Lecture 7, add. : Non-linear truss model

The question arose on the relation of the snap-through geometrically non-linear model to Strang's linear(ized) framework.
We did the snap-through in a way to get as quickly as possible to the properties, taking whatever short-cuts made possible by the simplicity.

- The coordinates /degrees of freedom were chosen (the angle) so the constraints necessary to remove rigid body motion are built in. The general truss model presented uses Cartesian coordinates of the joints, and needs explicit enforcement of constraints.
- With only one state variable the "incidence matrix" which forms differences is hard to spot: it is a number.
- We eliminated the "extension" variable immediately so the bar forces never appeared - the "Schur complement" $\mathbf{A}^T\mathbf{CA}$ came out immediately.

So let us redo the model and linearize it after the formulation; in the linear Strang truss model we linearized the relationship between joint position and bar length and used that in the Lagrange equations.

$$W = 2\frac{1}{2}Ke^2 + FL\tan\theta + \lambda\left(e - \left(\frac{L}{\cos\theta} - L^0\right)\right)$$

$$\frac{\partial W}{\partial\theta} = \frac{FL}{\cos^2\theta} - \lambda L\frac{\sin\theta}{\cos^2\theta}$$

$$\frac{\partial W}{\partial e} = 2Ke + \lambda, e = -\frac{\lambda}{2K}$$

$$\frac{\partial W}{\partial\lambda} = e - \left(\frac{L}{\cos\theta} - L^0\right);$$

$$\begin{cases} \dfrac{\lambda}{2K} + \dfrac{L}{\cos\theta} - L^0 = 0 : \; C^{-1}\lambda + a(\theta) = 0 \\ \lambda L\dfrac{\sin\theta}{\cos^2\theta} = \dfrac{FL}{\cos^2\theta} : A^T\lambda = f \end{cases}$$

Linearization around a solution θ^*, λ^* gives

$$\frac{\delta\lambda}{2K} + L\frac{\sin\theta}{\cos^2\theta}\delta\theta = 0, \lambda L\frac{\sin\theta}{\cos^2\theta} = \frac{FL}{\cos^2\theta} :$$

$$\begin{pmatrix} C^{-1} & A(\theta^*) \\ A^T(\theta^*) & \dfrac{\lambda L}{\cos\theta^*} \end{pmatrix}\begin{pmatrix} \delta\lambda \\ \delta\theta \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{L}{\cos^2\theta^*} \end{pmatrix} \cdot \delta F; A(\theta) = L\frac{\sin\theta}{\cos^2\theta}$$

which does fit Strang's framework, with A the derivative of a. The replacement of a zero by a non-zero block comes from the repeated differentiation.

Root following (Homotopy, continuation, ...)

The bifurcation analysis was illustrated by drawing curves (see last lecture notes). The linearized system brings the *implicit function theorem* to mind:

Let $\mathbf{f}(\mathbf{x};p)$ =0 be n nonlinear equations in the n variables x_i and p a parameter, and assume that we know a solution x* for $p = p^*$. Let the Jacobian matrix be **J**, $\mathbf{J}(\mathbf{x};p)_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x};p)$. Then, the equations define **x** as a function of p in a neighborhood of p^* if $\mathbf{J}(\mathbf{x}^*,p^*)$ is non-singular.

So we may naively try to follow the solution by solving the system of ODE,

$$\frac{d\mathbf{x}}{dp} = -\mathbf{J}^{-1}(\mathbf{x};p)\frac{\partial \mathbf{f}}{\partial p}; \mathbf{x}(p^*) = \mathbf{x}^*$$

and continue at least as long as the matrix is non-singular. We know where the singular points should be: the "turning points" of the curve where the solution snaps. Hopefully that will come out of this analysis, too:

$$\mathbf{J} = \begin{pmatrix} C^{-1} & A(\theta) \\ A(\theta) & \dfrac{\lambda L}{\cos\theta} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2K} & A(\theta) \\ A(\theta) & \dfrac{2KL(L_0 - L/\cos\theta)}{\cos\theta} \end{pmatrix}$$

where we used that (θ,λ) is a solution.

> Note: DO NOT replace λ by the seemingly simpler $F/\sin\theta$, because that (re-) introduces the varying parameter F; the expression above contains only the state variable θ and constants.

J is singular when $A^2(\theta) = \frac{L(L_0 - L/\cos\theta)}{\cos\theta} : \cos^3\theta^* = \frac{L}{L_0}$ which agrees with our earlier result.

Negotiation of turning points?

The naive root-follower will give up on approach to a turning point, or a pitchfork, or a Hopf bifurcation, or, indeed, on any interesting point where things happen quickly. One can make the root-follower negotiate turning points by introducing as independent variable the arclength along the curve traced by $(\mathbf{x},p)$ in R^n x R; This gives a differential-algebraic system which can be differentiated again to produce an ODE system.

In the **f**-example,

$$\mathbf{f_x}\dot{\mathbf{x}} + \mathbf{f}_p\dot{p} = 0 \Rightarrow \mathbf{f_x}\ddot{\mathbf{x}} + \mathbf{f_{xx}}\dot{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{f}_p\ddot{p} + \mathbf{f}_{pp}\dot{p}^2 + 2\mathbf{f}_{\mathbf{x}p}\dot{\mathbf{x}}\dot{p} = 0$$

$$\dot{\mathbf{x}}^T\dot{\mathbf{x}} + \dot{p}^2 = 1 \Rightarrow \dot{\mathbf{x}}^T\ddot{\mathbf{x}} + \dot{p}\ddot{p} = 0$$

$$\begin{pmatrix} \mathbf{f_x} & \mathbf{f}_p \\ \dot{\mathbf{x}} & \dot{p} \end{pmatrix}\begin{pmatrix} \ddot{\mathbf{x}} \\ \ddot{p} \end{pmatrix} = \ldots$$

so regularity of $\mathbf{f_x}$ is no longer required. But there are n n x n second derivative matrices involved so for $n > 2$ (1?) we need a symbolic differentiation package to make this practical. Numerical root followers do work in R^{n+1} using arclength, but attack a discretized version of the original problem and do not use second derivatives, except possibly for figuring out what sort of singular point is approaching.