Chapter 10: Linear And Nonlinear Conservation Laws

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Mathematical Models, Analysis and Simulation, Part I

Plan

- A simple upstream discretization of the transport equation
- Proof of convergence via Fourier transformations
- Nonlinear conservation laws
- Shock waves
- Unique solvability: viscosity solutions

The problem

- Consider the pure initial value problem

$$u_t = cu_x, \quad x \in \mathbb{R}, \quad u(x,0) = u^0(x)$$

  with $c > 0$.
- The discretization is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c\frac{u_{j+1}^n - u_j^n}{\Delta x}.$$

- Convergence:
  Q: How to compare the discrete solution u_j^n and the continuous solution $u(x,t_n)$?
  A: Use *interpolation* by trigonometric polynomials!

Trigonometric Interpolation

- Let $\mathbf{f} = \{f_j | j \in \mathbb{Z}\}$ be an infinite sequence:

$$\tilde{f}(x) := \sum_{j=-\infty}^{+\infty} \operatorname{sinc}\frac{\pi}{h}(x - x_j) f_j, \quad x \in \mathbb{R}.$$

- Because of

$$\operatorname{sinc}\frac{\pi}{h}(x - x_j) = \frac{\sin\frac{\pi}{h}(x_k - x_j)}{\frac{\pi}{h}(x_k - x_j)} = \begin{cases} 0, & k \neq j \\ 1, & k = j \end{cases}$$

  it holds

$$\boxed{\tilde{f}(x_j) = f_j}$$

  *$\tilde{f}$ is a trigonometric interpolant of **f**.*

Semi-Discrete Fourier Transform

- The *semi-discrete Fourier transform* is given by

$$\hat{f}(k) = h \sum_{j=-\infty}^{+\infty} f_j e^{-ikx_j}, k \in \mathbb{R}.$$

  Notes:
  - $\hat{f}$ is defined on all of $\mathbb{R}$.
  - Compared to the DFT, a different scaling is used.
- $\hat{f}$ is $2\pi/h$-periodic,

$$\hat{f}(k) = \hat{f}(k + 2\pi/h).$$

- The *band-limited* Fourier transform:

$$\breve{f}(k) = \begin{cases} \hat{f}(k), & |k| < \pi/h, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem.

$$\boxed{\hat{\hat{f}}(k) \equiv \breve{f}(k)}$$

Proof of Theorem

- Define, for $a > 0$,

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & \text{otherwise.} \end{cases}$$

- Fourier transform (Strang, p 310):

$$\widehat{\chi_a}(k) = 2a \frac{\sin ak}{ak}.$$

- Using the inversion theorem, one gets

$$\widehat{\left(\frac{\sin bx}{bx}\right)} = \frac{\pi}{b} \chi_b(x)$$

Proof: Cont

$$\begin{aligned}
\hat{\hat{f}}(k) &= \int_{-\infty}^{+\infty} \Big(\sum_{j=-\infty}^{+\infty} \frac{\sin \frac{\pi}{h}(x - x_j)}{\frac{\pi}{h}(x - x_j)} f_j \Big) e^{-ikx} dx \\
&= \sum_{j=-\infty}^{+\infty} f_j \int_{-\infty}^{+\infty} \frac{\sin \frac{\pi}{h}(x - x_j)}{\frac{\pi}{h}(x - x_j)} e^{-ikx} dx \\
&\overset{y = x - x_j}{=} \sum_{j=-\infty}^{+\infty} f_j e^{-ikx_j} \int_{-\infty}^{+\infty} \frac{\sin \frac{\pi}{h} y}{\frac{\pi}{h} y} e^{-iky} dy \\
&= \sum_{j=-\infty}^{+\infty} f_j e^{-ikx_j} h \chi_{\pi/h}(k) \\
&= \begin{cases} h \sum_{j=-\infty}^{+\infty} f_j e^{-ikx_j}, & |k| < \pi/h \\ 0, & \text{otherwise} \end{cases} \\
&= \breve{f}(k)
\end{aligned}$$

Convergence

$$\|u(\cdot, t_n) - \tilde{u}(\cdot, t_n)\| \xrightarrow{h, \Delta t \to 0} 0??$$

