Is the Clay Navier-Stokes Problem Wellposed?

JOHAN HOFFMAN[1] and CLAES JOHNSON[1]

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[1]*School of Computer Science and Communication*
Royal Institute of Technology
10044 Stockholm, Sweden.
email: jhoffman@csc.kth.se, cgjoh@csc.kth.se

Abstract

We discuss the formulation of the Clay Mathematics Institute Millennium Prize Problem on the Navier-Stokes equations in the perspective of Hadamard's notion of wellposedness.

1 The Clay Navier-Stokes Millennium Problem

The Clay Mathematics Institute Millennium Prize Problem on the *incompressible Navier-Stokes equations* [3, 7] asks for a proof of (I) global existence of smooth solutions for all smooth data, or a proof of the converse (II) non global existence of a smooth solution for some smooth data, referred to as *breakdown* or *blowup*.

In [10, 12, 13, 14] we have discussed the formulation of the Millennium Prize Problem and pointed to a possible reformulation and resolution. Central to our discussion is Hadamard's concept [8] of *wellposed* solution of a differential equation. Hadamard makes the observation that perturbations of data (forcing and initial/boundary values) have to be taken into account. If a vanishingly small perturbation can have a major effect on a solution, then the solution (or problem) is *illposed*, and in this case the solution may not carry any meaningful information and thus may be meaningless from both mathematical and applications points of view. According to Hadamard, only a *wellposed solution*, for which small perturbations have small effects (in some suitable sense), can be meaningful. Hadamard, thus makes a distinction between a wellposed and illposed solution through a *quantitative measure* of the effects of small perturbations: For a wellposed problem the effects are small and for an illposed large. A wellposed solution is meaningful, an illposed not.

In this perspective it is remarkable that the issue of wellposedness does not appear in the formulation of the Millennium Problem [7]. The purpose of this note is to seek an explanation of this fact, which threatens to make the problem formulation itself illposed in the sense that a resolution is either trivial or impossible [10].

2 Wellposedness vs Continuous Dependence

The notion of wellposedness is commonly expressed as existence and uniqueness of solutions with *continuous dependence on data*. In this formulation Hadamard's requirement of small effects of small perturbations of data, thus is phrazed as a dependence which is

(C) *continuous.*

This is an equivalent reformulation, if "continuous dependence" is defined as small effects of small perturbations, which however more precisely should be expressed as a dependence which is

(L) *Lipschitz continuous with a Lipschitz constant of moderate size.*

We recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *Lipschitz continuous with Lipschitz constant* L if for $x \in \mathbb{R}$

$$|f(x) - f(x + dx)| \leq L\,|dx| \quad \text{for (small) } dx \in \mathbb{R}. \tag{1}$$

If L is of moderate size, then a small perturbation dx of the argument x, will result in a small change of function value from $f(x)$ to $f(x + dx)$. On the other hand, if L is "large", then the effect of a small perturbation may not be small, because then $L|dx|$ can be large even if dx is small.

Comparing (C) and (L), we see that (L) contains the requirement that the Lipschitz constant L is of moderate size, whereas this information is not visible in (C). There is thus a risk that if (L) is replaced by (C), then the quantitative aspect in (L) gets lost and thus Hadamard's concept of wellposedness gets distorted and looses meaning. This happens if (C) is allowed to accommodate a large Lipschitz constant, which seems to be the case in [7].

It is thus necessary to make a distinction between continuous dependence with "moderate" and "large" Lipschitz constants. In the context of the Navier-Stokes equations and the Prize Problem, we shall see that this is not something very subtle: A large Lipschitz constant can be of size of googol $= 10^{100}$, while one of moderate size say $1 - 1000$. A distinction is thus made between 10^n with $n \leq 3$ and $n \geq 100$, say. Illposedness will then be represented by a Lipschitz constant of googol size, and wellposedness by a constant of moderate size, and thus there will be a very clear distinction between an illposed and a wellposed solution.

