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ASYMPTOTIC BEHAVIOR OF CENTRAL ORDER STATISTICS UNDER MONOTONE NORMALIZATION*

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Dedicated to the 100th anniversary of Gnedenko's birth

Abstract. Smirnov [*Trudy Mat. Inst. Steklov.*, 25 (1949), pp. 3–60 (in Russian); *Amer. Math. Transl.*, 67 (1952) (in English)] derived four limit types of distributions for linearly normalized central order statistics under the weak convergence. In this paper we investigate the possible limit distributions of the kth upper order statistics with central rank using monotone regular norming sequences and obtain 13 possible types.

Key words. kth upper order statistic, central rank, monotone normalization, regular norming sequence

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1. Introduction and preliminaries. Let $X_1, X_2, \ldots, X_n$ be a sample of independent identically distributed random variables with common continuous distribution function F. In our study all distribution functions are assumed to be left-continuous. We denote $X_{1,n} = \max\{X_i\}_{i\in\{1,2,\ldots,n\}}$, $X_{2,n} = \max\{X_i\}_{i\in\{1,2,\ldots,n\}\setminus L}$ for L the index of the first maximum, and $X_{n,n} = \min\{X_i\}_{i\in\{1,2,\ldots,n\}}$. The sequence $X_{n,n} \leqq \cdots \leqq X_{k,n} \leqq \cdots \leqq X_{1,n}$ is the ordered sample. The random variables $X_{n,n}, \ldots, X_{k,n}, \ldots, X_{1,n}$ are called upper order statistics and $X_{k,n}$ is the kth upper order statistic (u.o.s.).

Gnedenko's paper "Sur la distribution limite du terme maximum d'une série aléatoire," (*Ann. of Math.* (2), 44 (1943), pp. 423–453) marked the beginning of a new branch in modern stochastics, the extreme value theory. In this paper, he stated three limit distributions (Φ_α, $\alpha > 0$ — Fréchet, Ψ_α, $\alpha > 0$ — Weibull, Λ — Gumbel) for the normalized maxima of independent identically distributed random variables using linear normalization. He also gave the criteria for the domain of attraction. In [5], the author studied the limit distributions for normalized maxima using regular power transformation. The limit class in this case contains six distributions.

The first statements concerning the kth order statistic were made in 1949, by Smirnov [9]. More on their properties is given in [8] and [10]. The asymptotic theory for the order statistics on a high theoretical level is generalized in [1] and [2].

The case of increasing ranks (k_n), with the intermediate property $k_n/n \to 0$ as $n \to \infty$, was considered in [9] and [4]. Chibisov [4] derived the possible types of the limit distributions of the k_nth order statistics and their domain of attraction under linear normalization. In [9], Smirnov also considered the case of sequence (k_n) with the central order property $k_n/n \to \theta \in (0,1)$ as $n \to \infty$. The appropriate restrictions for the (k_n) are $\min\{k_n,\ n-k_n\} \to \infty$ and $(k_n - np_n)/(np_n(1-p_n))^{1/2} \longrightarrow \tau$. In that case, $F_{k_n,n}(u_n) := \mathbf{P}\{X_{k_n,n} < u_n\} \longrightarrow \Phi(\tau)$ as $n \to \infty$. In order to obtain a unique limit distribution, usually one assumes the second order condition $\sqrt{n}(k/n - \theta) \to 0$ as $n \to \infty$.

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Furthermore, in the case of the central order statistics, Smirnov [9] proved that the condition

$$\sqrt{n}\,\frac{F(a_nx+b_n)-\theta}{(\theta(1-\theta))^{1/2}} \longrightarrow \tau(x)$$

is equivalent to the asymptotic relation

$$F_{k_n,n}(a_nx+b_n) \xrightarrow{w} \Phi(\tau(x)).$$

By solving the functional equation

$$\tau(x) = \sqrt{n}\,\tau(\alpha_nx+\beta_n) \quad \text{for } \alpha_n>0,\ \beta_n\in\mathbf{R},\ n>1,$$

under certain conditions which hold for α_n, β_n (stated at the end of our section 4), Smirnov derived four different types of limit distributions $H = \Phi\circ\tau$. The first has a jump of 1/2 in the left end point of the support and the second has a jump of 1/2 in the right end point. The third is continuous, and the last is a two jump distribution with jumps of height 1/2.

The recent results given by Barakat and Omar [6] state limit theorems for order statistic under nonlinear normalization. The authors claim that they used some of the results of [5] and extended them to order statistics with increasing rank. But in fact, in [5] there are no results concerning $X_{k,n}$. The properties of the norming mappings are derived from the max-stability functional equation, which here we do not have. This is the main reason for us to fill these gaps, giving precise analysis of the solutions to the functional equation for $\tau(x)$ using regular norming mappings.

