Lab 8:   DN2266 Modeling projects

Deadline: Jan 11, 2009

The last lab of the course is a modeling exercise involving some well-known application. The solution requires numerical computing, often the solution of a PDE for which `COMSOL Multiphysics` is available. *You shall choose exactly one project!*

Contents

# The D'Alembert Paradox – 3D mesh refinement

Background

The paradox states that the force exerted by potential flow on a closed body vanishes. Your task is not to resolve the paradox but rather to verify its mathematical formulation, and to investigate a simple model devised by the NACA-scientist M.Munk in 1924 for forces and moments on slender bodies.
We consider the potential model for plane incompressible flow of negligible viscosity,

$$(u,v,w)=-\nabla\Phi, \ \Delta\Phi=0,$$
$$\nabla\Phi\cdot n=0 \text{ on the solid boundary } \Gamma,$$
$$\Phi\approx U(x\cos\alpha+z\sin\alpha) \text{ far away}$$

where Φ is the velocity potential, Γ is the surface of an object immersed in a wind of magnitude U at "angle of attack" α:
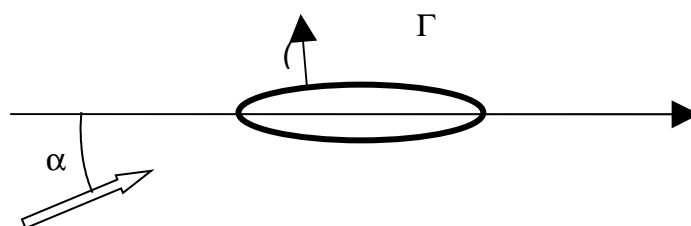
The pressure is computed by Bernoulli's law

$$p+1/2\rho\|(u,v,w)\|^2=const.=p_\infty+1/2U^2$$

where $p(x,y,z)$ is the (static) pressure (unit: N/m^2), p_{infty} is the undisturbed static pressure and ρ is the density (constant) of the fluid. The force on the solid boundary is directed along the normal and proportional to pressure: $p\mathbf{n}$.
Your task is to compute the force and the moment on the body, and compare your results with the Munk approximation for slender bodies: the moment is that of a pure couple and proportional to the volume in 3D of the body

$$M=qV\sin2\alpha\cos\alpha, q=1/2\rho U^2$$

where q is called the dynamic pressure.
Force and moment around point $\mathbf{r}_0 = (x_0,y_0,0)$:

$$F=\int_\Gamma pn\,dS, M=\int_\Gamma pn\times(r-r_0)\,dS,$$

Tasks

Choose some simple shape like an ellipsoid for the body, and a large sphere of radius about 10 times the body length. Use $U = 1$ and $\rho = 1$. Solve the PDE with COMSOL, use boundary integration to plot force and moment for a range of α-values from 0 to 45°. Try

different aspect ratios for the ellipsoid, from circle to thin cigar and comment on the accuracy of Munk's moment formula.

Hints: You need a fine grid close to the body! Use local mesh refinement and conduct a convergence study. Plot streamlines and modulus of the gradient to illustrate. Define α and the expression for the far away (Dirichlet) condition to make it simple to change U and α. Try the `parametric solver` (under `solver parameters`) to generate the solutions for α = 0:5:45 degrees at once.

Questions
Show that
- i) Constant pressure give zero force and moment, so you can choose pressure far away as you please, like = 0.
- ii) Pressure with constant gradient gives force directed along the gradient, proportional to the volume, discovered before BC 200 by ... whom?
- iii) For a given body, M depends on α, ρ, and U. Show, by using linearity of the flow equations, that
$$M(\alpha,\rho,U)=2qM(\alpha,1,1)$$
(Looks funny, but remember that q far away is 1/2). You can also use   dimensional analysis to derive the formula.

