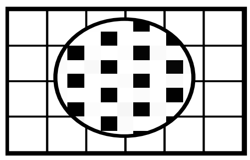
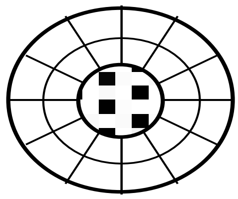
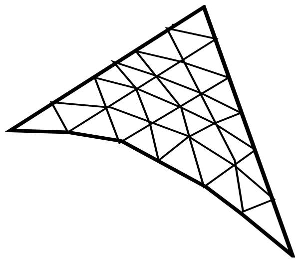
Lecture: Finite Element Methods for Time-Domain Problems

We have seen the Yee-scheme, and how to treat sources and boundary conditions. In all the Yee scheme is extremely successful. Its drawback is the necessity for a rectangular equi-sized grid: Non-rectangular geometry has to be approximated by LEGO-style bricks, sometimes also referred to as staircase effect.
Finite Volume and Finite Element methods for dealing with curvilinear geometry have been used for e.g. computational fluid dynamics since the sixties. Common grid types are shown here:

Cartesian (Yee) Mapped Cartesian Unstructured triangular

It is natural to investigate how such schemes can be used for the Maxwell equations as well: Existing pre- and post-processing software, and some of the solvers developed for CFD, could be re-used. This lecture gives an overview of how the various issues have been addressed.

Difficulties in standard approaches applied to the Maxwell equations

• There are waves running both right and left and symmetric schemes are natural. However standard central difference schemes are prone to odd/even decoupling, i.e., oscillatory numerical solutions that oscillate with the shortest wavelength representable on the grid. These waves are sometimes referred to as "spurious modes" and can also be associated with the dispersion, wavelength $2\Delta x$ perturbations travel at speed 0.
In steady CFD the remedy is to add a controlled amount of numerical dissipation, by artificial viscosity or diffusion, or even fourth order difference terms.

• An alternative is the use of (unsymmetric) upstream differences. This requires essentially that the system be diagonalized, which is easy when the coefficients are piecewise constant. But first order schemes are much too dissipative, and better than first order accuracy requires complicated schemes. Note that it is much easier to devise good schemes for the second order wave equation formulation because of the compactness of the second difference formula. But often both ***H*** and ***E*** are necessary, e.g. for modeling dispersive and non-linear media, and in such cases the wave equation formulation thus is slightly more cumbersome.

For spatially multidimensional cases there is also another source of spurious solutions:
• Artificial growth of non-solenoidal components of ***B*** and ***D***, i.e., violation of (proper discrete variants of) the Gauss laws. This does not happen in the Yee scheme but may in second order wave equation formulations, or in approximations to curl curl unless the scheme is properly designed – see Andre’s lecture Oct 9. This is related to "long time instability", which is often a linear growth of the error:

Exercise

a) Write out the curl curl equation for **H** in a TM 2D case in terms of d/dx and d/dy
and with **H** = (H_x, H_y,,0)^T show that it becomes (take $\varepsilon = \mu = 1$, $\sigma = 0$)

$$\begin{cases} \dfrac{\partial^2 H_x}{\partial t^2} = \dfrac{\partial^2 H_x}{\partial y^2} - \dfrac{\partial^2 H_y}{\partial x \partial y} \\ \dfrac{\partial^2 H_y}{\partial t^2} = -\dfrac{\partial^2 H_x}{\partial x \partial y} + \dfrac{\partial^2 H_y}{\partial x^2} \end{cases} \qquad (**)$$

b) Derive the dispersion relation for the system. You will find one (a double) eigenvalue zero, i.e., a "standing" wave. What is the interpretation of the corresponding eigenvector?

Exercise The curl Maxwell equations preserve the divergence of the initial field which should be zero to satisfy the Gauss laws, and the L2 norm of their solution is conserved (see Lect 1).

a) Consider non-solenoidal perturbations to (**). Show that the initial value problem is satisfied by

$$H_x, = d/dx\,(f + t\,g),\ H_y = d/dy\,(f + t\,g)$$

Hence, *gradient* components generically grow linearly with time. This is part of the connection between divergence preservation and long-time stability.

Exercise a) Derive the dispersion relation for a central difference ($\Delta x = \Delta y = h$) space-discretization using the ansatz

$$\begin{pmatrix} H_x \\ H_y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{i\omega t + ik(x\cos\alpha + y\sin\alpha)}$$

This is a $2\pi/k$-wavelength wave traveling in the direction ($\cos\alpha$, $\sin\alpha$). Plot phase speed ω/k as function of kh. For the wave equation it is $\omega/k = \pm 1$. Note that the space-discrete scheme has a *pair* of ± roots.

b) Then consider the second order time-discretization

$$d^2y/dt^2 = f(y) \Rightarrow (y^{n+1} - 2y^n + y^{n-1}) = \Delta t^2 f(y^n)$$

Numerical calculation of the ω-values is probably easiest.

