Near-Resonance with Small Damping

Claes Johnson [*]

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Abstract

We analyze a phenomenon of near-resonance in an oscillator with small damping and make connections to blackbody radiation and the acoustics of string instruments. We find that near-resonance with small damping is characterized by an efficiency index $E \approx 1$ as the quotient of the damping energy and the forcing energy, as a result of a phase shift of a quarter of a period beween forcing and velocity. Near-resonance is used in tuning the three string of a piano tone with an offset of about 0.5 Hz to generate longer sustain and a singing quality to the piano.

1 Resonance in Forced Damped Oscillators

The problem of resonance is of fundamental importance in the physics of absorption and emission of light/radiation and in the acoustics of string instruments. The analysis of blackbody radiation [3] shows the basic role of a phenomeon of *near-resonance* in a *resonator* with *small damping*. The basic phenomenon can be illustrated for a damped harmonic oscillator modeled by

$$\ddot{u}(t) + \nu^2 u(t) + \gamma \dot{u}(t) = f(t), \quad -\infty < t < \infty, \tag{1}$$

where $\dot{u} = \frac{du}{dt}$, $\ddot{u} = \frac{d2u}{dt^2}$, ν is a given moderate to large frequency, $\gamma > 0$ is a damping parameter and $f(t)$ is a periodic forcing. We seek periodic solutions and measure the relation between the forcing and the damping by the *efficiency* $E = \frac{F}{R}$ with

$$F = \int f^2(t)\, dt, \quad R = \int \gamma \dot{u}^2(t)\, dt\,, \tag{2}$$

[*]Computer Science and Communication, KTH, SE-10044 Stockholm, Sweden.

with integration over a time period. If the forcing $f(t)$ is periodic with the resonance frequency ν, referred to as *perfect resonance*, then $\dot{u}(t) = \frac{1}{\gamma}f(t)$, which gives $E = \gamma$ with $\dot{u}$ in phase with $f(t)$.

We shall distinguish two basic different cases with the forcing $f(t)$ balanced by the harmonic oscillator term $\ddot{u}(t) + \nu^2 u(t)$ and the damping term $\gamma\dot{u}(t)$ in two different ways:

1. $\gamma \approx 1$ with $\gamma\dot{u} \approx f(t)$ and $|\ddot{u}(t) + \nu^2 u(t)| << |f(t)|$,
2. $\gamma\nu < 1$ with $|\gamma\dot{u}(t)| << |f(t)|$ and $\ddot{u}(t) + \nu^2 u(t) \approx f(t)$,

with the case 2. representing near-resonance with small damping, as the case of most interest. We shall define near-resonance at a given frequency ν by flat spectrum centered at ν of width 1. A spectrum of width $\gamma << 1$ would then correspond to sharp resonance (with γ not very small this is sometimes referred to as broad resonance).

In case 1. the damping is large and the force $f(t)$ is balanced by the damping $\gamma\dot{u}(t)$ with $\dot{u}$ in phase with $f(t)$. In this case trivially $E \approx 1$.

In case 2. with near resonance and small damping, $f(t)$ is balanced by the oscillator with $\dot{u}$ out-of-phase with $f(t)$, and we shall see that also in this case $E \approx 1$. The case of near-resonance is to be compared with the case of perfect resonance with $f(t)$ again balanced by $\gamma\dot{u}$, with now $\dot{u}(t)$ in phase and $E = \gamma$.

If γ is small there is thus a fundamental difference betwen the case of near-resonance with $E \approx 1$ and the case of perfect resonance with $E = \gamma << 1$.

In applications to blackbody radiation we may view F as input and R as output, but it is also possible to turn this around view R as the input and F as the output, with $E = \frac{F}{R}$ then representing an efficiency index. In the case of small damping we then have $E \approx 1$ in the case of near-resonance and $E << 1$ in the case of perfect resonance.

In the case of near-resonance the force $f(t)$ is balanced mainly by the excited harmonic oscillator with a small contribution from the damping term, which gives $E \approx 1$.

In the case of perfect resonance the oscillator does not contribute to the force balance, which requires a large damping term leading to small efficiency.

The above discussion concerns time-periodic (equilibrium) states attained after a transient start-up phase, with the forcing now F in-phase with the velocity $\dot{u}$, in contrast to out-of-phase in equilibrium.

