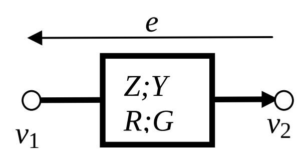
Lecture 8: Electric circuits; wrap-up of first part of course



Mathematical modeling by computer relies on development of a *systematic* way of deriving equations which can be solved numerically. Strang's $\mathbf{A}^T\mathbf{C}\mathbf{A}$ framework is such a formalization, which we have shown to work for materially linear trusses and electric circuits with discrete components. Let us re-iterate the components modeled: two-ports (two leads to solder), and Strang's notation *e* for potential difference over impedances.

An Impedance *Z* /Admittance $Y = 1/Z$ /Resistor *R* /Conductance $G = 1/R$ admits calculation of the current *i* from the potential difference *e*, for steady, time-harmonic ($j\omega$), and transient calculations:

$e = v_1 - v_2; Zi = e \quad / \quad i = Ye \quad / \quad Ri = e \quad / \quad i = Ge$

The relation can also be non-linear, e.g. for a diode,

$$i = f(e) = \left(\exp\left(\frac{e}{V_D}\right) - 1\right) I_D$$

A capacitor *C*:

Transient $\frac{dQ}{dt} = i, Q = Ce$ where the capacitance C is usually constant.

Time-harmonic $j\omega Ce = i : Z = \frac{1}{j\omega C}$

An inductor *L*:

Transient $\frac{d\Phi}{dt} = e, \Phi = Li$, *L* is the self-inductance.

Time-harmonic $j\omega Li = e : Z = j\omega L$

Coupling between inductors in different branches (transformers) can also be modeled:

$$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = e_1,$$

$$M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = e_2$$

In Strang's model definition, a branch contains an impedance and a voltage source, and the current sources are associated with the nodes. The "modified nodal analysis" described in Dr. Hanke's MNA notes uses components, each with a single task: current sources *I* and generators *V* are also components, defined by incidence matrices, $\mathbf{A}_I$, $\mathbf{A}_V$.

One must choose the state variables for the transient analysis. Capacitor voltages and inductor currents appear time-differentiated so MUST be included. The MNA model adds also currents through generators. UNFORTUNATELY the MNA uses *v* for potential differences and *e* for node potentials; a translation table MNA/Strang is provided in the notes.

Here is the final result

$$\begin{bmatrix} A_C C A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} e \\ i_L \\ i_V \end{bmatrix} + \begin{bmatrix} A_G G A_G^T & A_L & A_V \\ -A_L^T & L & 0 \\ A_V^T & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ i_L \\ i_V \end{bmatrix} = \begin{bmatrix} -A_I I \\ 0 \\ E \end{bmatrix}$$

- If mutual inductances are included, the L-matrix becomes non-diagonal.
- When non-linear conductances are included, $\mathbf{GA}^T{}_G\mathbf{e}$ becomes $\mathbf{f}(\mathbf{A}^T{}_G\mathbf{e})$:

$$\begin{bmatrix} A_C C A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} e \\ i_L \\ i_V \end{bmatrix} + \begin{bmatrix} 0 & A_L & A_V \\ -A_L^T & L & 0 \\ A_V^T & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ i_L \\ i_V \end{bmatrix} + \begin{bmatrix} A_G f(A_G^T e) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -A_I I \\ 0 \\ E \end{bmatrix}$$

Two-port Equivalents and Reciprocity

A “Reciprocity Theorem” is true for linear systems in some generality. For circuits it runs like this:

> Consider a *closed* circuit, and assume that a 1 V voltage generator added to branch a changes the current in b by i. Then, a voltage generator V added to b creates an extra current Vi in a.