Report
1. Description of COMSOL model, explanations on how the mesh was refined, etc.
2. Plots of the Munk result and your obtained results for M for a fat ellipsoid (sphere) and a thin cigar.
3. Answers to questions

The FitzHugh-Nagumo model for neural wave propagation

The model

Look up the FitzHugh-Nagumo model on Wikipedia. This URL is historical but writes the equations differently:
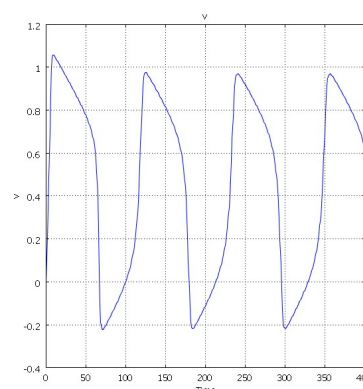
http://www.scholarpedia.org/article/FitzHugh-Nagumo_model
 The FHN-equations are

$$v_t = D\Delta v + f(v) - w + I_{app}$$

$$w_t = \varepsilon(v - \gamma w)$$

$$f(v) = v(v - \alpha)(1 - v)$$

and a time-plot of v at a point for a particular set of parameter values is shown right. The cubic non-linearity gives the system bifurcations. There are limit cycles, or stable equilibria, or "single pulse" solutions depending on the applied current I_{app}. We fix the parameters

$\alpha = 0.14$, $\varepsilon = 0.01$ and $\gamma = 0.254$

for the lab work and look for values of D and I_{app} which produce interesting results.

Make a model for the spike generation and propagation by diffusion of a nerve impulse on a plane domain as follows. The result of a similar simulation is in fig 2. The task is to find parameter values which produce the different types of solution and show simulations.

The excitation is an applied current in a small circular sub-domain, say at (x_0, y_0) with radius R. If the current is too small, nothing much happens.
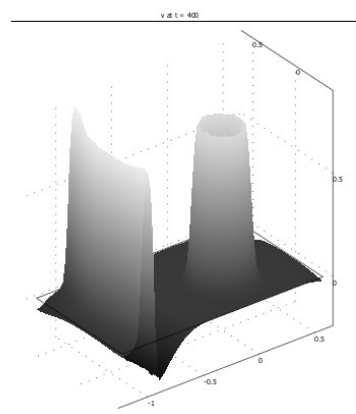
Tasks

1. Identify the parameter values with an ODE model (i.e. $D = 0$) for which you can run simulations quickly in MATLAB. That also gives information on the time-scale: the period and length of pulses. Find the threshold (actually, "quasi-threshold") current for the appearance of the limit cycle of spikes.

2. COMSOL modeling

Construct a "pepparkakehjärta"-shaped geometry and position the excitation electrode. Use homogeneous Neumann boundary conditions for *v* and *w*.

Try also Dirichlet for v – what happens if you try Dirichlet for *w*?

Hints: Choose the `PDE Modes/PDE Coefficient form/Time-dependent analysis` mode, 2D, in the Model Navigator, and name two variables `v` and `w`. Create constants `alpha, epsil, gamma, D, x0, y0, R` and `Iapp` (and maybe a few more) under `Options`

3. The ODE runs will give an idea of the period of the spikes. Once you know the time scale, you can choose a value for *D* which gives steep fronts without too slow wave speed. Measure the wave speeds for two different *D*. Theory predicts that wave speed $c \approx const.\sqrt{D}$, assuming that the *v*-wave has an amplitude of about 1.

4. *For enthusiasts only*: Suppose the impulses travel in a thin layer on the surface of a simple 3D object like an ellipsoid – the heart, for instance, or a doughnut. Then, there are no boundary conditions to worry about. For a spherical surface of radius *R*, one can use spherical coordinates (θ,ϕ), and the Laplace (-Beltrami) operator for surface diffusion is

$$R^2 \Delta_S u = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial\varphi^2}$$

which of course is not so good at $\theta = 0$, but .... Propose how to solve the FHN-equations on a spherical surface in COMSOL.

