Chapter 9: Stability of Difference Schemes

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Mathematical Models, Analysis and Simulation, Part I

Hadamard's Concept of Well-Posedness

A given problem is well-posed if its solution depends continuously on the data.

- Note: This notion depends essentially on
  - the allowed data,
  - the type of solutions searched for,
  - the measure of continuity ("the norm").
- Consider a linear homogeneous initial value problem,

$$\frac{d}{dt}u = Lu, \quad u(0) = u_0.$$

  L may be a matrix, a linear differential operator w r t space variables etc.
- Consider only initial values as data.

Definition. *The IVP is well-posed w r t the initial data, if there exist constants K, C independent of u_0 such that the IVP is uniquely solvable, and*

$$\|u(t)\| \leq Ke^{Ct}\|u_0\|.$$

Example: Advection Equation

- Consider

$$u_t = cu_x, \quad x \in \mathbb{R}, \quad t > 0.$$

  - Special case of the transport equation in 1D.
  - Did we already meet in connection with d'Alembert's solution of the wave equation.
  - It is a *hyperbolic equation*.
- Multiply by u and integrate:

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} (uu_t - cu_x u)\,dx \\ &= \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}u^2 dx - \underbrace{c\frac{1}{2}u^2\Big|_{-\infty}^{+\infty}}_{=0} \\ &= \frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 \end{aligned}$$

Hence:

$$\boxed{\|u(\cdot,t)\| = \|u_0\|}$$

Example (cont.)

- Growth estimate: $\|u(\cdot,t)\| \leq 1 \cdot e^{0 \cdot t}\|u_0\|$
- Well-posedness with
  - $K = 1$ and $C = 0$
  - data from $L^2(\mathbb{R})$, solutions from $C^1([0,\infty), L^2(\mathbb{R}))$.
- *Hyperbolic equations are well-posed.*

Well-Posedness for Nonlinear Systems of ODEs

- Consider

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}^0, \quad 0 \le t \le T.$$

- *Lipschitz condition*: There exists a L such that

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \le L\|\mathbf{y} - \mathbf{x}\|$$

  for all $\mathbf{y}, \mathbf{x}$ in a neighborhood of $\mathbf{y}_0$.
- *Theorem of Picard-Lindelöf*: For a sufficiently small T there is a unique solution which depends continuously on the data $\mathbf{y}_0$.
- Limit on the growth rate:

$$\boxed{\|\mathbf{y}(t) - \mathbf{y}^0\| \le te^{Lt}\|\mathbf{f}(\mathbf{y}^0)\|.}$$

- Consider now two solutions $\mathbf{y}, \mathbf{x}$ subject to initial conditions $\mathbf{y}^0, \mathbf{x}^0$:

$$\boxed{\|\mathbf{y} - \mathbf{x}\| \le e^{Lt}\|\mathbf{y}^0 - \mathbf{x}^0\|.}$$

  These inequalities are a consequence of *Gronwall's lemma*.

Euler Discretizations

Read: Strang, p 461–466

- Consider

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}^0, \quad t_n = n\Delta t.$$

- Let first $\mathbf{y} = y \in \mathbb{R}^1$, $\mathbf{A} = a$,

$$y(t_n) = e^{t_n a} y^0.$$

- Explicit Euler reads

$$\frac{y^n - y^{n-1}}{\Delta t} = ay^{n-1}, \text{ hence } y^n = (1 + \Delta t a)^n y^0.$$

- Since

$$\lim_{n\to\infty}(1 + \Delta t a)^n = \lim_{n\to\infty}(1 + \frac{t_n}{n}a)^n = e^{at_n},$$

  it holds

$$e_n = y(t_n) - y^n \to 0,$$

  that is *convergence*.

Euler Discretizations: Cont

- Observe

$$|e^{\Delta t a}| < 1 \text{ iff } a < 0$$

- This shall be modeled by the discrete system:

$$a < 0 \Longrightarrow |1 + \Delta t a| < 1.$$

  This holds true only if

$$\boxed{-a\Delta t < 2}$$

  Notation: *asymptotic stability*.
- If y^n shall *non-oscillatory*, $1 + \Delta t a > 0$ must hold such that even

$$\boxed{-a\Delta t < 1}$$

  is required!