The discussion in this note connects to apects of wave vs particle modeling of light and sound [4, 1, 2, 7, 8, 9, 11].

2 Fourier Analysis of Near-Resonance

Although (3) is a maybe the most studied model of all of physics, it appears that the phenomenon of near-resonance has received little attention. We use Fourier transformation in t of (3), writing

$$u(t)=\int_{-\infty}^{\infty}\hat{u}(\omega)e^{i\omega t}d\omega,\quad \text{with}\quad \hat{u}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(t)e^{-i\omega t}\,dt,$$

to get

$$(-\omega^2+\nu^2)\hat{u}(\omega)+i\gamma\omega\hat{u}(\omega)=\hat{f}(\omega).$$

We then use Parseval's formula, to seek a relation between the mean value of $u^2(t)$ and $f^2(t)$, assuming $\hat{f}(\omega)$ is support around $\omega=\nu$:

$$\overline{u^2}\equiv\int_{-\infty}^{\infty}|u(t)|^2\,dt=2\pi\int_{-\infty}^{\infty}|\hat{u}(\omega)|^2\,d\omega=2\pi\int_{-\infty}^{\infty}\frac{|\hat{f}(\omega)|^2\,d\omega}{(\nu-\omega)^2(\nu+\omega)^2+\gamma^2\omega^2}$$

$$\approx\frac{2\pi}{\nu^2}\int_{-\infty}^{\infty}\frac{|\hat{f}(\omega)|^2\,d\omega}{4(\nu-\omega)^2+\gamma^2}=\frac{2\pi}{\gamma\nu^2}\int_{-\infty}^{\infty}\frac{|\hat{f}(\nu+\gamma\bar{\omega})|^2)\,d\bar{\omega}}{4\bar{\omega}^2+1},$$

where we used the change of integration variable $\omega=\nu+\gamma\bar{\omega}$.

We now assume that $|\hat{f}(\omega)|^2\sim\overline{f^2}$ for $|\nu-\omega|\leq\frac{1}{\pi}$ as an expression of near-resonance, and that $|\hat{f}(\omega)|$ is small elsewhere. With this assumption we get

$$\gamma\overline{\dot{u}^2}\sim\gamma\nu^2\overline{u^2}\sim\overline{f^2},$$

that is $R\sim F$ and thus $E\sim 1$, with a constant of proportionality ≈ 1 independent of ν and γ. Here, we use that in near resonance $|\dot{u}|\approx|\nu u|$.

3 Application to Blackbody Radiation

The model of blackbody radiation studied in [3] is the following variant of (3)

$$\ddot{u}(t)+\nu^2u(t)-\gamma\dddot{u}(t)=f(t),\quad -\infty<t<\infty, \tag{3}$$

with the damping term $\gamma\dot{u}$ replaced by $-\gamma\dddot{u}$. The analysis is analogous and shows in the case of near resonance an efficiency $E=\frac{F}{R}\sim 1$ with now $R=\int\gamma\ddot{u}^2(t)\,dt$, which gives the Rayleigh-Jeans Radiation Law, and Planck's Radiation Law with a finite precision cut-off [5, 6, 4].

4 Application to Acoustical Resonance

A musical string instrument consists in principle of a vibrating string and a resonating body or soundboard, where we model the resonator with input from the string, as the force f during start-up and as a viscous force in equlibrium.

In the case of near resonance in equilibrium the input from the vibrating string is amplified by the resonator to an efficiency index $E \sim 1$, while perfect resonance would give $E << 1$, with small damping.

During start-up we consider the forcing f to be given by the vibrating string (without damping) and acting in-phase with the velocity $\dot{u}$ thus is pumping vibrational energy from the string into the body. Once equilibrium is reached, we shift view and consider f as the output from the body, which is sustained by a still vibrating string generating the viscous force. This mean that during both start-up and equilibrium the string vibrates in-phase with the body, by pumping energy into the body during start-up, and sustaining the output from the body in equilibrium.

The importance of near-reonance forcing is well-known to a piano-tuner, who tunes the three strings of a tone (except single stringed bass tones) at slightly different pitches (of about 0.5 Hz), which gives a longer sustain and a singing quality to the piano.