Let (k_n) be a sequence of integers such that $k_n/n\to\theta\in(0,1)$ and $X_{k_n,n}$ the k_nth upper order statistic. We denote by GMA the group of max-automorphisms on $\mathbf{R}$, i.e., strictly increasing, continuous mappings that preserve the operation maximum. Also let $C(f)$ be the set of all continuity points of f. We use the abbreviation $\overline{F} = 1-F$.

THEOREM 1. *Let H be a nondegenerate distribution function, $k_n/n\to\theta\in(0,1)$, and let $\{G_n\}$ be a sequence of norming mappings in GMA. Then*

$$F_{k_n,n}(G_n(x)) = \mathbf{P}\{{G_n}^{-1}(X_{k_n,n})<x\} \xrightarrow[n]{w} H(x) \tag{1}$$

if and only if

$$\sqrt{n}\,\frac{\theta-\overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}} \xrightarrow[n]{w} \tau(x), \tag{2}$$

where $\tau(x)$ is a nondecreasing function uniquely determined by the equation

$$H(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tau(x)} e^{-x^2/2}\,dx = \Phi(\tau(x)). \tag{3}$$

Remark. Let $l_H = \inf\{x\in\mathbf{R}\,|\,H(x)>0\}$ be the left end point of the support of H and $r_H=\sup\{x\in\mathbf{R}\,|\,H(x)<1\}$ be the right end point. From $H=\Phi\circ\tau$ it follows that $\operatorname{supp}H=\operatorname{supp}\tau$ and $C(H)=C(\tau)$. Let us denote $S=(l_H,r_H)$ and notice that τ is defined on S.

DEFINITION 1. *The sequence $\{G_n\}$ is a regular norming sequence on the set $\mathbf{X}\times\mathbf{T}$ if for $x\in\mathbf{X}$ and $t\in\mathbf{T}$, there exists $\lim_{n\to\infty}{G_{[nt]}}^{-1}\circ G_n(x) = g_t(x)\in$ GMA and the correspondence $t\to g_t$ is a continuous 1-1 mapping.*

Here $[x]$ is the integer part of x.

For our purposes, $\mathbf{X} = (l_H, r_H)$, $\mathbf{T} = (0, 1]$. Clearly, for $t = 1$, ${G_n}^{-1} \circ G_n(x) = x = g_1(x) \in \mathrm{GMA}$.

Lemma 1. *Let the sequence* $\{G_n\} \subset \mathrm{GMA}$ *be a regular norming sequence on* $(l_H, r_H) \times (0, 1]$. *Then the limit mapping* $g_t(x)$ *satisfies the functional equation*

$$g_t \circ g_s(x) = g_{t \cdot s}(x), \qquad s, t \in (0, 1]. \tag{4}$$

Definition 2. *A distribution function* F *belongs to the normal domain of* θ*-attraction of* H *if there exists a norming sequence* $\{G_n\} \subset \mathrm{GMA}$ *and the second order condition* $(k_n/n - \theta)\sqrt{n} \to 0$ *holds such that* $F_{k_n,n}(G_n(x)) \xrightarrow[n]{w} H(x)$.

Theorem 2. *If a distribution function* H *has a normal domain of* θ*-attraction with respect to a regular norming sequence* $\{G_n\} \subset \mathrm{GMA}$, *then its corresponding function* τ *satisfies the following functional equation*:

$$\sqrt{t} \cdot \tau(x) = \tau(g_t(x)) \qquad \forall t \in (0, 1], \quad x \in C(\tau). \tag{5}$$

In view of Theorems 1 and 2, one observes that the class of all possible limit distribution functions $H(x)$ is entirely determined by the possible forms of the functions $\tau(x)$ and $g_t(x)$. Thus, in section 2 we consider the properties of the functions $\tau(x)$ and $g_t(x)$ and several consequences of the functional equations (4) and (5). In section 3 we give the solution of the functional equation (4), and section 4 concerns the solution of functional equation (5). In section 5 one example for convergence of central upper order statistics is demonstrated. The proofs of Lemma 1 and Theorems 1 and 2 are given in the appendix.

2. Properties of $\tau(x)$ and $g_t(x)$.

A. Properties of $\tau(x)$ derived from (2), (3), and (5). (i) Obviously $\tau(x)$ is nondecreasing and left-continuous in x function since $\tau(x) = \Phi^{-1}(H(x))$.