Finite Element Schemes

(see also CEMbook Ch6)

We consider now the second order formulation, assuming no currents,

$$\begin{cases} \mu \dfrac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \left(\dfrac{1}{\varepsilon} \nabla \times \mathbf{H} \right) & \text{in } V \subseteq \mathbf{R}^3 \\ \nabla \cdot \mathbf{H} = 0 \end{cases}$$

Weak (or variational) form:

$$(\mathbf{v}, \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} + \nabla \times \left(\frac{1}{\varepsilon} \nabla \times \mathbf{H} \right)) = 0 \text{ for "all" } \mathbf{v}(\mathbf{x});$$

Inner product:

$$(\mathbf{f}, \mathbf{g}) = \int_V \mathbf{f}^H(\mathbf{x}) \mathbf{g}(\mathbf{x}) dV$$

Integration by parts:

$$\frac{\partial^2}{\partial t^2}(\mathbf{v}, \mu \mathbf{H}) - \left(\frac{1}{\varepsilon} \nabla \times \mathbf{H}, \nabla \times \mathbf{v} \right) + \int_{\partial V} \mathbf{v}^H ((\frac{1}{\varepsilon} \nabla \times \mathbf{H}) \times \mathbf{n}) dS = 0 (*)$$

The last formulation makes it clear what we are looking for:

• $\mathbf{H}(\boldsymbol{x},t)$ twice differentiable in time, with div $\mathbf{H} = 0$,

• $\mathbf{H}(.,t)$ in $[H^1(V)]^3$: first derivatives must be square-integrable over V.
Any solution of the original equation then satisfies the weak formulation, and any weak solution, if sufficiently differentiable, satisfies the strong form.

The "method of lines", Galerkin approximation in space, now proceeds by seeking a numerical solution $\mathbf{H}_h(\boldsymbol{x},t)$ as a linear combination of a set of selected basis functions $\mathbf{N}_j$, j = 1,…,N, with time-dependent coefficients c_j, and to produce a set of equations by requiring (*) to hold for $\mathbf{v} = \mathbf{N}_1, \ldots, \mathbf{N}_N$.

$$\mathbf{H}_h(x,t) = \sum_{j=1}^{N} c_j(t)\mathbf{N}_j(x)$$

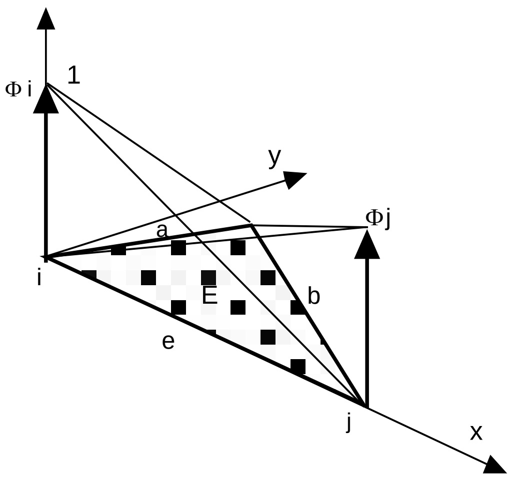
The standard (*scalar*) nodal elements are piecewise linear functions Φj over triangles (2D) (tetrathedra in 3D) and would be used like this:

$$\mathbf{H}_h(\mathbf{x},t) = \sum_{j=1}^{n} \mathbf{H}_j \Phi_j(\mathbf{x})$$

Let the vertices of the elements be $\mathbf{x}_i$, i = 1,…,n. Then Φ_j is non-zero only over the elements which have $\mathbf{x}_j$ as a vertex, and $\Phi_j(\mathbf{x}_k) = \delta_{kj} = 1$, k = j, = 0, other k. Thus, the unknown $\mathbf{H}_j$ is the (*vector*) value at $\mathbf{x}_j$ of $\mathbf{H}_h$. The Φj are used to discretize the scalar Helmholtz equation in 2D.

