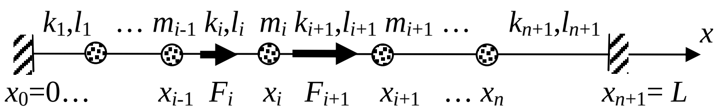
Lecture 7: Strang's framework for Applied Math.: Graph models

We have seen the model

$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{f} \end{pmatrix} \Rightarrow -\mathbf{A}^T\mathbf{CAx} = \mathbf{f} - \mathbf{A}^T\mathbf{Cb}$$

for mechanical equilibrium, with **C** a positive definite (often diagonal: spring constants) **f** external "forces", **y** the internal "forces" or Lagrange multipliers. The matrix **A** will now be studied in more detail.

A line of springs



For each mass point m_i, Newton's law:

$$m_i \ddot{x}_i = F_{i+1} - F_i + f_i^e, i = 1,...,n \qquad (1)$$

where f^e are the external forces. The springs are assumed linearly elastic (Hookean) so

$$F_i = k_i(l_i - l_i^0) \qquad (2)$$

where l^0 is the length of the unloaded spring, and finally, the relation between spring length and the coordinates x_i

$$l_1 = x_1; l_i = x_i - x_{i-1}, i = 2,...,n; l_{n+1} = L - x_n \qquad (3)$$

The equations are of very different origin: (1) is a "law of Nature", (3) is an obvious consequence of how we choose to parametrize the model, and (2) captures the physics. The relation between extension and force is an expression of observations, and holds only for limited extension, etc. This is an example of an empirical "constitutive equation" needed to close the system of equations, and it can of course be challenged, refined, etc. But (1) and (3) are not subject to discussion.
Making the obvious vectors out of the l, x, and F,

$$\mathbf{M\ddot{x}} = -\mathbf{A}^T\mathbf{F} + \mathbf{f}^e = 0$$

$$\mathbf{F} = \mathbf{C}(\mathbf{l} - \mathbf{l}^0), \mathbf{C} = diag(k_i) \Rightarrow \mathbf{l} = \mathbf{l}^0 + \mathbf{C}^{-1}\mathbf{F}$$

$$\mathbf{l} = \mathbf{Ax} + \mathbf{b} : \mathbf{C}^{-1}\mathbf{F} - \mathbf{Ax} = \mathbf{b} + \mathbf{l}^0$$

where **A** is the *difference matrix* (shown for n = 4, so there are 5 springs)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L \end{pmatrix}$$

The block system becomes

$$\begin{pmatrix} \mathbf{C}^{-1} & -\mathbf{A} \\ -\mathbf{A}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{F} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{b} + \mathbf{l}^0 \\ \mathbf{f}^e \end{pmatrix}$$

A is also the edge-node incidence matrix for the directed graph whose vertices are the mass points and the edges are the springs. The Schur complement system to solve, after we eliminate the forces, has coefficient (= stiffness) matrix $\mathbf{K} = \mathbf{A}^t\mathbf{CA}$. It is symmetric and positive definite, because its columns are independent (easy to see). Here are its elements:

- If an edge, say number k, runs between nodes i and j then $K_{ij} = -k_k$.
- All rows but the first and last sum to zero: **A** takes differenes, so $\mathbf{A1} = 0$, except in rows 1 and n+1.
- Indeed, K_{jj} = sum of the spring constants of all edges connected to node j.
- It is tri-diagonal

$$\mathbf{A}^t\mathbf{CA} = \begin{pmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{pmatrix}$$

- **K** is an *M*-matrix: Its inverse has all positive elements.

Exercise: Prove the *M*-matrix property by considering $(\mathbf{I}-\mathbf{B})^{-1} = \sum_{k=0}^{\infty}\mathbf{B}^k$ which is true whenever $\|\mathbf{B}\| < 1$.