Continuous Fourier transform of $u_t = c u_x$, $u(\cdot, 0) = u^0$ yields

$$\frac{d\hat{u}}{dt} = ick\hat{u}, \quad \hat{u}(\cdot, 0) = \widehat{u^0}.$$

Hence,

$$\hat{u}(k, t + \Delta t) = e^{ick\Delta t} \hat{u}(k, t).$$

Finally,

$$\boxed{\hat{u}(\cdot, t) = H(\Delta t)^n \hat{u}(\cdot, 0) = H(k, \Delta t)^n \widehat{u^0}}$$

$$H(k, \Delta t) = e^{ick\Delta t}.$$

Convergence: Cont

Discrete Write the discretization like

$$u_j^{n+1} = u_j^n + r(u_{j+1}^n - u_j^n), \quad r = \frac{c\Delta t}{\Delta x}.$$

Semi-discrete Fourier transform

$$\hat{u}^{n+1} = \hat{u}^n + r(e^{ikh}\hat{u}^n - \hat{u}^n).$$

Finally,

$$\boxed{\begin{aligned}\hat{u}^n &= G^n \hat{u}^0\\ \check{u}^n &= G^n \check{u}^0, |kh| < \pi\end{aligned}}$$

$$G = 1 + r(e^{-ikh} - 1).$$

The initial value can be chosen in different ways. We use the most obvious

$$u_j^0 = u^0(x_j).$$

Convergence: Cont

Remember Plancherel's and Parseval's identities:

$$\|\hat{u}\|^2 = 2\pi\|u\|^2, \quad 2\pi\|\widetilde{u^0}\|^2 = \|\check{u}^0\|^2.$$

We obtain

$$\begin{aligned}\|u(\cdot,t_n) - \tilde{u}(\cdot,t_n)\| &= \frac{1}{\sqrt{2\pi}}\|\hat{u}(\cdot,t_n) - \check{u}^n\|\\ &= \frac{1}{\sqrt{2\pi}}\|H^n\widehat{u^0} - G^n\check{u}^0\|\\ &= \frac{1}{\sqrt{2\pi}}\|H^n\widehat{u^0} - H^n\check{u}^0 + H^n\check{u}^0 - G^n\check{u}^0\|\\ &\le \frac{1}{\sqrt{2\pi}}\left(\|H^n(\widehat{u^0} - \check{u}^0)\| + \|(H^n - G^n)\check{u}^0\|\right)\end{aligned}$$

- The first term contains the propagation of the initial error (in frequency space). In the present case, $|H| = 1$.
- The first term becomes small if the approximation of the initial values is consistent:

$$\boxed{\|\widehat{u^0} - \check{u}^0\| \to 0 \text{ for all } u^0 \in L^2(\mathbb{R}).}$$

The Second Term

- Assume stability:

$$|G| < 1 \Leftrightarrow 0 < c\Delta t < h.$$

- Split the factors:

$$H^n - G^n = (H - G)\sum_{j=0}^{n-1} H^{n-j-1}G^j.$$

- Because of stability, it holds

$$\left|\sum_{j=0}^{n-1} H^{n-j-1}G^j\right| \le \sum_{j=0}^{n-1} 1\cdot|G^j| < n = \frac{t_n}{\Delta t}.$$

The Second Term: Cont

- Fix the wave number k now. Then

$$\begin{aligned}H - G &= e^{ick\Delta t} - (1 + r(e^{ikh} - 1))\\ &= 1 + ick\Delta t + O\big((\Delta t)^2\big) - (1 + \frac{c\Delta t}{h}(1 + ikh + O(h^2) - 1))\\ &= 1 + ick\Delta t + O\big((\Delta t)^2\big) - (1 + ick\Delta t + O\big((\Delta t)^2\big))\\ &= O\big((\Delta t)^2\big))\end{aligned}$$

This is a consequence of first order consistency: $|H - G| = O\big((\Delta t)^{p+1}\big)$

- Putting everything together,

$$\boxed{|H^n - G^n| = t_n \cdot O(\Delta t).}$$

Conclusions

- We have just seen:

  consistency + stability ⇒ convergence.

- Previous chapter:

  no stability ⇒ no convergence.

- This is obvious:

  convergence ⇒ consistency.