Note the difference between: Convergence, asymptotic stability, non-oscillation!

Explicit Euler For Systems

- Define the matrix exponential by

$$e^{\mathbf{B}} = \sum_{j=0}^{\infty} \frac{\mathbf{B}^j}{j!}$$

- Solutions:

$$\mathbf{y}(t_n) = e^{t_n \mathbf{A}}\mathbf{y}^0, \quad \mathbf{y}^n = (1 + \Delta t\mathbf{A})^n \mathbf{y}^0$$

- Convergence can be proved as before!
- “Growth factor”:

$$\mathbf{G} = \mathbf{I} + \Delta t\mathbf{A}.$$

Explicit Euler for Systems: Cont

Q: Under which conditions holds: $\|\mathbf{G}\| < 1$?

- Let $\mathbf{A}$ be *symmetric* with eigenvalues λ_i. Then:
  - Eigenvalues of $\mathbf{G}$: $\mu_i = 1 + \Delta t \lambda_i$.
  - $\|\mathbf{G}\| = \max_i |\mu_i|$.
  - $\mathbf{y}(t) \to 0$ iff $-\mathbf{A}$ is positive definite.
  - For spd $-\mathbf{A}$: $\|\mathbf{G}\| < 1$ iff

$$\boxed{\Delta t |\lambda_{\max}| < 2}$$

- For general $\mathbf{A}$:

$$\mathbf{y}(t) \to 0 \text{ iff } \Re(\lambda_i) < 0$$
$$\mathbf{y}^n \to 0 \text{ iff } \max_i |1 - \Delta t \lambda_i| < 1.$$

Explicit Euler For PDE Discretizations

- Consider a parabolic problem

$$u_t - \operatorname{div}(k \nabla u) = 0, \quad k > 0$$
$$u_\Gamma = 0, \quad u(\cdot, 0) = u^0.$$

- Galerkin approximation:

$$\mathbf{M} \frac{d}{dt} \tau_h + \mathbf{A} \tau_h = 0, \tau_h^0 = \Pi u^0.$$

- Need the eigenvalues of $\mathbf{M}^{-1}\mathbf{A}$. It holds
  - This matrix is pd.
  - $0 < \lambda_{\min} = O(1)$ and $\lambda_{\max} = O(h^{-2})$.
- Asymptotic stability requirement $\Delta t \lambda_{\max} < 2$ becomes

$$\boxed{\Delta t < 2ch^2}$$

This requirement makes explicit Euler rather expensive!

Implicit Euler Discretization

- Discretization

$$\frac{\mathbf{y}^n - \mathbf{y}^{n-1}}{\Delta t} = \mathbf{A}\mathbf{y}^n, \text{ hence } \mathbf{y}^n = (\mathbf{I} - \Delta t \mathbf{A})^{-n} \mathbf{y}^0.$$

- Growth factor:

$$\mathbf{G} = (\mathbf{I} - \Delta t \mathbf{A})^{-1}$$

  - For a symmetric $\mathbf{A}$: $\mu_i = (1 - \Delta t \lambda_i)^{-1}$.
  - Asymptotic stability iff

$$\boxed{\max_i \frac{1}{|1 - \Delta t \lambda_i|} < 1}$$

- *If $-\mathbf{A}$ is spd, there are no restrictions on Δt!*
- The price to pay: Solve a linear system in each step:

$$(\mathbf{I} - \Delta t \mathbf{A}) \mathbf{y}^n = -\mathbf{y}^{n-1}.$$

Detailed Stability Analysis

Read: Strang, p 481–482

Theorem. [Lax Equivalence Theorem] *Consistency and stability are necessary and sufficient for convergence.*

	equilibrium problems	evolution problems
continuous	$Lu = f$	$u_t = Lu$
discrete	$L_h u_h = f_h$	$u_h^n = G_h u_h^{n-1}$
example	FEM	MOL/Rothe

Convergence Does the discrete *solution* converge towards the continuous one?

