Chapter 7: Time-Dependent Problems

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Mathematical Models, Analysis and Simulation, Part I

Plan

- Conservation laws
- Flux vector Φ
- Fourier's law $\Phi = -k\nabla T$
- The 1D heat equation by separation of variables
- Transport equation, characteristics
- Second order wave equation in 1D; d'Alembert's solution
- The wave equation and Newton's law; spring-mass oscillator

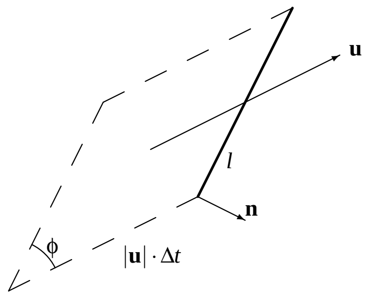
Conservation Laws

- Consider the flow of a fluid with velocity field

$$\mathbf{u}(x,y,t) = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix}$$

- Let $c(x,y,t)$ denote the concentration of a species (e.g., number of molecules per m^2 in 2D)
- **Q:** How many molecules pass in time Δt across a line of length l with normal $\mathbf{n}$?

Conservation Laws (Cont)



- $A =$ area of parallelogram $= l|\mathbf{u}|\Delta t \cos\phi$
- number of molecules: cA
- At every instant (number of molecules per second and meter):

$$c|\mathbf{u}|\cos\phi = c\mathbf{u}\cdot\mathbf{n}$$

Definition. *The* flux vector *is Φ if the number of molecules per length and time unit is $\Phi\cdot\mathbf{n}$.*

In case of (passive) advection in velocity field $\mathbf{u}$

$$\boxed{\Phi = c\mathbf{u}}$$

Conservation Laws (Cont)

- Consider the number c of molecules in a fixed *control volume* Ω.
- c can only change if molecules are going through (in/out) of the boundary $\partial\Omega$:

$$\frac{d}{dt}\int_{\Omega} c d\Omega = -\int_{\partial\Omega} \Phi \cdot \mathbf{n} d\Gamma.$$

- Divergence theorem provides:

$$\int_{\Omega} \frac{\partial}{\partial t} c d\Omega = -\int_{\Omega} \operatorname{div} \Phi d\Omega.$$

  This equation holds for *any* control volume.
- Conservation law (continuity equation)

$$\boxed{\frac{\partial}{\partial t} c + \operatorname{div} \Phi = 0}$$

Conservation Laws (Cont)

- Continuity equation:

$$\frac{\partial}{\partial t} c + \operatorname{div} \Phi = 0$$

- In integral form:

$$\boxed{\int_{\Omega} \frac{\partial}{\partial t} c d\Omega + \int_{\partial\Omega} \Phi \cdot \mathbf{n} d\Gamma = 0}$$

- In our case:

$$\frac{\partial}{\partial t} c + \operatorname{div}(c\mathbf{u}) = 0.$$

The Heat Equation

- Consider heat conduction: temperature T.
- Conserved quantity is energy:

$$\Delta H = \rho c_p \Delta T$$

  (change in energy per volume).
- *Fourier's law:*

$$\boxed{\Phi = -k\nabla T}$$

- Conservation of energy:

$$\frac{\partial}{\partial t}(\rho c_p T) - \operatorname{div}(k\nabla T) = 0.$$

- For constant ρ, c_p, k:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

  with $\Delta \equiv \operatorname{div} \nabla$ is the Laplacian operator.
  This is the celebrated *heat equation*.

The 1D Heat Equation

- Let $\Omega = (0,1)$:

$$u_t = \alpha u_{xx}, x \in (0,1).$$

- Boundary conditions:
  - prescribed temperature: Dirichlet condition (e.g., $u(0) = 0$)
  - isolated boundary: Neumann condition (no flux, e.g., $\partial u / \partial \mathbf{n} = 0$)
  - partly isolated boundary: Robin condition
- Initial condition: $u(x,0) = u_0(x)$ for $x \in (0,1)$.

