l^2 \sin^2\theta,$$

Kinetic energy:

$$T = 1/2ml^2(\mathrm{d}\theta/\mathrm{d}t)^2$$

Equations of motion:

$$\frac{d}{dt}\frac{\partial(T-W)}{\partial\dot{\theta}} = \frac{\partial(T-W)}{\partial\theta} : ml^2\ddot{\theta} = -mgl\sin\theta + m\omega^2 l^2 \sin\theta\cos\theta,$$

$$\ddot{\theta} = \sin\theta(\lambda\cos\theta - 1), \lambda = \frac{\omega^2 l}{g}$$

where the time units were chosen to match the natural frequency of small oscillations.

The state vector is $\mathbf{u} = (\theta, d\theta/dt)^T$.
The critical points are $d\theta/dt = 0$,

$\sin(\theta) = 0$: $\theta = k\pi$, $k = 0,+-1,+-2,\ldots$,
and if $\lambda > 1$, also $\theta = \arccos(1/\lambda)$.

Stability is determined by the second derivative of W,

$$\frac{d^2W}{d\theta^2} = \lambda\cos 2\theta - \cos\theta:$$

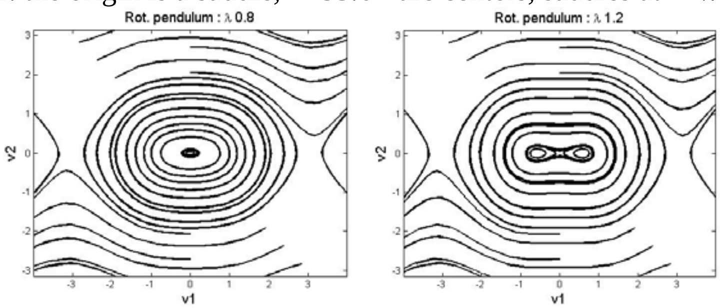
$\theta = 0: \lambda - 1$, all $\lambda. < 0$, neutrally stable, for $\lambda < 1, > 0$, unstable for $\lambda > 1$

$\theta = \pm\pi: \lambda + 1$, all $\lambda. > 0$, unstable

$\lambda\cos\theta = 1: \frac{1-\lambda^2}{\lambda}, \lambda \geq 1. < 0$, neutrally stable.

Phase portraits

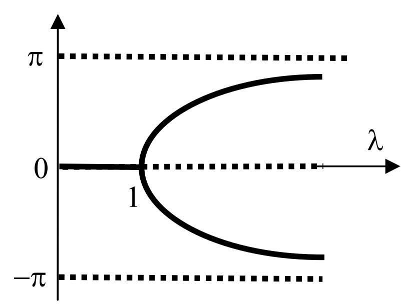
Note that the θ-axis is periodic, so $+\pi$ and $-\pi$ are identical.
Left, $\lambda = 0.8$: the origin is a center, saddles at $+-\pi$.
Right, $\lambda = 1.2$: the origin is a saddle, $+-33.6^{\circ}$ are centers, saddles at $+-\pi$



Exercise:

Check the claims about the phase portraits by computing the Jacobian and its p and q!

The equilibria are continuous functions of λ, as shown in the *bifurcation* diagram of the roots (critical points, equilibria, …), right.
Dotted: unstable, continuous line: center.
A *pitchfork bifurcation.*



At the bifurcation, the equilibrium depends very sensitively on λ.

$\cos\theta = 1 - \theta^2/2 + \ldots$, $\lambda = 1 + \delta$:

$\theta = \sqrt{2\delta} + \ldots$

so the root is *not* differentiable as function of the parameter λ there.

Hopf bifurcation

The last example is the famous Hopf bifurcation: at the critical parameter value the root becomes unstable, but there is no other (finite) critical point, and the solution becomes a *limit cycle*, approaching a periodic solution. The screaming noise emitted by a PA-system when the microphone is brought close to the loudspeaker is a dynamical system of this kind. We refrain from more concrete modelling and stay with the abstract, noting that the equation admits a very simple complex form,

$z = x + iy$,

$$\dot{z} = z(\lambda + (\mu + i\omega)|z|^2)$$

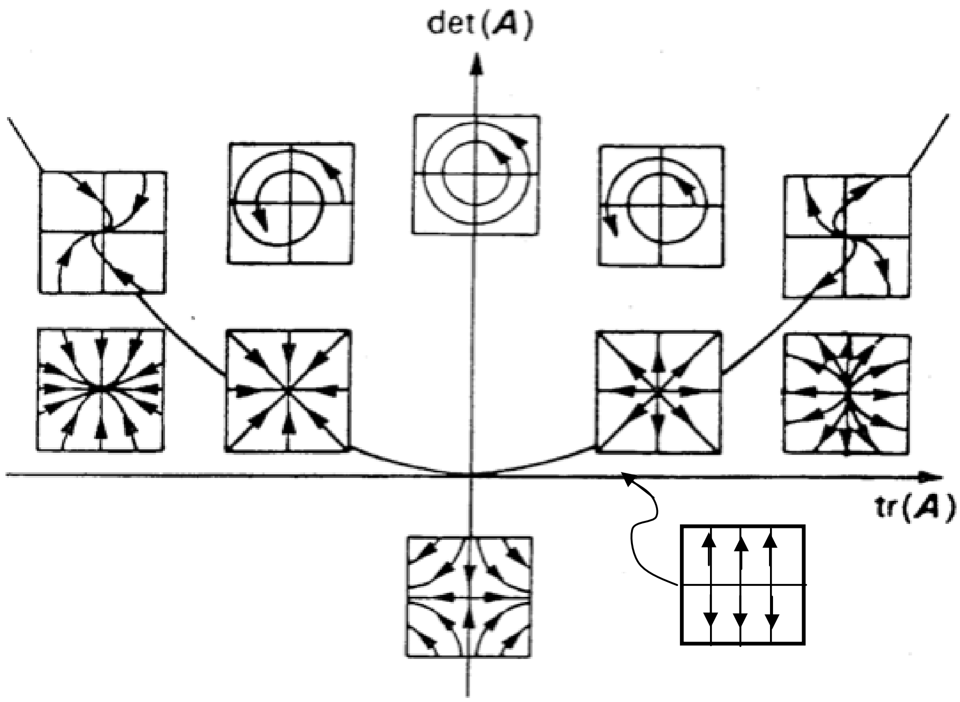
$$\dot{x} = \lambda x + (x^2 + y^2)(\mu x - \omega y)$$

$$\dot{y} = \lambda y + (x^2 + y^2)(\omega x + \mu y)$$

The origin is an (isolated) critical point with Jacobian $\lambda\mathbf{I}$, a *star*, stable if $\lambda < 0$, unstable if $\lambda > 0$, and no other, if ω is non-zero, as can be seen e.g. by writing the equations in polar coordinates (r,ϕ):

$$\frac{1}{2}\frac{d}{dt}r^2 = (\lambda + r^2\mu)r^2, \dot{\phi} = \omega r^2$$

So if $\mu < 0$, $\lambda > 0$ the r-equation has a positive stable critical point (like the logistic equation), $r \to \sqrt{\frac{\lambda}{-\mu}}$, but ϕ increases monotonically. The trajectory approaches circular motion with angular velocity $\omega\lambda/|\mu|$: a limit cycle.



The last picture is the artist's impression of the possible cases, copied from lecture notes C.Trygger, unknown source