Consistency Does the discrete *equation* approximate the continuous counterpart? $\Rightarrow$ Easy!
Example:

$$L_h u_h \to Lu, \quad f_h \to f?$$

Stability Need to distinguish:
- equilibrium: Is L_h^{-1} uniformly bounded?
- evolution: Is the discrete evolution uniformly bounded, i.e., $|G_h^n| \le e^{Kn\Delta t}$?

This is the hard part!

Detailed Stability Analysis: Cont

For PDEs, this coarse description is not sufficient. We need a more detailed study.

Tool Fourier analysis

Result von Neumann stability analysis

Why Does it Work? An Example

Read: Strang, Ch. 6.3

- Consider the hyperbolic Cauchy problem

$$u_t = cu_x, \quad x \in \mathbb{R},$$
$$u(x,0) = u^0(x), \quad u^0 \in L^2(\mathbb{R}).$$

- Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx, \quad k \in \mathbb{R}$$

- Plancherel's identity

$$\|\hat{f}\|^2 = 2\pi \|f\|^2.$$

- Taking the Fourier transform in x:

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial t} u(x,t) e^{-ikx} dx = c \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} u(x,t) e^{-ikx} dx$$
$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} u(x,t) e^{-ikx} dx = -c \int_{-\infty}^{+\infty} u(x,t) \frac{\partial}{\partial x} e^{-ikx} dx + \text{b.t.}$$
$$\hat{u}_t(k) = cik\hat{u}(k)$$

An Example: Cont

- The transformed equation is a set of ode's w r t t parametrized by k:

$$\hat{u}_t = cik\hat{u} \Longrightarrow \hat{u}(k) = e^{cikt}\hat{u}^0(k).$$

- Taking norms:

$$\|\hat{u}\|^2 = \int_{-\infty}^{+\infty} |e^{cikt}\hat{u}^0(k)|^2 dk = \|\hat{u}^0\|^2.$$

- Using Plancherel's identity, we obtain the stability estimate (slide 3) once again:

$$\|u(t)\| = \|u^0\|.$$

Example: The Discrete Version

- We use the direct *complete* discretization:

$$x_j = j \cdot \Delta x, t_n = n \cdot \Delta t, u_j^n \approx u(x_j, t_n).$$

- First-order accurate difference approximations ("explicit Euler"):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

  More explicit:

$$u_j^{n+1} = u_j^n + r(u_{j+1}^n - u_j^n), \quad r = \frac{c\Delta t}{\Delta x}.$$

  r is called the *Courant number*.
- Interpolate u_j^n by a smooth function $v(x,t)$. Then

$$v(x, t+\Delta t) - v(x,t) - r(v(x+\Delta x, t) - v(x,t)) = 0$$

The Discrete Version: Cont

- It holds

$$\widehat{f(\cdot+\Delta x)} = \int_{-\infty}^{+\infty} f(x+\Delta x)e^{-ikx}dx = \int_{-\infty}^{+\infty} f(x+\Delta x)e^{-ik(x+\Delta x)}e^{ik\Delta x}dx = \hat{f}e^{ik\Delta x}$$

- Applied to our recursion:

$$\hat{v}(k,t+\Delta t) - \hat{v}(k,t) - r(e^{ik\Delta x}\hat{v}(k,t) - \hat{v}(k,t)) = 0$$

$$\boxed{\hat{v}(k,t+\Delta t) = G(\theta,r)\hat{v}(k,t)}$$

with the growth factor

$$\boxed{G(\theta,r) = 1 + r(e^{i\theta} - 1)}$$

depending on
  - $\theta = k\Delta x$, the phase shift per cell,
  - $r = c\Delta t/\Delta x$, the Courant number.