Denote $\mathcal{D} := \{x \,|\, \tau(x) \neq 0,\ \tau(x) \neq \pm\infty\}$. Let $x \notin \mathcal{D}$. For $\tau(x) = \Phi^{-1}(H(x)) = \infty$, $H(x) = 1$; hence $x \geqq r_H$. For $\tau(x) = -\infty$, $H(x) = 0$; hence $x \leqq l_H$. And for all x such that $\tau(x) = 0$, $H(x) = 1/2$. Defining the median of H by $m_H = \sup\{x \,|\, H(x) < 1/2\}$, we reach the uniqueness of the median. One may notice that $\tau(m_H+) = 0$.

Let $\mathcal{D} \neq \varnothing$ and $x \in S \cap \mathcal{D}$. Then

(ii) $\tau(x) < 0$ for all $x \in (l_H, m_H)$ and $\tau(x) > 0$ for all (m_H, r_H).

As direct consequences of the functional equation (5), we have

(iii) $x \in \mathcal{D} \Leftrightarrow g_t(x) \in \mathcal{D}$ for all $t > 0$, and hence the trajectory $\Gamma_x = \{g_t(x) \,|\, t \in (0, 1]\} \subset \mathcal{D}$;

(iv) for $x \in \mathcal{D}$, x and $g_t(x)$ cannot belong to the same interval of the constancy of τ;

(v) for $x \in \mathcal{D}$, the function $\tau(g_t(x))$ is continuous and strictly monotone in t. Hence, along the trajectory Γ_x, τ is strictly increasing and continuous.

Proposition 1. *The function* $\tau(x)$ *is continuous in all* $x \in \mathcal{D}$.

Proof. First, we prove that if τ has a discontinuity point x_0, then all $y \in \Gamma_{x_0}$ would be discontinuity points of τ. Namely, let x_0 be a discontinuity point and let the jump of τ in x_0 have the height $\beta - \alpha$. Then for $\varepsilon > 0$

$$\sqrt{t} \cdot \tau(x_0 - \varepsilon) \leq \sqrt{t}\alpha < \sqrt{t}\beta \leq \sqrt{t} \cdot \tau(x_0 + \varepsilon) \qquad \forall t \in (0, 1].$$

Hence there exists $\delta(\varepsilon) > 0$ such that

$$\tau(g_t(x_0) - \delta(\varepsilon)) \leq \sqrt{t}\alpha < \sqrt{t}\beta \leq \tau(g_t(x_0) + \delta(\varepsilon)),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, τ is discontinuous at $g_t(x_0)$ for all $t \in (0, 1]$.

As a monotone function τ can only have countable many jumps, whereas the curve Γ_{x_0} is continuous. This is a contradiction.

PROPOSITION 2. *The function $\tau(x)$ is strictly increasing on $\mathcal{D}$.*

Proof. Assume $\mathcal{D} \neq \varnothing$. We have already noted that $\tau(x)$ is a nondecreasing function in x. Let $x_1, x_2 \in \mathcal{D}$, $x_1 \neq x_2$, $\tau(x_1) = \tau(x_2)$. Then the interval $\alpha = [x_1, x_2]$ is an interval of constancy for $\tau(x)$. Since $g_t \in$ GMA it is clear that $g_t(x_1) \neq g_t(x_2)$. From $\tau(g_t(x_1)) = \tau(g_t(x_2))$ it follows that $\alpha_t = [g_t(x_1), g_t(x_2)]$ is also an interval of constancy for τ for all $t \in (0, 1]$. Hence, there are uncountable many intervals of constancy for τ, which contradicts the monotonicity of τ.

Other properties of τ connected with the function $g_t(x)$ will be discussed in the following subsection.

B. Properties of $g_t(x)$ derived from (4) and (5). By Definition 1, $g_t(x)$ is continuous and strictly increasing in x and continuous in t. The functional equation (5) written as

$$\tau({g_t}^{-1}(x)) = \frac{1}{\sqrt{t}} \cdot \tau(x), \qquad t \in (0, 1],$$

allows us to expand the half group $\{g_t \,|\, t \in (0, 1]\}$ (with respect to the composition) to a continuous one parameter group $\{g_t \,|\, t > 0\}$ with a unit element $g_1(x) = x$ and ${g_t}^{-1}(\bullet) = g_{1/t}(\bullet)$ for $t > 0$. Now, abusing the previous notation, $\Gamma_x = \{g_t(x) \,|\, t > 0\}$ is the whole trajectory of the mapping g_t through the point x, and (5) can be considered as true for all $t > 0$ and $x \in C(\tau)$. For thoroughly determining the asymptotic of Γ_x we assume that the following boundary conditions hold for all $x \in C(\tau) \cap S$:

$$\text{(BC1)} \quad \lim_{t \downarrow 0} g_t(x) := g_0(x) = m_H;$$

$$\text{(BC2)} \quad \lim_{t \uparrow \infty} g_t(x) := g_\infty(x) = \begin{cases} r_H & \text{if } x > m_H, \\ l_H & \text{if } x < m_H. \end{cases}$$