This connects to the meaning of "smooth solution" or "C^∞-solution" in the Prize Problem formulation. Does a "smooth function" have derivatives of any order which are Lipschitz continuous with Lipschitz constants which are of moderate size, or can they be large? Of course, this is a matter of definition of "smooth" or "continuous" derivatives of any order. The Prize Problem formulation is not clear on this point, but it appears that a definition allowing arbitrarily large Lipschitz constants is used. Thus no distinction seems to be made between wellposedness and illposedness. This is like making no distinction between 1 and 10^{100}, only between 10^{100} and ∞.

Defining wellposedness as existence and uniqueness coupled with continuous data dependence, may give the impression that uniqueness in principle can be separated from continuous dependence. Doing so would allow a mathematical proof of existence and uniqueness without assessment of continuous data dependence according (L), only according to (C) with arbitrarily large Lipschitz constants. We argue that this runs the risk of being meaningless from both mathematical and physical point of view.

In the formulation of the Millenium Problem, uniqueness is not explicitly mentioned, only existence of a smooth solution, the reason being that uniqueness with continuous dependence according to (C) is known to be a direct consequence of smoothness. This is another indication that no distinction is made between wellposedness and illposedness.

3 Illposed into Wellposed by Regularization

Even if Hadamard defined certain problems as being illposed, as e.g. the backward heat equation or many so-called inverse problems, such problems remained as a challenge in applications, and techniques for their solution were developed based on regularization. By regularization certain solution outputs, typically mean-values, can become wellposed, but regularization does not remove all aspects of illposedness. Thus, it is still necessary to make a distinction between wellposed and illposed solution outputs. We meet this aspect in turbulent flow with pointvalues being illposed while mean-values may be wellposed.

4 The Incompressible Navier-Stokes Equations

The incompressible Navier-Stokes equations express conservation of momentum and mass of an incompressible Newtonian fluid enclosed in an open domain Ω in $\mathbb{R}^3$ with boundary Γ: Find the velocity $u = (u_1, u_2, u_3)$ and pressure p depending on $(x,t) \in \Omega \cup \Gamma \times I$, such that

$$\begin{array}{rcll} \dot{u} + (u \cdot \nabla)u + \nabla p - \nu\Delta u &=& f & \text{in } \Omega \times I, \\ \nabla \cdot u &=& 0 & \text{in } \Omega \times I, \\ u &=& 0 & \text{on } \Gamma \times I, \\ u(\cdot, 0) &=& u^0 & \text{in } \Omega, \end{array} \tag{2}$$

where $\nu > 0$ is a constant viscosity, f is a given volume force, u^0 is a given initial condition, $\dot{u} = \frac{\partial u}{\partial t}$ and $I = (0, T]$ a given time interval. It is generally believed that the Navier-Stokes equations is a good mathematical model for a wide range of flows including in particular turbulent flows generically occuring when ν is small. We focus here on this case and thus expect mathematical solutions of (2) to generically be turbulent. To indicate the dependence on the viscosity ν, we denote a solution of (2) by (u_ν, p_ν).

The *Reynold's number* is defined b y $Re = \frac{UL}{\nu}$ where U is a representative velocity and L a length scale. We assume $U = L = 1$ and thus $Re = \nu^{-1}$ is large. In typical applications in aero and fluid dynamics, $Re \geq 10^6$ or $\nu \leq 10^{-6}$.

The assumption of constant viscosity ν is central in the Millenium Problem formulation. If the viscosity is increased by $h^2|\nabla u|$, where $|\cdot|$ is a matrix norm, and $h > 0$ is an arbitarily small parameter signifying a smallest scale, then existence and uniqueness of a smooth solution can be proved by standard mathematical techniques, without assessment of wellposedness according to (L). With this modification of the viscosity, we can thus for the discussion assume the existence of smooth solutions to the Navier-Stokes equations, and as a main objective subject these solutions to a mathematical analysis including in particular wellposedness according to (L).