Report

1. The results of ODE-simulations – parameter values and a few plots with explanations.
2. Surface plots of *v* and *w* in the COMSOL solution for a representative time, measurements of wave speed, explanations; plot of *v*(*t*) and *w*(*t*) at a point close to the excitation electrode. Don't forget axis labelling etc.!

Shortest escape times by a PDE

Background

Suppose a particle wants to escape from a bounded 2D domain Ω as quickly as possible. It knows that the speed is limited by the speed of light $c(x,y)$ and can choose direction freely. The task is to compute the fastest escape route for all points in W. One solution comes from defining $T(x,y)$ as the shortest travel time from (x,y) to the boundary where, of course, $T = 0$. It follows that

$$|\nabla T| = 1/c \qquad\qquad (1)$$

the Hamilton-Jacobi equation for the optimization problem.
Your task is to set up such a problem and solve it by COMSOL, and to compare the solution with a known solution computed by geometrical optics. The domain is the unit square, and $c = 1$ everywhere except in a circular disk centred at (x_0,y_0) with radius R where $c = c_2$.

Geometrical Optics

We know that
i) when c is constant, the fastest route is a straight line and light travels so.
ii) If c is piecewise constant, the rays are refracted at the interface to follow Snell's law,

$$\frac{c_1}{\sin\theta_1} = \frac{c_2}{\sin\theta_2}$$

usually written in terms of the index of refraction, $n = c_{\text{inf}}/c(x,y)$, where θ is the angle between the ray and the interface normal, and c_{inf} is a reference speed (e.g speed in vacuum). These facts allow you to compute and plot fastest routes which don't cross the circle or once (or twice) by a little matlab programming.

COMSOL modeling

Equation (1) is very non-linear and its solutions are not expected to be differentiable. This is clear from the known solution to the problem with $c_2 = 1$ (no interface) where T has discontinuous gradient at the square diagonals. Let us look at the time-dependent variant of (1) with homogeneous initial data

$$T_t - |\nabla T| = 1/c , T(x,y,0) = 0 \qquad\qquad (2)$$

Linearization of (2) indicates that small perturbations travel with velocity 1 (yes, 1, not c) along the characteristics which are the gradients of the steady solution, i.e., the fastest escape routes. This leads us to believe that the steady solution will be reached in finite time = length of longest fastest escape route.
The non-differentiability must be relieved by addition of artificial viscosity,

$$T_t - |\nabla T| = D\Delta T + 1/c , T(x,y,0) = 0 \qquad\qquad (3)$$

Tasks

1. Implement (3) in COMSOL on the geometry suggested.
2. Use a "large" D and $c_2 = 1$ and plot:
   a. the gradient as arrows
   b. escape routes as streamlines
3. Compare with the exact solution and choose a D and element sizes to give 1% max. error in T. Plot the l2 norm of dT/dt as function of t. Is it true that the steady solution is reached after finite time?
4. Now choose $c_2 > 1$ and solve and plot. Compare to the exact solution computed by ray-tracing. *Hint:* The ray-tracer find fastest paths easily from the boundary, but it is hard to find the fastest path from a given (x,y). So choose an (x,y) *on* a ray-traced path and compare the T so obtained with COMSOLs which is easy to get for any (x,y) by e.g the `post processing/data display menu`)
5. For $c_2 < 1$ the paths are quite different. Explain.

Report

1. Motivate equation (1), assuming that T is differentiable.
2. Explain your matlab-program and show plots of the rays computed for $c_2 > 1$.
3. Carry out the linearization indicated of the equation, written out for your convenience as
$$T_t - \sqrt{T_x^2 + T_y^2} = 1/c(x,y),$$
$$T = u(x,y,t) + T_\infty(x,y), \sqrt{T_{\infty,x}^2 + T_{\infty,y}^2} = 1/c(x,y),$$
Hint:You should arrive at
$$u_t - v \cdot \nabla u = 0, v = \frac{1}{|\nabla T_\infty|} \nabla T_\infty$$
4. Include the plots indicated in the tasks 1-5 above, with explanations, proper   axis labelling, etc.