As a consequence of the boundary conditions we conclude that every trajectory Γ_x starts from m_H and increases to r_H if $x > m_H$, or decreases to l_H if $x < m_H$. One can easily see that these conditions are fulfilled for $x \in S \cap \mathcal{D}$.

Let us now consider the set $S \cap \mathcal{D}^c$. Here S is the open interval (l_H, r_H) and $\mathcal{D}^c$ consists of all points $x \leqq l_H$, $x \geqq r_H$ and all x with $\tau(x) = 0$. Hence, the set $S \cap \mathcal{D}^c$ consists either of only one point m_H or is a connected interval $I \subseteq S$ whose left end point is m_H and on which $\tau \equiv 0$.

PROPOSITION 3. $I = (m_H, r_H)$.

Proof. Assume $I = (m_H, a)$, $a < r_H$. For arbitrary $x_0 \in I \cap C(\tau)$, from the functional equation (5), it follows that $\tau(g_t(x_0)) = 0$ for all $t > 0$. Hence $g_t(x_0) \in (m_H, a)$ for all $t > 0$. This contradicts boundary condition (BC2).

COROLLARY 1. *Let $I = (m_H, r_H)$. Then*

$$\tau(x) = \begin{cases} -\infty & \text{if } x \leqq l_H = m_H, \\ 0 & \text{if } x \in (l_H = m_H, r_H), \\ +\infty & \text{if } x \geqq r_H. \end{cases}$$

This case will be referred to as a singular case. From the limit relation (2), one observes that there is no need for the norming sequence $\{G_n\}$ to be regular. The only condition here is $G_n(x) \to x_\theta$, where x_θ is determined by $\overline{F}(x_\theta) = \theta$. In this case the corresponding limit distribution H is the two jumps distribution, with jumps of height 1/2 in l_H and r_H, and is constant on $(l_H = m_H, r_H)$. In [3] there are examples of discrete limit distributions obtained using monotone nonregular norming sequences.

Further on, we ask for the explicit form of τ in the remaining case, $I = \{m_H\}$, which we assume to hold in all of the following analysis.

As a consequence of Proposition 1, τ (respectively, H) might have jumps only at l_H, m_H, and r_H. The following three cases are the only possible cases:

(i) $S = (l_H = m_H, r_H)$, i.e., H jumps at l_H by 1/2 and H might have a jump at r_H;

(ii) $S = (l_H, m_H = r_H)$, i.e., H jumps at r_H by 1/2 and H might have a jump at l_H;

(iii) $S = (l_H, r_H)$ and $l_H < m_H < r_H$, i.e., H might have jumps at r_H, l_H, and m_H.

Proposition 4. *The three points l_H, m_H, and r_H are the only possible fixed points of g_t.*

Proof. Recall that x_0 is a fixed point for g_t if and only if $g_t(x_0) = x_0$. Using (5), it is easy to check that l_H, m_H, and r_H are fixed points for g_t. We need to prove that those are the only ones. Indeed, if x_0 is a fixed point for g_t, from (5) one has

$$\sqrt{t} \cdot \tau(x_0) = \tau(g_t(x_0)) = \tau(x_0).$$

Hence the only possible solutions for $\tau(x_0)$ are $\tau(x_0) = 0$ and $\tau(x_0) = \pm\infty$, i.e., $x_0 \notin \mathcal{D}$ and, therefore, $x_0 \in \{l_H, m_H, r_H\}$. The proposition is proved.

In view of (5), we also observe that the mapping g_t is expanding, i.e., $g_t(x) > x$, for $x \in (l_H, m_H)$ and $t \in (0, 1)$, and g_t is contracting, i.e., $g_t(x) < x$ for $x \in (m_H, r_H)$ and $t \in (0, 1)$. For $t > 1$, g_t is contracting for $x \in (l_H, m_H)$ and expanding for $x \in (m_H, r_H)$.