5 Turbulent Solutions

The basic *energy estimate* for (2) is obtained by multiplying the momentum equation with u_ν and integrating in space and time, to give in the case $f = 0$,

$$K(u_\nu; t) + D(u_\nu; t) = K^0, \quad t > 0, \tag{3}$$

which expresses a balance between instantaneous *kinetic energy*

$$K(u_\nu; t) = \frac{1}{2}\int_\Omega |u_\nu(t,x)|^2 dx, \quad K^0 = \frac{1}{2}\int_\Omega |u^0|^2 dx,$$

and accumulated *viscous dissipation*

$$D(u_\nu; t) = \int_0^t \int_\Omega \nu|\nabla u_\nu(s,x)|^2 dxds,$$

with any loss in kinetic energy appearing as viscous dissipation, and vice versa. Obviously $K(u_\nu; t) \geq 0$, $D(u_\nu; t) \geq 0$ and $D(u_\nu; t) \leq K^0$.

Turbulent solutions of (2) are characterized by *substantial turbulent dissipation* defined by $D(u_\nu; t) \sim 1$ if $K^0 \sim 1$, which is consistent with Kolomogorov's conjecture that locally

$$|\nabla u_\nu| \sim \frac{1}{\sqrt{\nu}}, \tag{4}$$

In turbulent flow, thus a significant part of the kinetic energy is transformed into viscous dissipation, and turbulence is defined this way. Conversely, *laminar flow* is defined as smooth flow with small viscous dissipation.

Massive evidence from computational and physical experiments indicate that slightly viscous initially smooth laminar flow invariably becomes turbulent. There is thus considerable evidence that for slightly viscous flow kinetic energy and viscous dissipation balance with $K(u_\nu; t) \sim D(u_\nu; t) \sim 1$ forcing $|\nabla u_\nu|$ to be large in turbulent regions typically according to (4).

The appearance of turbulence in slightly viscous flow can be seen as a result of pointwise instability or illposedness generating strong velocity gradients which are tamed by viscous dissipation. Since the instability is so strong, substantial viscous dissipation is required to maintain the flow, because blowup is not an option.

6 Wellposedness

The standard method to study wellposedness of the Navier-Stokes equations (2) is to subtract these equations for two solutions (u, p) and $(\bar{u}, \bar{p})$ with corresponding (slightly) different data, to obtain the following *linearized equation* for the difference $(v, q) \equiv (\bar{u} - u, \bar{p} - p)$:

$$\begin{array}{rcll} \dot{v} + (\bar{u}\cdot\nabla)v + (v\cdot\nabla)u + \nabla q - \nu\Delta v & = & 0 & \text{in } \Omega\times I, \\ \nabla\cdot v & = & 0 & \text{in } \Omega\times I, \\ v & = & 0 & \text{on } \Gamma\times I, \\ v(\cdot,0) & = & v^0 & \text{in } \Omega, \end{array} \tag{5}$$

where $v(0) = v^0$ is the perturbation in initial data, and for simplicity we assume the the perturbation of the force f is zero. With u and $\bar{u}$ given, this is a linear convection-reaction problem for (v, q) with the reaction term given by the 3×3 matrix ∇u. By the incompressiblity, the trace of ∇u is zero, which shows that in general ∇u has eigenvalues with real value of both signs, of the size of $|\nabla u|$ (with $|\cdot|$ som matrix norm), thus with at least one exponentially unstable eigenvalue (domainating the stablizing contribution from the Laplacian in the present case with ν small. Thus, exponential local perturbation growth with exponent $|\nabla u|$ can be expected. Observations of turbulent flow and computations of turbulent Navier-Stokes solutions [10] give concrete evidence of strong local perturbation growth making pointvalues of turbulent solutions very sensitive to perturbations and thus illposed [1].

On the other hand we present evidence in [10] that global mean-values such as drag and lift are wellposed, which we associate with cancellation effects in the linearized problem with a rapidly oscillating reaction coeffient ∇u for a turbulent velocity u. Thus, there is strong evidence that pointvalues of turbulent Navier-Stokes solutions are illposed, while mean-values can be wellposed.