Solution by Separation of Variables

- To be specific assume $u(0,t) = u(1,t) = 0$.
- Ansatz: Find solutions of the form $u(x,t) = T(t)X(x)$.
- $T'X - \alpha TX'' = 0$ yields

$$\frac{T'}{T} = \alpha \frac{X''}{X}$$

- *Since this identity must be fulfilled for all* (x,t)*, the quotient must be constant.* Denote this constant by λ:

$$T' = \lambda T, \quad X'' - \frac{\lambda}{\alpha} X = 0.$$

- Solutions

$$T(t) = ce^{\lambda t}, \quad X(x) = Ae^{\mu x} + Be^{-\mu x} \quad (\mu = \sqrt{\lambda/\alpha})$$

- We must have $X(0) = X(1) = 0$, hence A and B must fulfil

$$\begin{pmatrix} 1 & 1 \\ e^{\mu} & e^{-\mu} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  In order to obtain nontrivial solutions, the system matrix must be singular.

Separation of Variables (Cont)

- This leads to the (imaginary!!) values $\mu = n \cdot 2\pi i$,

$$\boxed{\lambda = -n^2\pi^2\alpha, n = \pm 1, \pm 2, \ldots}$$

- Solutions to the odes:

$$T_n(t) = c_n e^{-\alpha n^2 \pi^2 t}, \quad X_n(x) = \sin n\pi x.$$

- Superposition principle:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha n^2 \pi^2 t} \sin n\pi x}$$

- Theorem: If u_0 permits a convergent Fourier series, then this representation is the solution of the heat equation.
- The coefficients c_n can be determined from u_0.

The Heat Equation: Comments

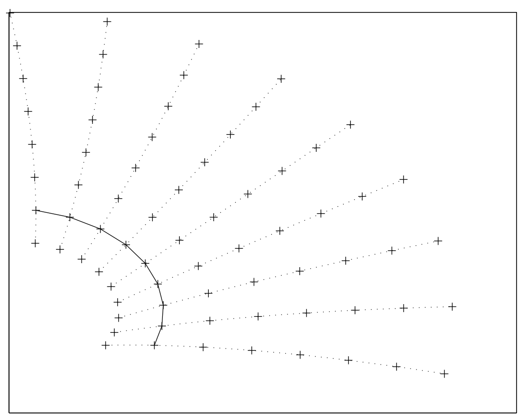
- – small n: low frequency components
  – large n: high frequency components
- High frequency components decay very rapidly: like $\exp(-\alpha n^2 \pi^2)$.
- After some time, only the component for $n = 1$ is left.
- The smoothing property is expected from intuition about heat conduction.
- The strong smoothing property makes solving the heat equation *backwards* very difficult.
  The initial value problem $u_t = -u_{xx}$ is *ill-posed*, i.e., perturbations of the initial state grow exponentially fast.
- Heat conduction has a wide range of time scales $\exp(-\alpha^2\pi^2)$. Even on a grid (which limits n to a finite number) it requires *small* time steps to follow the fastest variations whereas the solution changes globally only like $\exp(-\alpha\pi^2 t)$. (A very stiff problem!)

The Transport Equation

- Consider (in 2D)

$$c_t + \operatorname{div}(c\mathbf{u}) = 0.$$

- Assume that c is given on a curve Γ in the (x,y)-plane.



Q: Can we determine $c(x,y,t)$ in some region in (x,y,t) from this?
A: In general, *yes*!

The Material Derivative

The transport equation can be written as

$$c_t + uc_x + vc_y = -c\mathrm{div}\,\mathbf{u}.$$

Definition. *The expression*

$$\frac{Dc}{Dt} = c_t + uc_x + vc_y$$

is the material derivative *of* c*.*

The material derivative is the time rate of change of a fluid particle which moves with the stream.

There is a nice discussion of the material derivative in Strang!

The Transport Equation: Characteristics

Read: Strang, p 472–474

- Taylor expansion:

$$c(x+\Delta x, y+\Delta y, t+\Delta t) = c(x,y,t) + c_t\Delta t + c_x\Delta x + c_y\Delta y + \text{h. o. t..}$$

- Let $(x(t), y(t))$ be the trajectory of a particle following the stream:
$$\frac{dx}{dt} = u(x,y,t), \quad \frac{dy}{dt} = v(x,y,t).$$
This curve is called a *characteristic*.
- In differentials: $\Delta x = u\Delta t, \Delta y = v\Delta t$.
- Thus, on a characteristic it holds

$$\Delta c = \Delta t\underbrace{(c_t + uc_x + vc_y)}_{-c\mathrm{div}\,\mathbf{u}} + \text{higher order terms}$$

- Going to the limit:

$$\boxed{\begin{array}{c} \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v \\ \frac{Dc}{Dt} = -c\mathrm{div}\,\mathbf{u} \end{array}}$$

This is a system of ordinary differential equations (on the characteristic!)

