ROYAL INSTITUTE OF TECHNOLOGY

Royal Institute of Technology, KTH
Department of High Performance Computing and Visualization
Bärbel Janssen, barbel@kth.se

Numerical Linear Algebra

Prestudy No. 4 (Autumn 2014)

Problem 4.1: The unit sphere with respect to a vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is defined by

$$S := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Sketch the spheres in $\mathbb{R}^2$ corresponding to the l_1-norm, the euclidian norm and the l_∞-norm:

$$\|x\|_1 = |x_1| + |x_2|, \quad \|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2}, \quad \|x\|_\infty = \max\{|x_1|, |x_2|\}.$$

How do the unit spheres corresponding to the general l_p-norms look like?

Problem 4.2: Write a program in MATLAB which implements the Gram-Schmidt algorithm for orthonormalizing a set of linearly independent vectors.

1. Apply your program to the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ -4 \end{pmatrix}, v_2 = \begin{pmatrix} -9 \\ 4 \\ 4 \\ 1 \\ 9 \end{pmatrix}, v_3 = \begin{pmatrix} 6 \\ 1 \\ -5 \\ -2 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} -8 \\ 3 \\ 2 \\ 3 \\ 9 \end{pmatrix}, v_5 = \begin{pmatrix} -6 \\ 2 \\ 3 \\ 1 \\ 5 \end{pmatrix}.$$

   Make sure they are linearly independent before you try to apply the Gram-Schmidt algorithm.

2. Check the orthogonality after you applied the algorithm for each pair of vectors:

$$(w_i, w_j) \overset{??}{=} 0 \quad \text{for } i \neq j, \quad i, j = 1, \dots, 5,$$

   where $w_i,\ i = 1, \dots, 5$ are the vectors after orthonormalizing.

Problem 4.3: For the matrices

$$A_1 = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -2 \\ 2 & 2 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 5 & 5 & 0 \\ -1 & 5 & 4 \\ 2 & 3 & 8 \end{pmatrix},$$

compute

1. $\|A_1\|_1, \|A_2\|_1$,
2. $\|A_1\|_2, \|A_2\|_2$,
3. $\|A_1\|_\infty, \|A_2\|_\infty$.

Problem 4.4: Given the matrix

$$A = \begin{pmatrix} 1 & 0.1 & -0.2 \\ 0 & 2 & 0.4 \\ -0.2 & 0 & 3 \end{pmatrix},$$

1. apply the Gerschgorin-Hadamard theorem to obtain the Gerschgorin circles,
2. draw a sketch of the circles you got in 1,
3. derive the eigenvalues of A.

Problem 4.5: Consider the linear system

$$\begin{aligned} .550x_1 + .423x_2 &= .127 \\ .484x_1 + .372x_2 &= .112, \end{aligned}$$

which corresponds to

$$Ax = b \quad \text{with} \quad A = \begin{pmatrix} .550 & .423 \\ .484 & .372 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} .127 \\ .112 \end{pmatrix}.$$

Suppose that you are given two candidate solutions,

$$\tilde{x} = \begin{pmatrix} 1.7 \\ -1.91 \end{pmatrix} \quad \text{and} \quad \bar{x} = \begin{pmatrix} 1.01 \\ -.99 \end{pmatrix}.$$

1. Decide depending on the residual $b - Ax$ which of the two candidates is a "better" solution.
2. Compute the exact solution.
3. Compute ther erros to the exact solution:

$$\|\tilde{x} - x\|_\infty, \quad \|\bar{x} - x\|_\infty.$$

Send your solutions to `barbel@kth.se` until October 11th, 2014.