The issue is how to satisfy the Gauss law exactly. Vector basis functions with div = 0 have been invented e.g. by Nédelec and (in a similar but different context) by Rao, Glisson, and Wilson (RGW). The piecewise linear *edge* or *vector element* $\varphi_e(\mathbf{x})$ which will be our $\mathbf{N}_e$, one for each edge of the triangle mesh, can be defined for triangles as follows: it is constructed from an anti-symmetric combination of the Φj in such a manner that div $\varphi_e(\mathbf{x}) = 0$: Let e be the edge between nodes i and j in the triangle. Then take

$$\varphi_e = \Phi_i \nabla \Phi_j - \Phi_j \nabla \Phi_i \qquad (**)$$



Note that
• along $\mathbf{e}$, assumed directed from i to j – this is where ordering of the triangle vertices becomes important! - , the tangential component is the same for the two triangles sharing the edge, namely

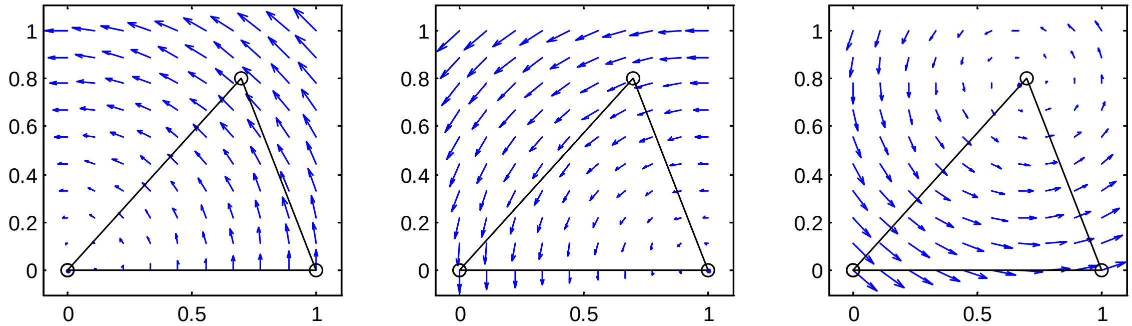
$$\varphi_e \cdot \mathbf{e} = \Phi_i \nabla \Phi_j \cdot \mathbf{e} - \Phi_j \nabla \Phi_i \cdot \mathbf{e} = \Phi_i 1/e + \Phi_j 1/e = 1/e$$

• the tangential components along *a* and *b* are zero: at *b*, Φi = 0 and

$$\varphi_e = -\,\Phi j\; 1/h_b\; \mathbf{n}$$

where **n** is the outward unit normal to *b* and h_b is the length of the normal from i to *b*.
and thus

• c_e/e is the value of the tangential component of **H** along the edge, hence constant.
• the *tangential* component of **H** is continuous across element boundaries.
• the *normal* component of **H** is in general discontinuous across the edge, and this is necessary to allow piecewise constant μ.
• div = 0 from the antisymmetry and second derivatives being zero.
Here is a plot of the three edge basis functions for a triangle:



by these m-files:

```
function [f,fx,fy] = nodfi(x,y,xi,yi)
% computes the three nodal basis functions f
% and their gradients fx, fy
% for the (x,y) triangle at points (xi,yi)
n    = length(xi);
one  = ones(1,n);
zero = 0*one;
B  = inv([x;y;ones(1,3)]);
f  = B*[xi;yi;one];
fx = B(:,1)*one;
fy = B(:,2)*one;
```

```
function [f1,f2,f3] = edgefi(x,y,xi,yi)
% computes the three edge basis functions f1,f2,f3
% for the (x,y) triangle at points (xi,yi)
[fi,fix,fiy] = nodfi(x,y,xi,yi);
f1 = [fi(2,:).*fix(3,:)-fi(3,:).*fix(2,:);...
      fi(2,:).*fiy(3,:)-fi(3,:).*fiy(2,:)];
f2 = [fi(3,:).*fix(1,:)-fi(1,:).*fix(3,:);...
      fi(3,:).*fiy(1,:)-fi(1,:).*fiy(3,:)];
f3 = [fi(1,:).*fix(2,:)-fi(2,:).*fix(1,:);...
      fi(1,:).*fiy(2,:)-fi(2,:).*fiy(1,:)];
```

Exercise: It is a necessary condition for convergence of the finite element solution that any *constant* **H**-field can be represented exactly by the element. Is that true? Can any solenoidal field (i.e. divergence free) with all components first degree polynomials in (*x*,*y*) be represented?

Exercise: Investigate how a vector element can be constructed over quadrilaterals. Can the bilinear basis functions over quadrilaterals be used as Φj in (**)?

It is now easy to derive formulae for the element mass- and stiffness matrices,

$$\begin{cases} m_E = \int_E \varphi_{e1} \cdot \varphi_{e2} dS, \\ s_E = \int_E (\nabla \times \varphi_{e1}) \cdot (\nabla \times \varphi_{e2}) dS \end{cases} , e1, e2 = a, b, e$$

e.g. by using geometry and $\mathbf{a} \times \mathbf{b} = ab \sin \phi \mathbf{e}_z$ when **a** and **b** are vectors in the xy-plane. Also study the CEM book code. Note that although φ is a first degree polynomial, curl φ is not necessarily constant over each triangle, but grad Φ is constant.