- Conclusion:

  consistency + stability ⇔ convergence

  This is one instance of the *Lax Equivalence Theorem*.

Nonlinear Conservation Laws: An Example

Read: Strang, Ch 6.6

Problem: Determine the flow of cars on a narrow street.

- Density ρ of cars: number of cars per unit length.
- Velocity of cars v depends on density: $v = v(\rho)$.

  ρ small ⇒ v large

  ρ large ⇒ v low

  Normalization:
  - Largest possible velocity: $v = 1$.
  - Road full means $v = 0$. Let this be at $\rho = 1$.
- Simple velocity model: $v(\rho) = 1 - \rho$.
- Conservation: Flux is $\Phi = v\rho = (1-\rho)\rho$,

$$\rho_t + \mathrm{div}((1-\rho)\rho) = 0.$$

- Initial data: $\rho(x,0) = \rho^0(x)$.
- Solution by characteristics:

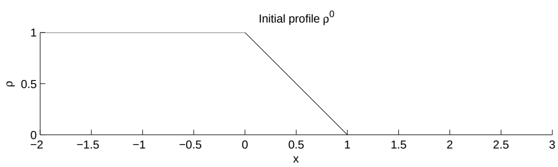
$$\frac{dx}{dt} = 1 - 2\rho, \quad \frac{d\rho(x(t),t)}{dt} = 0.$$

  - ρ is constant along each characteristic.
  - Since dx/dt is constant, the characteristics are *straight lines*.

Example: Smooth Solutions

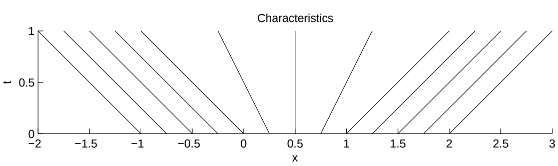
- Assume initial data:

$$\rho^0(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$$



- Characteristics:

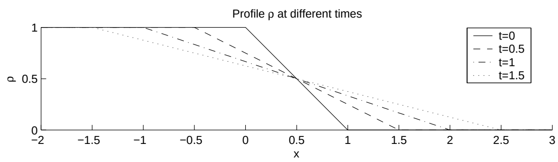
$$x(t) = \left.\frac{dx}{dt}\right|_{x_0} t + x_0 = (1 - 2\rho^0(x_0))t + x_0.$$



Example: Cont

- The following possibilities arise:

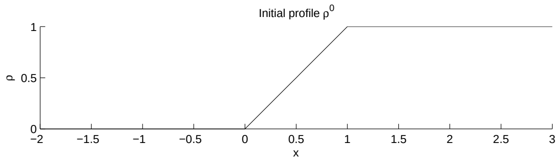
case	density	solution
$x \ge t+1$	$\rho^0 = 0$	$\rho = 0$
$x \le -t$	$\rho^0 = 1$	$\rho = 1$
$-t < x < t+1$	$\rho^0 = 1 - x_0$	$x = (1-2(1-x_0))t + x_0$ $x_0 = \frac{x+t}{2t+1}$ $\rho = 1 - x_0 = \frac{1+t-x}{2t+1}$



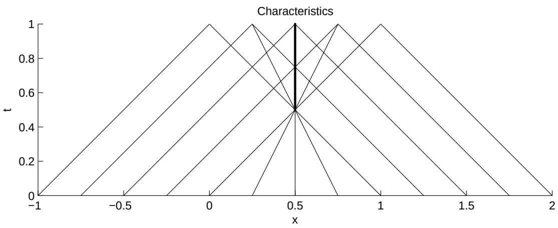
Example: Shocks

- Initial data:

$$\rho^0(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$



- Characteristics:



  – What is the correct solution at crossing characteristics?
  – A *shock* will form. **Where?**

Shocks: Cont

Compute the shock location $s(t)$: Integrate the equation on $[s-\varepsilon, s+\varepsilon]$:

$$\int_{s(t)-\varepsilon}^{s(t)+\varepsilon} \frac{\partial\rho}{\partial t} dx + \Phi(\rho(s(t)+\varepsilon)) - \Phi(\rho(s(t)-\varepsilon)) = 0$$