Equilibrium problems described by directed graphs

One may think of an electric circuit as the prototype model in the following: current flows in its branches, driven by electric potential differences between the vertices; Ohm's law for a resistive branch is $V = RI$.
Consider a directed graph with n vertices and m edges, with "flow" y_i in edge i, and "potential" x_k at vertex k.
The graph edge-node incidence matrix is $\mathbf{A}_0$ (the reason for the 0 will be apparent shortly): edge k from node i to node j means $A_{ki} = -1$, $A_{kj} = +1$, the rest zeros.
The vector of potential differences e_j, $j = 1,\ldots,m$, across the edges is given by

$$\mathbf{e} = \mathbf{A}_0\mathbf{x}$$

$\mathbf{A}_0\mathbf{1} = 0$ (row sums zero), so $\mathbf{A}_0$ has at most n-1 linearly independent columns. Since only potential differences matter, we may assign 0 to an arbitrary x, say x_k Then only n–1 x-values are unknown and we can remove column k from $\mathbf{A}_0$ to produce the reduced incidence matrix **A**, still $\mathbf{e} = \mathbf{Ax}$.
The transpose $\mathbf{A}^t$ acts on the flow variables **y**:

$\mathbf{A}^t\mathbf{y}$ = net flow into the vertices from the branches = external sinks

In other words, *Kirchhoff's current law* is

$$\mathbf{A}^t\mathbf{y} = \mathbf{f}$$

where **f** is the external "current" source. This is a *conservation law* stating that electric charge is neither created nor destroyed.
The system is closed by the flow vs. potential difference relation for each branch:

$$\mathbf{e} = \mathbf{b} - \mathbf{C}^{-1}\mathbf{y}$$

allowing for external driving "forces" (think of batteries) **b**. Putting it together:

$$\mathbf{Ax}+\mathbf{C}^{-1}\mathbf{y} = \mathbf{b}$$
$$\mathbf{A}^t\mathbf{y} = \mathbf{f}$$

…

Small strain deformation of trusses

A structure built from linear elements ("sticks"), joined so the joints cannot support any torque, is called a truss (Sv. "Fackverk"). The elements are compressed or extended and respond by forces, which we assume to be Hookean: Force is proportional to extension. "CEINOSTUV" was the anagram, for: *Ut Tensio, Sic Vis*. Let the vertices have coordinates $\mathbf{X}_j = (x_j, y_j)$ (2D plane) when no loads are present, and $\mathbf{X}_j + \mathbf{x}_j$ under load. The notation which will be used is that the change in distance between nodes i and j under loading is d_{ij}, which is small compared to the undeformed length L_{ij} of the bar:

$$d_{ij} = \left\|\mathbf{X}_j + \mathbf{x}_j - (\mathbf{X}_i + \mathbf{x}_i)\right\|_2 - \left\|\mathbf{X}_j - \mathbf{X}_i\right\|_2 = \|\mathbf{Y} - \mathbf{y}\|_2 - \|\mathbf{Y}\|_2 =$$

$$= \sqrt{(\mathbf{Y}+\mathbf{y})^t(\mathbf{Y}+\mathbf{y})} - \sqrt{\mathbf{Y}^T\mathbf{Y}} = \sqrt{\mathbf{Y}^T\mathbf{Y} + 2\mathbf{y}^T\mathbf{Y} + O(\|\mathbf{y}\|^2)} - \sqrt{\mathbf{Y}^T\mathbf{Y}} =$$

$$= \sqrt{\mathbf{Y}^T\mathbf{Y}}\left(1 + \frac{\mathbf{y}^T\mathbf{Y}}{\mathbf{Y}^T\mathbf{Y}}\right) - \sqrt{\mathbf{Y}^T\mathbf{Y}} + O(\|\mathbf{y}\|^2) = \frac{\mathbf{y}^T\mathbf{Y}}{\sqrt{\mathbf{Y}^T\mathbf{Y}}} = (\mathbf{x}_j - \mathbf{x}_i)^T\,\hat{\mathbf{Y}}_{ij}$$

where the $\mathbf{Y}_{ij}$ is the unit vector $(\cos\theta_{ij}, \sin\theta_{ij})$ along the bar between i and j.