Conclusions

- The norm estimate becomes

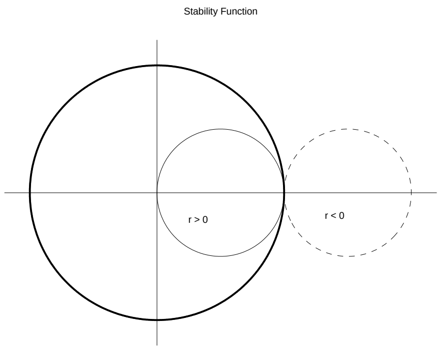
$$\|\hat{v}(\cdot,t+\Delta t)\| \le \|G\|_\infty \|\hat{v}(\cdot,t)\|.$$

- Taking into account the stability properties of the continuous problem, we want to model it discretely. This leads to the *stability requirement*

$$\|G\|_\infty \le 1$$

- The actual necessary and sufficient condition is $|G| \le 1 + O(\Delta t)$.
- Fixing r, $G(.,r)$ describes a circle with center $1-r$ and radius r.

Conclusions: Cont



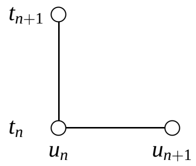
- Hence:
  - $c > 0$ is necessary for stability.
  - $r \le 1$ is necessary, which is equivalent to

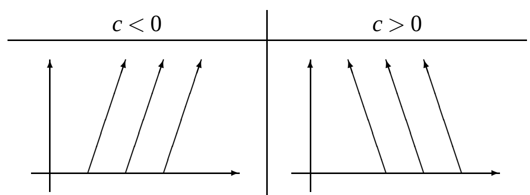
$$\boxed{\frac{c\Delta t}{\Delta x} \le 1}$$

- This condition is the celebrated *Courant-Friedrichs-Lewy condition* (short: CFL condition):
  If it is violated, then there cannot be convergence.
- Here, it is a necessary (and sufficient) condition for *stability*.

A Geometrical Interpretation

Difference stencil:





$c < 0$	$c > 0$
Waves come from the *left* but difference scheme uses only data from the *right*.	Here, the waves come from the right. **Upstream** or **upwind scheme**

Higher Order Schemes

- Try a scheme *forward in time, centered in space* (FTCS):

$$u_j^{n+1} - u_j^n - \frac{r}{2}(u_{j+1}^n - u_{j-1}^n) = 0$$

It holds $G(\theta, r) = 1 + ir\sin\theta$ such that the scheme is *unstable*,

$$\boxed{|G| \geq 1}$$

- *Lax-Friedrichs scheme* (first order)

$$\boxed{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) = \frac{r}{2}(u_{j+1}^n - u_{j-1}^n)}$$

Amplification factor $G(\theta, r) = \cos\theta + ir\sin\theta$. This is an ellipse with half axes 1 and r. So

$$\boxed{|G| \leq 1 \text{ iff } |r| \leq 1}$$

This scheme works independent of the sign od c!
Necessary for the wave equation with two characteristics of opposite directions.

Higher Order Schemes: Cont

- *Lax-Wendroff scheme* (second order)
Start by using the Taylor expansion:

$$u(t + \Delta t) = u(t) + \Delta t u_t(t) + \frac{(\Delta t)^2}{2} u_{tt}(t) + O\big((\Delta t)^3\big).$$

u_t can be replaced bu cu_x using the differential equation.
What to do with u_{tt}? Use the equation once again:

$$u_{tx} = cu_{xx} \text{ and } u_{tt} = cu_{xt} \Rightarrow u_{tt} = c^2 u_{xx}.$$

Hence,

$$u(t + \Delta t) \approx u(t) + \Delta t c u_x + \frac{(\Delta t)^2}{2} c^2 u_{xx}.$$

The last term is additional diffusion which has a stabilizing effect!
Discretization:

$$\boxed{u_j^{n+1} - u_j^n = \frac{r}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{r^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}$$

Amplification factor $G(\theta, r) = 1 + ir\sin\theta - 2r^2\sin^2\theta$.
One can prove that

$$\boxed{|G| \leq 1 \text{ iff } |r| \leq 1}$$