5 Computational Resonance

We show in Fig. 1-5 some computations with $\gamma = 0.001$, $\nu = 20$ and $\nu = 100$ with the following near-resonance forcing:

$$f(t) = \sum_{k=-5, k\neq 0}^{5} \sin((n + \frac{k}{10})t)\, 0.1 \tag{4}$$

starting with the intial data $u(0) = 0$ and $\dot{u}(0) = 15$ and computing for $t > 0$. The efficiency index is computed as the mean value over the entire time interval. We see as expected from the Fourier analysis that the efficiency index $E \approx 1$ and that the forcing is out-of-phase with the damping, as the main characteristics of near-resonance with small damping.

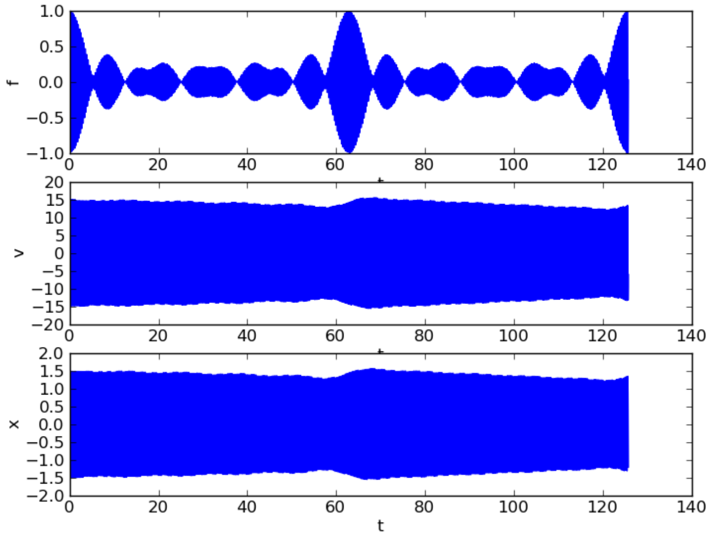

Figure 1: Position $x = u$, velocity $v = \dot{u}$ and forcing f with $n = 20$ over scaled time.

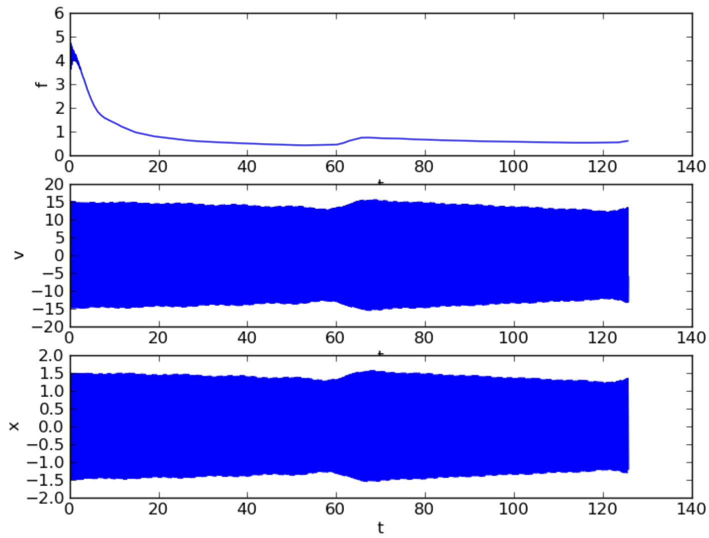

Figure 2: Position $x = u$, velocity $v = \dot{u}$ and efficiency $f = E$ with $n = 20$ over scaled time.