It holds

$$\frac{d}{dt}\int_{s(t)-\varepsilon}^{s(t)+\varepsilon} \partial\rho dx = \int_{s(t)-\varepsilon}^{s(t)+\varepsilon} \frac{\partial\rho}{\partial t} dx + \rho(s(t)+\varepsilon)\frac{ds}{dt} - \rho(s(t)-\varepsilon)\frac{ds}{dt}$$

Taking the limit $\varepsilon \to 0$, this leads to

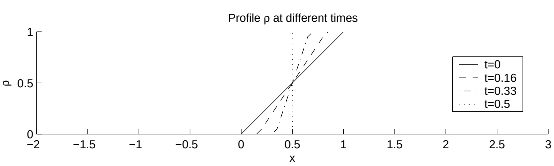
$$0 = \lim_{\varepsilon\to 0}\int_{s(t)-\varepsilon}^{s(t)+\varepsilon} \frac{\partial\rho}{\partial t} dx + \rho(s(t)+)\frac{ds}{dt} - \rho(s(t)-)\frac{ds}{dt}.$$

Rankine-Hugoniot Jump Condition

Result:

$$\boxed{\frac{ds}{dt} = \frac{\Phi(\rho^+) - \Phi(\rho^-)}{\rho^+ - \rho^-}}$$

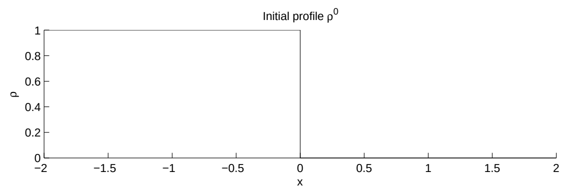
This is the famous *Rankine-Hugoniot jump condition*.



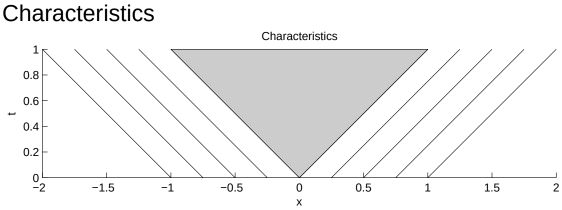
Entropy Solutions, Rarefaction Waves

- Consider initial data

$$\rho^0(x) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases}$$



- Characteristics



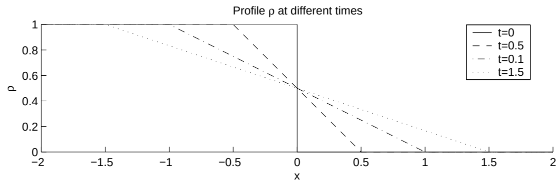
Entropy Solutions, Rarefaction Waves: Cont

There are (at least) two solutions which satisfy the Rankine-Hugoniot condition:

- $\rho(x,t) = \rho^0(x)$ for all $t > 0$.
- Rarefaction wave

$$\rho(x,t) = \begin{cases} 1, & x \le -t \\ 0, & x \ge t \\ \frac{1}{2}(1 - \frac{x}{t}), & -t < x < t \end{cases}$$

Which one is the "correct" one?



In order to select one solution, we must use physical considerations.

Entropy Solutions, Rarefaction Waves: Cont

Idea: Modify the equation by adding small artificial diffusion:

$$v = 1 - \rho - \varepsilon\rho_x/\rho, 0 < \varepsilon \ll 1$$

(Say, drivers look ahead.)

Modified equation:

$$\boxed{\rho_t + (\rho(1-\rho))_x = \varepsilon\rho_{xx}}$$

- *Entropy solution*: The limiting function $\rho = \lim_{\varepsilon\to 0} \rho_\varepsilon$
- All solutions are smooth. However, gradients may become large.
- The added diffusion is also necessary for numerical schemes.

Entropy Solutions, Rarefaction Waves: Cont

Numerical approximations: How to choose ε?

1. $\varepsilon = O(h)$ gives nice, non-wiggly solutions, but only first-order accurate.
2. Choose $\varepsilon = \varepsilon(\rho_x)$, where

$$\varepsilon \sim \begin{cases} h, & \text{if } \rho_x \text{ is large} \\ 0, & \text{if } \rho_x \text{ is small} \end{cases}$$

   This is called *switched artificial diffusion*.
   - ON: near shocks,
   - OFF: where the solution is smooth.