The equilibrium is formulated by the Lagrange equations. Let there be external forces $\mathbf{f}^e_j$ on vertex j:

$$\min\ W = \sum_{k,m}\frac{1}{2}k_{km}d_{km}^2 - \sum_j \mathbf{x}_j^T\mathbf{f}_j^e,$$

$$\text{subject to } d_{km} - (\mathbf{x}_k - \mathbf{x}_m)^T\,\hat{\mathbf{Y}}_{km} = 0, k, m = 1,2,\ldots,n$$

$$L = \sum_{k,m}\frac{1}{2}k_{km}d_{km}^2 - \sum_j \mathbf{x}_j^T\mathbf{f}_j^e + \sum_{k,m}\lambda_{km}(d_{km} - (\mathbf{x}_k - \mathbf{x}_m)^T\,\hat{\mathbf{Y}}_{km})$$

$$\frac{\partial L}{\partial \mathbf{x}_r} = -\mathbf{f}_j^e - \sum_{k,m}\lambda_{km}(\delta_{kr} - \delta_{mr})\hat{\mathbf{Y}}_{km} = 0 \quad (-\mathbf{f}^e = \mathbf{A}^T\boldsymbol{\lambda})$$

$$\frac{\partial L}{\partial d_{rs}} = k_{rs}d_{rs} + \lambda_{rs} = 0 \Rightarrow d_{rs} = -\frac{1}{k_{rs}}\lambda_{rs}$$

$$\frac{\partial L}{\partial \lambda_{rs}} = d_{rs} - \hat{\mathbf{Y}}_{rs}^T(\mathbf{x}_r - \mathbf{x}_s) = 0 : -\frac{1}{k_{rs}}\lambda_{rs} - \hat{\mathbf{Y}}_{rs}^T(\mathbf{x}_r - \mathbf{x}_s) = 0 \quad (\mathbf{C}^{-1}\boldsymbol{\lambda} + \mathbf{A}\mathbf{x} = 0)$$

Most of the possible $n(n\text{-}1)/2$ edges do not exist, so the double sums over "k,m" are actually a sum over only the m edges.
To make that "*Yhat*" formula a little more transparent, write the derivatives for both x and y,

$$\frac{\partial L}{\partial x_r} = -f_r^{e,x} - \sum_m \lambda_{rm}\cos\theta_{rm} + \sum_k \lambda_{kr}\cos\theta_{kr} = 0$$

$$\frac{\partial L}{\partial y_r} = -f_r^{e,y} - \sum_m \lambda_{rm}\sin\theta_{rm} + \sum_k \lambda_{kr}\sin\theta_{kr} = 0$$

$$\frac{\partial L}{\partial \lambda_{rs}} = -\frac{1}{k_{rs}}\lambda_{rs} - (x_r - x_s)\cos\theta_{rs} - (y_r - y_s)\sin\theta_{rs} = 0$$

Leaving aside for the moment the "grounding" necessary to make a non-singular system, we let **A** be the edge-node incidence matrix for the graph, and replace the +-1's by +-cosθ to produce the $\mathbf{A}^{\cos}$ matrix, and $\mathbf{A}^{\sin}$, analogously. Sorting the unknowns in the order **λ**, **x**, **y** gives the matrix:

$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A}^{\cos} & \mathbf{A}^{\sin} \\ \mathbf{A}^{\cos,T} & 0 & 0 \\ \mathbf{A}^{\sin,T} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}^{e,x} \\ \mathbf{f}^{e,y} \end{pmatrix}$$

Is this system non-singular? Elimination of **λ** leads to

$$\begin{pmatrix} \mathbf{A}^{\cos,T} C \mathbf{A}^{\cos} & \mathbf{A}^{\cos,T} C \mathbf{A}^{\sin} \\ \mathbf{A}^{\sin,T} C \mathbf{A}^{\cos} & \mathbf{A}^{\sin,T} C \mathbf{A}^{\sin} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{e,x} \\ \mathbf{f}^{e,y} \end{pmatrix}$$

and the diagonal blocks are definite, if the columns of **A** are independent. Clearly, $(\mathbf{1},\mathbf{0})^T$ and $(\mathbf{0},\mathbf{1})^T$ are in the null-space: translation in *x*- and *y*-direction do not deform the truss and create no forces. But also rotation of the graph as a rigid body gives no deformation. What (*x*,*y*) corresponds to (infinitesimal) rigid body rotation ε around, say, the point **P**?