This is easy to prove, using MNA notation but with all currents as unknowns (Strang). **i** and **e** are linear functions of the applied voltage (generators) **E**, so changes δ**i**, δ**e**, and δ**E** satisfy (no changes to current generators)

$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta\mathbf{i} \\ \delta\mathbf{e} \end{pmatrix} = \begin{pmatrix} \delta\mathbf{E} \\ 0 \end{pmatrix} : \delta\mathbf{e} = \mathbf{K}^{-1}\mathbf{AC}\delta\mathbf{E}, \mathbf{K} = \mathbf{ACA}^T$$

$$\delta\mathbf{i} = (\mathbf{C} - \mathbf{CA}^T\mathbf{K}^{-1}\mathbf{AC})\delta\mathbf{E}, QED : \text{the } m \times m \text{ matrix } \frac{\partial \mathbf{i}}{\partial \mathbf{E}} \text{ is symmetric,}$$

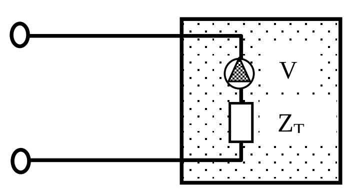
$$\text{and } \delta i_a = \frac{\partial i_a}{\delta E_b} V = \frac{\partial i_b}{\delta E_a} V = Vi$$

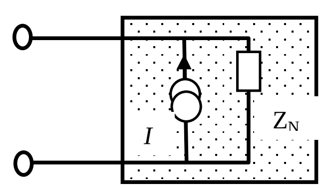
Another cornerstone tool for manual circuit analysis is the

Equivalence Theorem **- for static and $j\omega$-analysis**

Any linear circuit, connected to the outside by only two wires, can from the outside’s viewpoint be replaced by a two-port with

1) a current source and an impedance in parallel, a Norton equivalent, or
2) a voltage source and an impedance in series, a Thévenin equivalent.





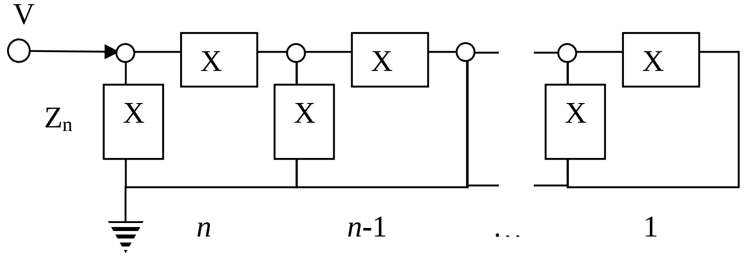
Thévenin Norton

Exercise: Write the relations between the impedances and sources which make the two equivalent.

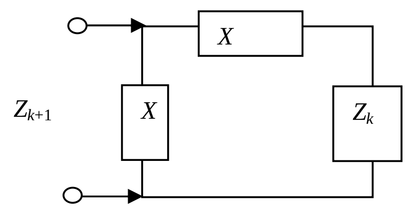
Like reciprocity, this can be proved by considering how changes in currents and voltages are connected by the linear system above. A special case is the parallel/series connection of impedances: two impedances Z_1 and Z_2 in parallel and series, viz., can be replaced by

$$Z_{par} = \frac{1}{1/Z_1 + 1/Z_2}, Z_{ser} = Z_1 + Z_2$$

Example: A ladder network.
The ladder consists of *n* identical links and the task is to compute its impedance seen from the generator *V*.



Let the impedance of the first *k* links, defined as above, be Z_k. Z_{k+1} is then X in parallel with $X + Z_k$:



$$Z_{k+1} = \frac{1}{1/X + 1/(X + Z_k)}; Z_0 = 0. \text{With } x_k = \frac{Z_k}{X}:$$

$$x_{k+1} = \frac{1 + x_k}{2 + x_k}, x_0 = 0.$$

$$or:$$

$$x_{k+1} = \frac{1}{1 + \frac{1}{1 + x_k}}$$

So the ladder computes the golden section ratio $\phi = \frac{\sqrt{5} - 1}{2}$ by the continued fraction

$$\phi = 1/(1 + 1/(1 + 1/(1 + ...)))$$

Exercise
Recall the fixed point theorem for iterations $x_{k+1} = G(x_k)$ and draw the relevant sketch with curves $y = x$ and $y = (x+1)/(x+2)$ to conclude that

i) the sequence converges to ϕ for any $x_0 > -(\phi + 1)$

ii) $|x_{k+1} - \phi| \le \frac{c}{1-c}|x_k - \phi|$ with *c* approx. = $\frac{2}{7 - \sqrt{5}} \approx 0.42$

iii) and show that the limit is actually ϕ.