Indeed, if $x \in (l_H, m_H)$, then $\tau(x) < 0$ and $\sqrt{t} \cdot \tau(x) = \tau(g_t(x))$ implies $g_t(x) > x > g_{1/t}(x)$ for $t \in (0, 1)$. Conversely, if $x \in (m_H, r_H)$, then $\tau(x) > 0$, and for $t \in (0, 1)$ the functional equation (5) implies $g_t(x) < x < g_{1/t}(x)$.

Finally, let us discuss the domains of definition of $\tau(x)$ and $g_t(x)$. Since $\tau(x) = \Phi^{-1} \circ H(x)$, we obtain

(i) $\tau(x) \uparrow \Phi^{-1}(H(r_H-))$ for $x \uparrow r_H$. If H is continuous at the right end point r_H, then $\tau(x) \uparrow \infty$;

(ii) $\tau(x) \downarrow \Phi^{-1}(H(l_H+))$ for $x \downarrow l_H$. If H is continuous at the left end point l_H, then $\tau(x) \downarrow -\infty$, and hence

$$\tau\colon\ (l_H, r_H) \to \Big(\Phi^{-1}(H(l_H+)), \Phi^{-1}(H(r_H-))\Big).$$

Since $g_t(\bullet)$ is continuous and l_H, r_H are fixed points of g_t, for fixed t,

$$g_t(\bullet)\colon\ (l_H, r_H) \to (l_H, r_H).$$

Now let x be fixed and consider $g_{(\bullet)}(x)$ as a function of $t \in (0, \infty)$:

(i) For $x \in (m_H, r_H)$, $g_{(\bullet)}(x)\colon\ (0, \infty) \to (m_H, r_H)$ is increasing in t;

(ii) for $x \in (l_H, m_H)$, $g_{(\bullet)}(x)\colon\ (0, \infty) \to (l_H, m_H)$ is decreasing in t.

3. Solution of the functional equation (4). In this section we give the solution of functional equation (4) according to the approach of Aczel [7].

Further on, we use the notation $g(t,x) := g_t(x)$. Now the functional equation (4), $g_t \circ g_s(x) = g_{t\cdot s}(x)$, $x \in C(\tau) \cap \mathcal{D}$, $t, s > 0$, obtains the form

$$g(t, g(s,x)) = g(st, x). \tag{6}$$

Denote $\psi(t) = d \cdot \log t \in (-\infty, \infty)$ and

$$S(\psi(t), x) := g(t,x). \tag{7}$$

Equation (6) becomes

$$S(u, S(v,x)) = S(u+v, x), \tag{8}$$

where $u = \psi(t)$, $v = \psi(s)$, $u, v \in (-\infty, \infty)$. We analyze two different cases:

(a) For fixed $x \in (l_H, m_H)$, we consider the correspondence $v \to y_v := S(v,x)$ only as a function of v. Let $y_v = h^*(v)$. The set $\{y_v \,|\, v \in (-\infty,\infty)\}$ coincides with the interval (l_H, m_H), having in mind the boundary conditions. The correspondence

$$(-\infty, \infty) \ni v \to y_v = h^*(v) \in (l_H, m_H)$$

is continuous and 1-1 (notice that $g(t,x)$ is continuous and strictly monotone in t for $x \in \mathcal{D}$). Thus, there exists a continuous strictly increasing mapping $h\colon (l_H, m_H) \to (-\infty, \infty)$ such that $h(y_v) = h(h^*(v)) = v$, and relation (8) becomes

$$S(u, y_v) = y_{v+u}. \tag{9}$$

At the same time, $y_{v+u} = h^*(v+u) = h^*(h(y_v)+u)$, and for $h^{-1} = h^*$, relation (9) becomes

$$S(u, y_v) = h^{-1}(h(y_v) + u). \tag{10}$$

From (7) we have $g(t,x) = S(d \log t, x)$, and hence

$$g(t,x) = h^{-1}(h(x) + d \log t), \qquad t > 0, \quad x \in (l_H, m_H).$$

Recall, for $x \in (l_H, m_H)$, $t \in (0,1)$, $g(t,x)$ is expanding; hence $d < 0$. Now

$$g(t,x) = h^{-1}(h(x) - c \log t), \qquad c = -d > 0.$$

For $t > 1$, $g(t,x) = g^{-1}_{1/t}(x) = h^{-1}(h(x) + c \log 1/t) = h^{-1}(h(x) - c \log t)$ is contracting. Consequently, $g_t(x) = h^{-1}(h(x) - c \log t)$, $c > 0$, $x \in (l_H, m_H)$, $t > 0$.