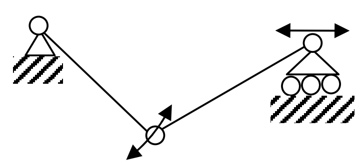
$$\mathbf{x}_j = \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathbf{X}_j - \mathbf{P}); \mathbf{x}_k - \mathbf{x}_j = \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathbf{X}_k - \mathbf{X}_j) = \varepsilon L_{kj} \begin{pmatrix} -\sin\theta_{kj} \\ \cos\theta_{kj} \end{pmatrix}$$

makes the $(\mathbf{x},\mathbf{y})^T$ orthogonal to all rows of the matrix.

The null space is three-dimensional so at least three constraints must be invoked, by setting selected displacements to zero ("grounding") and removing the corresponding column(s) of $\mathbf{A}^{\cos}$ and $\mathbf{A}^{\sin}$.

Several cases:
a) Stable, with unique solution, non-singular matrix
1. Statically determinate: the force equation $\mathbf{A}^T\boldsymbol{\lambda} = \mathbf{f}$ can be solved for **λ**.
2. Indeterminate, the "standard" case, must solve for (**x**,**y**).

b) Unstable, singular matrix
1. Rigid body motions allowed: Solutions exist only if the net force and moment vanish.
2. Mechanism: Constraints rule out rigid body motions, still there are deformations which do not change the length of the bars. Example, a slider mechanism, used in reciprocating engines to convert linear motion into rotation.



Dynamics

The equations are perfectly valid also for a moving truss, when we include the inertial and damping forces, as long as the assumption on *small deformations* of the bars is valid. In particular, there is no assumption of small displacements (**x**,**y**). We obtain

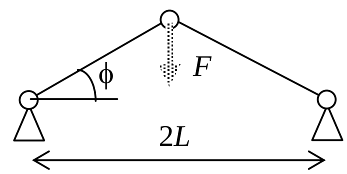
$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A}^{\cos} & \mathbf{A}^{\sin} \\ \mathbf{A}^{\cos,T} & 0 & 0 \\ \mathbf{A}^{\sin,T} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}^{e,x} + \mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} \\ \mathbf{f}^{e,y} + \mathbf{M}\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} \end{pmatrix}$$

$$\mathbf{A} = \mathbf{A}(\mathbf{X} + \mathbf{x}, \mathbf{Y} + \mathbf{y})$$

where the mass matrix **M** is diag(m_i) if the masses are point masses at the vertices, and the damping matrix **D**, for a simplistic force proportional to the mass point velocity, is diag(D_i). The state vector is (**x**,**y**) and (**u**,**v**), the velocities in *x*- and *y*-directions. The algorithm is as follows: From **x** and **y** compute the **A**-matrix, and then the bar forces from the first *m* rows, and the summed *x*- and *y*-forces on the vertices from the last rows; Then compute the accelerations, and take a time-step.

Equilibrium of non-linear truss: bifurcation, hysteresis

The non-linear truss-model can be used also for static analysis, and it is easy to find examples with non-unique solutions, and bifurcations. Here is a snap-through case: Two bars, both grounded, and a force acting perpendicularly to the line between the supports. Let the unloaded length of the bars be L_0, and the spring constant *K*.



Then, with $p = F/(KL)$, $a = L_0/L > 1$

$$W = (1/\cos\phi - a)^2 + p\tan\phi; W' = 2(1/\cos\phi - a)\frac{\sin\phi}{\cos^2\phi} + p\frac{1}{\cos^2\phi}$$

$$p/2 = a\sin\phi - \tan\phi$$

The graph of the solution, right: Starting at *p* small, with ϕ close to 1, when *p* increases to 0.33, ϕ decreases to 0.606, and then snaps through to –1.1; further increase of *p* decreases ϕ towards $-\pi/2$. If *p* is *de*creased again, the solution point follows the lower curve until $p = -0.33$, $\phi = -0.606$ and then ϕ snaps through to +1.1: *Hysteresis*.
The part of the solution curve marked ‘o’ is unstable, as the sign of the second derivative of *W* tells.