The single-wave rectifier: example of non-linear conductance.

Consider the one-way (?) rectifier, p4 of Lab 4 handout, but turn the diode the other way. The capacitor voltage is called $V(t)$. The diode voltage-current relation is very non-linear so we try to model an ideal diode:

zero resistance when voltage $e(t) = E(t) - V(t) >= 0$, actually then $V = E$.
infinite for $e < 0$.

So there are two cases,

The diode conducts: $e = 0$, when $i > 0$
The diode is blocked: $i = 0$, when $e < 0$

This is a complementarity relation, $e \cdot i = 0$, $i >=0$, $e >= 0$ and needs special algorithms to avoid "chattering": turning on and off frequently at $e = 0$ when the diode should be conducting continuously. So we may as well stay with a simple, explicit voltage-current relation which requires small timesteps, see below.

But it is easy to construct the periodic solution analytically for the simple case here. A blocked diode gives

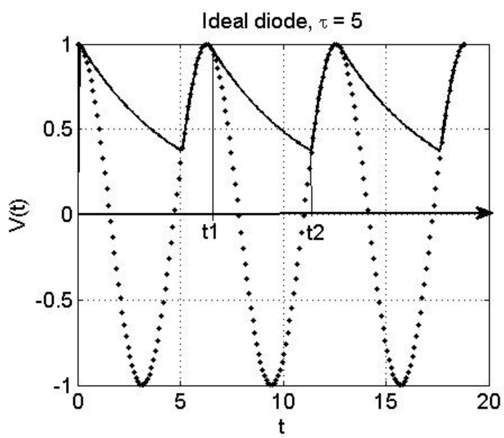
$$V(t_0 + t) = V(t_0)e^{-t/\tau}, \tau = RC$$

and we take the generator to give

$E(t) = \cos t$

(… non-dimensional time …)

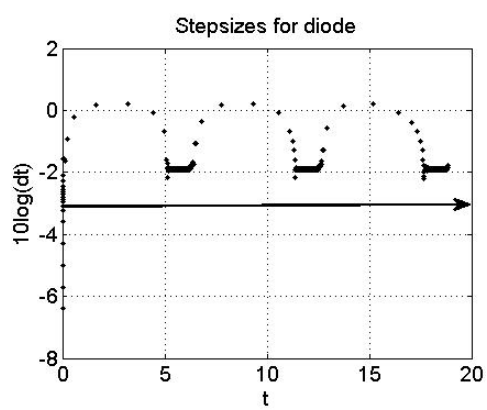
The plot shows three periods (E, dotted) and the capacitor voltage V for $RC = 5$. Times t_1 and t_2 are determined by



$$V(t_1) = \cos t_1$$
$$-1/\tau \cdot V(t_1) = -\sin t_1$$
$$V(t_1)e^{-(t_2 - t_1)/\tau} = \cos t_2$$

so $t_1 = \arctan(1/\tau)$. For t_2 we must solve a non-linear equation.

The plot was made by taking the diode current $i_D = \max(0, (E - V)/r_D)$ and a very small diode forward resistance r. The resulting ODE system is then Lipschitz continuous but not continuously differentiable. The ODE solver will reduce the timesteps substantially at the t-values when the max function turns off and on, and also in the *conducting* state when, for instance in the Runge-Kutta steps off the solution, it evaluates a $V > E$. The blocked state does not chatter when V is substantially greater than E.



Here is a plot of the size of timesteps for the default tolerance 10^{-3} used in `ODE23`. The steps in the blocked state with $i_D = 0$ are $O(1)$ but when the diode is *on* Δt drops to 0.01. The mean stepsize was about 0.05.