(b) For $x \in (m_H, r_H)$, g_t is contracting for $t \in (0,1)$ and expanding for $t > 1$. Analogously to case (a), one can show that there exists a continuous strictly increasing function $l\colon (m_H, r_H) \to (-\infty, \infty)$ such that

$$g_t(x) = l^{-1}(l(x) + c \log t), \qquad c > 0, \quad t > 0.$$

Remark. Denote $h_1(x) = c^{-1} \cdot h(x)$. One may notice that

$$g(t,x) = h^{-1}(h(x) - c \log t) = {h_1}^{-1}(h_1(x) - \log t).$$

Therefore, abusing the previous notation, further on we shall use the expression $g(t,x) = h^{-1}(h(x) - \log t)$ for $x \in (l_H, m_H)$ and $g(t,x) = l^{-1}(l(x) + \log t)$ for $x \in (m_H, r_H)$.

Finally, we give the following survey of the form and properties of $g_t(x)$ derived from *both* functional equations (4) and (5):

I. $x \in (l_H, m_H)$. In this interval, $\tau(x) < 0$ and $g(t,x) = h^{-1}(h(x) - \log t)$. Equation (5) implies that for $t \in (0,1)$, $g_t(x)$ is expanding, and for $t \in (1,\infty)$, $g_t(x)$ is contracting.

The function $g_t(x)$ decreases in t for fixed x.

II. $x \in (m_H, r_H)$. In this interval $\tau(x) > 0$ and $g(t,x) = l^{-1}(l(x) + \log t)$. Equation (5) implies that for $t \in (0,1)$, $g_t(x)$ contracts, and for $t \in (1,\infty)$, $g_t(x)$ expands. The function $g_t(x)$ increases in t for fixed x.

The analysis from this section can be generalized into the following lemma.

LEMMA 2. *Let* $g_t\colon (l_H, r_H) \to (l_H, r_H)$, $\{g_t \mid t > 0\}$ *be a continuous one parameter group in GMA satisfying* $g_t \circ g_s = g_{t\cdot s}$ *for all* $t, s > 0$. *We suppose that*

(i) g_t *is expanding for* $x \in (l_H, m_H)$, $t \in (0,1)$ *and for* $x \in (m_H, r_H)$, $t \in (1,\infty)$;

(ii) g_t *is contracting for* $x \in (m_H, r_H)$, $t \in (0,1)$ *and for* $x \in (l_H, m_H)$, $t \in (1,\infty)$.

Then there exist continuous and strictly increasing mappings

$$h\colon (l_H, m_H) \to (-\infty,\infty) \quad \text{and} \quad l\colon (m_H, r_H) \to (-\infty,\infty)$$

such that for $t > 0$

$$g_t(x) = \begin{cases} h^{-1}(h(x) - \log t), & x \in (l_H, m_H), \\ l^{-1}(l(x) + \log t), & x \in (m_H, r_H). \end{cases}$$

4. Solution of the functional equation (5). Let $S_1 = (l_H, m_H)$, $S_2 = (m_H, r_H)$. Using the previous analysis we can formulate the following lemma.

LEMMA 3. *Let* $\{g_t \mid t > 0\}$ *be the continuous one parameter group from Lemma* 2. *Suppose* τ *satisfies the functional equation* $\sqrt{t}\cdot\tau(x) = \tau(g_t(x))$, *for* $t > 0$ *and* $x \in S\cap\mathcal{D}$, *given* $\tau(x) > 0$ *on* $S_2 \cap \mathcal{D}$ *and* $\tau(x) < 0$ *on* $S_1 \cap \mathcal{D}$. *Then*

$$\tau(x) = \begin{cases} -c_1 e^{-h(x)/2},\ c_1 > 0 & \text{on } S_1 \cap \mathcal{D}, \\ c_2 e^{l(x)}/2,\ c_2 > 0 & \text{on } S_2 \cap \mathcal{D}. \end{cases}$$

Proof. I. Let $x \in S_2 \cap \mathcal{D}$, where $\tau(x) > 0$ and $g_t(x) = l^{-1}(l(x) + \log t)$. The functional equation (5) obtains the form

$$\sqrt{t}\cdot \tau \circ l^{-1}(l(x)) = \tau(l^{-1}(l(x) + \log t)), \qquad t > 0.$$