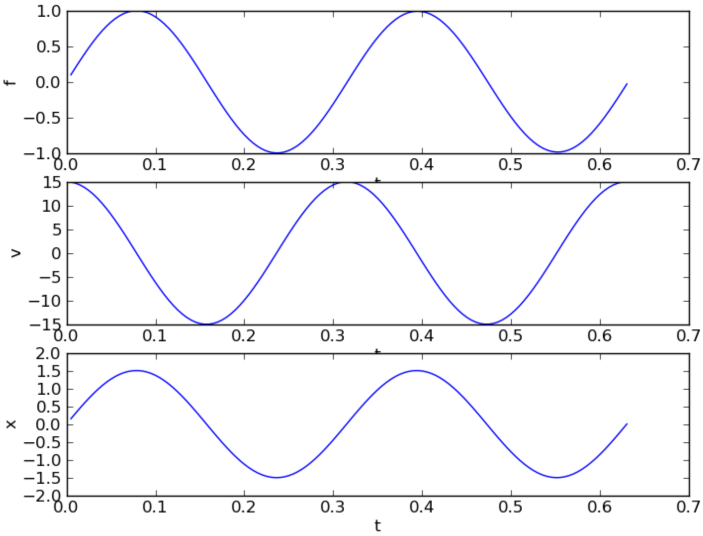

Figure 3: Position $x = u$, velocity $v = \dot{u}$ and forcing $f = E$ with $n = 20$ over ahort time. Notice a time lag of a quarter of a period between $\dot{u}$ and forcing f, representing out-of-phase forcing.

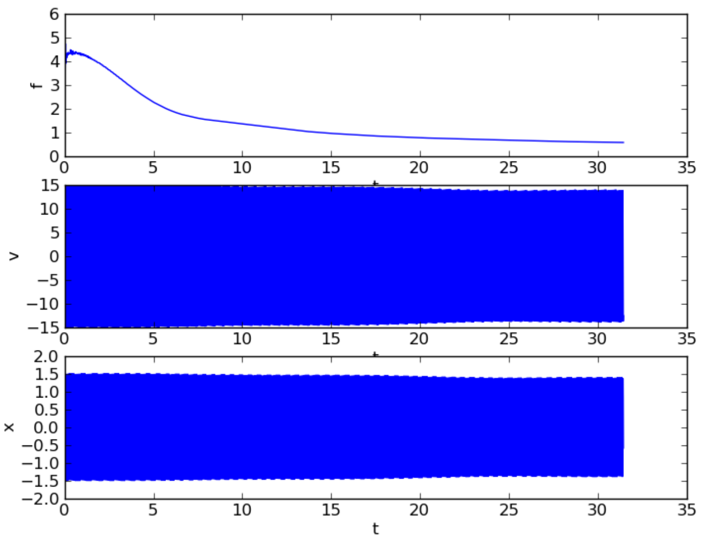

Figure 4: Position $x = u$, velocity $v = \dot{u}$ and efficiency $f = E$ with $n = 100$ over scaled time.

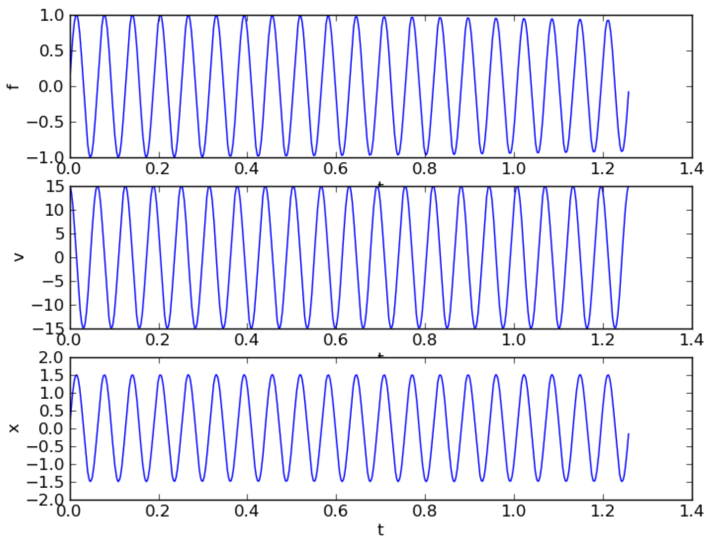

Figure 5: Position $x = u$, velocity $v = \dot{u}$ and forcing $f = E$ with $n = 100$ over short time.

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[2] Einstein: *I consider it quite possible that physics cannot be based on the field concept, i.e., on continuous structures. In that case, nothing remains of my entire castle in the air, gravitation theory included, and of the rest of physics*. (Einstein 1954)

[3] C. Johnson, Computational Blackbody Radiation, in *Slaying the Sky Dragon: Death of the Greenhouse Gas Theory*, Stairways Press, 2010.

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