7 Gronwall Stability Estimates

The standard way to assess the wellposedness of a Navier-Stokes solution (u, p) is to estimate the solution (v, q) of the linearized equation (5) in terms of the initial perturbation $v(0)$, via multiplication of the momentum equation by v and integration together with a Gronwall estimate bounding the non-linear term pointwise by $|\nabla u||v|^2$, which gives

$$\|v(T)\| \leq \|v(0)\| \exp(\int_0^T \|\nabla u(t)\|_\infty dt) \equiv \|v(0)\| L_1, \tag{6}$$

where $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the $L_2(\Omega)$ and $L_\infty(\Omega)$-norms. Alternatively, using a Sobolev inequality as in [9] in combination with the viscous term, one can similarly show that

$$\|v(T)\| \leq \|v(0)\| \exp(\frac{C}{\nu^3} \int_0^T \|\nabla u(t)\|^4 dt) \equiv \|v(0)\| L_2, \tag{7}$$

where C is a constant of moderate size depending on Ω. In the standard analysis, as presented e.g in [9], estimates of the form (6) and (7) would be taken as assessment of continuous dependence on initial data, if only the corresponding Lipschitz constants $L_i < \infty$, without quantitative estimation of their size.

Regularity estimates bounding a Sobolev norm of $\|u(t)\|_{H^s}$ in terms of $\|u^0\|_{H^s}$, are derived similarly by differentiating the momentum equation, and involve similar multiplicative exponentials. On the basis of such estimates it is claimed that for any $s \geq 0$, $u(t) \in H^s$ if $u^0 \in H^s$, if the corresponding exponentials are not infinite.

However, if $|\nabla u| \sim \nu^{-1/2}$ and $\nu \sim 10^{-6}$ as indicated, then the exponential factors

$$L_1 \sim \exp(T10^3), \quad L_2 \sim \exp(T10^{30})$$

which are both way bigger than googol as soon as T is not small. Effectively this indicates illposeness, rather than wellposedness, and non-regularity rather than regularity. If an initial smooth velocity profile with slope of size 1 over time develops into a profile with slope of size googol, then it cannot be argued that the velocity remains smooth, unless all meaning of smoothness has been given up.

We also note that since L_2 is much larger than L_1, the idea of using a Sobolev inequality combined with the viscous term, does not seem to be constructive, in the case of small viscosity.

We thus argue that Gronwall type estimates with crude estimates of the non-linear term, possibly combined with Sobolev estimates, cannot be used in case of large velocity gradients $|\nabla u|$ occuring in turbulent flow. Assessing continuous dependence with Lipschitz constants of size googol cannot be meaningful, neither from mathematics nor from applications point of view. We find support of this standpoint in [9] stating: *What would be useful in applications, and should be sought, is a theorem that gives an estimate for the continuous dependence of solutions on the prescribed data.* However, we have also met strong resistance from mathematicians to the idea of continuous dependence according to (L). A clarification of this point seems to be essential for the discussion.

On the other hand, we show in [10] that because of cancellation effects in the non-linear term, mean-value outputs such as drag and lift can be wellposed with Lipschitz constants of moderate size.

8 EG2 Regularization

We compute in [10] solutions of the Navier-Stokes equations using a least squares stabilized finite element method referred to as G2, presented in detail in [10]. A G2 solution (U, P) on a mesh with local mesh size $h(x, t)$ according to [10], satisfies the following energy estimate (with $f = 0$ and $g = 0$):

$$K(U(t)) + D_h(U; t) = K(u^0), \tag{8}$$

where

$$D_h(U; t) = \int_0^t \int_\Omega (h|R(U, P)|^2 + \nu|\nabla U|^2)\, dxdt, \tag{9}$$