Let $\tau^* := \tau \circ l^{-1}$, $l(x) = y$, $x = l^{-1}(y)$. The equation above becomes

$$\sqrt{t}\cdot \tau^*(y) = \tau^*(y + \log t).$$

Now denote $(\tau^*)^2 =: \hat{\tau}$ and observe that

$$t \cdot \hat{\tau}(y) = \hat{\tau}(y + \log t).$$

For $\log t = s$, $t = e^s$, $s \in (-\infty,\infty)$, we have

$$e^s \cdot \hat{\tau}(y) = \hat{\tau}(y + s),$$

which is a functional equation with solution $\hat{\tau}(y) = ce^y$. Here $c > 0$ since $(\tau^*)^2 = \hat{\tau}$. Now, going back to τ^* we find that $\tau^*(y) = c_2 e^{y/2} = c_2 e^{l(x)/2} = \tau(x)$, where $c_2 = \sqrt{c} > 0$. Thus, for $x \in S_2 \cap \mathcal{D}$, the solution of the functional equation (5) is

$$\tau_2(x) = c_2 e^{l(x)/2}, \qquad c_2 > 0, \quad l\colon\ S_2 \to (-\infty, \infty).$$

II. Let $x \in S_1 \cap \mathcal{D}$, where $\tau(x) < 0$ and $g_t(x) = h^{-1}(h(x) - \log t)$. The functional equation (5) obtains the form

$$\sqrt{t} \cdot \tau \circ h^{-1}(h(x)) = \tau(h^{-1}(h(x) - \log t)).$$

Again for $\tau_* := \tau \circ h^{-1}$, $h(x) = y$, $x = h^{-1}(y)$, the previous equation becomes

$$\sqrt{t} \cdot \tau_*(y) = \tau_*(y - \log t).$$

Now let $(\tau_*)^2 = \bar{\tau}$, and transform the equation above into

$$t \cdot \bar{\tau}(y) = \bar{\tau}(y - \log t).$$

For $t = e^{-s} \in (0, \infty)$, $s = -\log t \in (-\infty, \infty)$, we have $e^{-s} \cdot \bar{\tau}(y) = \bar{\tau}(y + s)$, and therefore $\bar{\tau}(y) = ce^{-y}$. Here c is a positive constant, since $(\tau_*)^2 = \bar{\tau}$. Then $\tau_*(y) = -c_1 e^{-y/2} = \tau(x) < 0$, where $c_1 = \sqrt{c} > 0$. Thus, for $x \in S_1 \cap \mathcal{D}$, the solution of the functional equation (5) is $\tau_1(x) = -c_1 e^{-h(x)/2}$, $c_1 > 0$, $h\colon\ S_1 \to (-\infty, \infty)$.

Let us summarize:

$$\tau(x) = \begin{cases} \tau_1(x) = -c_1 e^{-h(x)/2},\ c_1 > 0, & \text{on } S_1 \cap \mathcal{D}, \\ \tau_2(x) = c_2 e^{l(x)/2},\ c_2 > 0, & \text{on } S_2 \cap \mathcal{D}. \end{cases}$$

The lemma is proved.

Using all previous results, after obtaining the explicit form of $\tau(x)$, we can now state the characterization theorem for the limit distribution H.

THEOREM 3. *The nondegenerate distribution function H in the limit relation* (1) *may take one of the following four explicit forms*:

$$(1) \quad H_1(x) = \begin{cases} 0, & x < m_H = l_H, \\ \dfrac{1}{2}, & x = m_H = l_H, \\ \Phi(\tau_2(x)), & x \in (m_H, r_H), \\ 1, & x \geqq r_H, \end{cases}$$

$$\textit{for } \tau(x) = \begin{cases} -\infty, & x < m_H = l_H, \\ 0, & x = m_H = l_H, \\ \tau_2(x), & x \in (m_H, r_H), \\ \infty, & x \geqq r_H; \end{cases}$$

$$(2) \quad H_2(x) = \begin{cases} 0, & x \leqq l_H, \\ \Phi(\tau_1(x)), & x \in (l_H, m_H), \\ \dfrac{1}{2}, & x = m_H = r_H, \\ 1, & x > m_H = r_H \end{cases}$$

$$\textit{for } \tau(x) = \begin{cases} -\infty, & x \leqq l_H, \\ \tau_1(x), & x \in (l_H, m_H), \\ 0, & x = m_H = r_H, \\ \infty, & x > m_H = r_H; \end{cases}$$

$$(3) \quad H_3(x) = \begin{cases} 0, & x \leqq l_H, \\ \Phi(\tau_1(x)), & x \in (l_H, m_H), \\ \Phi(\tau_2(x)), & x \in (m_H, r_H), \\ 1, & x \geqq r_H \end{cases}$$

$$\textit{for } \tau(x) = \begin{cases} -\infty, & x \leqq l_H, \\ \tau_1(x), & x \in (l_H, m_H), \\ \tau_2(x), & x \in (m_H, r_H), \\ \infty, & x \geqq r_H; \end{cases}$$

$$(4) \quad H_4(x) = \begin{cases} 0, & x < l_H, \\ \frac{1}{2}, & x \in [l_H, r_H), \\ 1, & x \geqq r_H \end{cases}$$

$$\textit{for } \tau(x) = \begin{cases} -\infty, & x \leqq l_H = m_H, \\ 0, & x \in (l_H = m_H, r_H), \\ +\infty, & x \geqq r_H. \end{cases}$$