Characteristics: Consequences

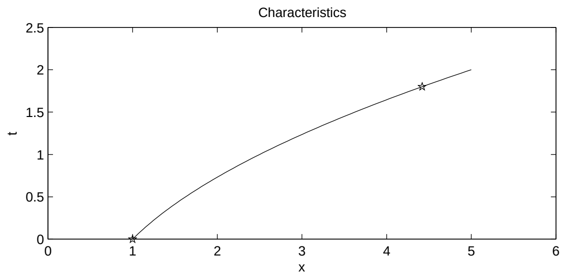
- If we know c at $(x_0, y_0) \in \Gamma$, the values of $c(x,y,t)$ can be determined on the characteristic.
- If Γ is *not* a characteristic, the solution is determined in a region of (x,y,t)-space.
- *Incompressible flow* means $\mathrm{div}\,\mathbf{u} = 0$.
For an incompressible flow, c is constant along characteristics.
- If there are sources and sinks $Q(x,y,t)$, the transport equation reads
$$\frac{Dc}{Dt} = -c\mathrm{div}\,\mathbf{u} + Q.$$
- c will be smooth along trajectories, but can be discontinuous *across*.
Such a solution does not satisfy the differential equation, but of course the integral form,

$$\frac{d}{dt}\int_{\Omega} c\,d\Omega + \int_{\partial\Omega} c\mathbf{u}\cdot\mathbf{n}\,d\Gamma = Q.$$

Method of Characteristics: Example

- Burger's equation:

$$u_t + uu_x = 1 \text{ for } t \geq 0,$$
$$u(x,0) = x \text{ for } x \in \mathbb{R}.$$



- Here, we have $c = u$. Hence, $\frac{Du}{Dt} = u_t + uu_x$ such that the system becomes
$$\frac{dx}{dt} = u, \frac{Du}{Dt} = 1.$$
- Solutions (note: $u(x_0,0) = x_0$):

$$u = x_0 + t, x = x_0 + x_0 t + \frac{1}{2}t^2.$$

Example (Cont)

- Material derivative $\equiv$ moving with the particle. Hence, we must go back in time to find the initial point on the characteristics:
$$x_0 = \frac{x - t^2/2}{1+t}.$$
- Insert this into the expression for u:
$$\boxed{u(x,t) = t + \frac{x - t^2/2}{1+t}}.$$
Exercise: Verify this!
- Note: This is a pure initial-value problem, or *Cauchy problem*.

A Vibrating Bar: The Wave Equation

Read: Strang, p 546–548

- Consider longitudinal vibrations of a bar. The displacement is u, assumed to be small.
- Let S denote the tension in the bar. Consider a small element of length dx.
Force over that element: dS
- Newton's law: mass times acceleration equals force:
$$\rho dx \cdot \frac{\partial^2 u}{\partial t^2} = dS$$
Going to differentials: $\rho u_{tt} = S_x$.
- Hooke's law (modulus of elasticity E):
$$S = E\frac{\partial u}{\partial x}.$$

The Vibrating Bar (Cont)

- Putting everything together,
$$\boxed{u_{tt} = c^2 u_{xx}}$$
where $c^2 = E/\rho$.
- This is the *wave equation*. Since it is *second* order in time, we need two initial conditions,
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = v_0(x).$$

Solution of The 1D Wave Equation

Read: Strang, p 485–486

- The standard way to solve the wave equation would be to use the separation of variables approach.
- Here, we observe
$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u$$
The equation $u_t \pm c u_x = 0$ can be solved by the method of characteristics which provides $F(x \mp ct)$.
- This motivates the change of variables
$$\xi = x + ct, \eta = x - ct.$$
Exercise: Show that the wave equations is $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$!
The general solution is (called *d'Alembert's solution*)
$$\boxed{u = F_1(\xi) + F_2(\eta) = F_1(x+ct) + F_2(x-ct)}$$

Consequences of d'Alembert's Solution

- The solution consists of one wave traveling *right* $F_2(x-ct)$ and one wave travelling *left* $F_1(x+ct)$.
- F_1 is the general solution of $u_t - cu_x = 0$, while F_2 is the general solution of $u_t + cu_x = 0$.
- The wave equation (scalar of second order) has *two* sets of characteristics,

$$\frac{dx}{dt} = +c \text{ and } \frac{dx}{dt} = -c.$$