is an analog of $D(u_\nu; t)$, where $R(U, P)$ is the Navier-Stokes residual. In applications to turbulent flow $h >> \nu$ and the viscous dissipation in G2 is dominated by the least squares stabilization $h|R(U, P)|^2$. Thus the G2 viscosity arises from penalization of a non-zero Navier-Stokes residual $R(U, P)$ with the penalty directly connecting to the violation (according the theory of criminology). A turbulent G2 solution is characterized by substantial dissipation $D_h(U; t)$ with $|R(U, P)| \sim h^{-1/2}$ locally. Furthermore,

$$\|R(U, P)\|_{-1} \leq \sqrt{h}. \tag{10}$$

where $\|\cdot\|_{-1}$ is the $H^{-1}(Q)$ norm. A turbulent G2 solution (U,P) is thus characterized by a residual $R(U,P)$ which is large in $L_2(Q)$ but small in $H^{-1}(Q)$.

9 Wellposedness of Mean-Value Outputs

Let $M(v) = \int_Q v\psi dxdt$ be a *mean-value output* of a velocity v defined by a smooth weight-function $\psi(x,t)$, and let (u,p) and (U,P) be two G2-solutions on two meshes with maximal mesh size h. Let (φ,θ) be the solution to the *dual linearized problem*

$$\begin{aligned} -\dot\varphi - (u\cdot\nabla)\varphi + \nabla U^\top\varphi + \nabla\theta &= \psi && \text{in } \Omega\times I,\\ \nabla\cdot\varphi &= 0 && \text{in } \Omega\times I,\\ \varphi\cdot n &= g && \text{on } \Gamma\times I,\\ \varphi(\cdot,T) &= 0 && \text{in } \Omega, \end{aligned} \tag{11}$$

where $\top$ denotes transpose. Multiplying the first equation by $u-U$ and integrating by parts, we obtain the following output error representation [10, 11]

$$M(u) - M(U) = \int_Q (R(u,p) - R(U,P))\cdot\varphi\, dxdt \tag{12}$$

from which follows the *a posteriori error estimate*

$$|M(u) - M(U)| \le S(\|R(u,p)\|_{-1} + \|R(U,P)\|_{-1}), \tag{13}$$

where the stability factor

$$S = S(u,U,M) = S(u,U) = \|\varphi\|_{H^1(Q)}. \tag{14}$$

In [10] we presented a variety of evidence, obtained by computational solution of the dual problem, that for global mean-value outputs such as drag and lift, $S << 1/\sqrt{h}$, while $\|R\|_{-1} \sim \sqrt{h}$. This allows an G2 solution (U,P) to pass a wellposedness test of the form

$$S(U,U)\|R(U,P)\|_{-1} \le TOL \tag{15}$$

for tolerances $TOL > 0$ and mesh sizes h of interest, because $S(U,U)$ shows to be of moderate size.

As above, a crude analytical stability analysis of the dual linearized problem (11) using Gronwall type estimates, indicates that the dual problem is pointwise exponentially unstable because the reaction coefficient ∇U is locally very large, in which case effectively $S(U,U) = \infty$. This is consistent with massive observation that point-values of turbulent flow are non-unique or illposed.

On the other hand we observe computationally that S is not large for mean-value outputs of turbulent solutions. We explain in [10] this remarkable fact as an effect of *cancellation* from the following two sources:

(i) rapidly oscillating reaction coefficients of turbulent solutions,

(ii) smooth data in the dual problem for mean-value outputs.

We remark that in the mathematics education reform project Body and Soul [2, 6], we consistently use the concept of Lipschitz continuity, and we show advantages of this approach. It is seen as a basic element of *Computational Calculus* as the modern computer-age form of classical Calculus.

10 Summary

We have pointed to the fact that in assessing continuous dependence of solutions to the Navier-Stokes equations in the case of small viscosity, it seems to be necessary to define continuity as Lipschitz continuity with specific consideration of the size of the Lipschitz constant. We have shown that standard Gronwall can give estimates of Lipschitz constants of size googol, which seem to have little informative value.

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