Remark. Let H and G be nondegenerate distribution functions. Recall that in the extreme value theory we say that $H \in \text{type}\,(G)$ if there exists a mapping $\varphi \in GMA$ such that $H = G \circ \varphi$. In Theorem 3 above, we speak of four different explicit forms of H and not of types. If we count the types, in the sense above, we have to acknowledge that there are 13 possible types. Namely, H_1 gives rise to two different types: one with one jump, and another one with two jumps; the same holds for H_2; H_3 gives rise to eight types with mostly three jumps. The singular H_4 determines only one type, the two jump distribution.

We complete this section by comparing our results with Smirnov's results for linear norming mappings. In [9], the sequence of norming mappings $\{G_n\}$, $G_n(x) = a_n x + b_n$, $a_n > 0, b_n \in \mathbf{R}$, satisfies the regularity condition if there exist

$$\lim_{n\to\infty} \frac{a_n}{a_{[nt]}} = \alpha_t > 0, \quad \lim_{n\to\infty} \frac{b_n - b_{[nt]}}{a_{[nt]}} = \beta_t \in \mathbf{R},$$

namely,

$$\begin{aligned} {G_{[nt]}}^{-1} \circ G_n(x) &= \frac{G_n(x) - b_{[nt]}}{a_{[nt]}} \\ &= \frac{a_n}{a_{[nt]}} \cdot x + \frac{b_n - b_{[nt]}}{a_{[nt]}} \to \alpha_t x + \beta_t \in \text{GMA}, \qquad \alpha_t > 0, \quad \beta_t \in \mathbf{R}. \end{aligned}$$

So the continuous one parameter group $\{g_t \,|\, t > 0\}$ consists of mappings $g_t(x) = \alpha_t x + \beta_t$, $\alpha_t > 0$, $\beta_t \in \mathbf{R}$. Because of the half group property $g_t \circ g_s(x) = g_{ts}(x)$, one can easily see that $\alpha_t = t^\gamma$, $\gamma > 0$, $\beta_t = 0$. For our purposes we put $\gamma := 1/2\alpha$ and get $g_t(x) = \alpha_t x = x t^{1/2\alpha}$, $\alpha > 0$. For $x > 0$, $g_t(x) = l^{-1}(l(x) + \log t)$, where $l(x) =$

$\log x^{2\alpha}$. Then $\tau_2(x) = c_2 e^{l(x)/2} = c_2 x^\alpha, c_2 > 0$. For $x < 0, g_t(x) = h^{-1}(h(x) - \log t)$, where $h(x) = -\log|x|^{2\alpha}$. Then $\tau_1(x) = -c_1 e^{-h(x)/2} = c_1|x|^\alpha$, $c_1 > 0$. Thus, the solutions of the functional equation $\sqrt{t}\cdot\tau(x) = \tau(g_t(x))$ are

$$(1)\quad {\tau_1}^{(S)}(x) = \begin{cases} -\infty, & x < 0, \\ cx^\alpha,\ \alpha, c > 0, & x \geqq 0, \end{cases}$$

$$(2)\quad {\tau_2}^{(S)}(x) = \begin{cases} -c|x|^\alpha,\ \alpha, c > 0, & x \leqq 0, \\ +\infty, & x > 0, \end{cases}$$

$$(3)\quad {\tau_3}^{(S)}(x) = \begin{cases} -c_1|x|^\alpha,\ \alpha, c_1 > 0, & x < 0, \\ c_2 x^\alpha,\ \alpha, c_2 > 0, & x \geqq 0, \end{cases}$$

$$(4)\quad {\tau_4}^{(S)}(x) = \begin{cases} -\infty, & x \leqq l_H, \\ 0, & x \in (l_H, r_H), \\ +\infty, & x \geqq r_H. \end{cases}$$

We observe that ${\tau_1}^{(S)}(x)$ cannot jump at r_H, ${\tau_2}^{(S)}(x)$ cannot jump at l_H, and ${\tau_3}^{(S)}(x)$ is continuous. Thus, each of the four explicit forms of $\tau(x)$ defines only one type of H. The four types of Smirnov's limit distribution H are as follows:

(1) ${H_1}^{(S)} = \Phi({\tau_1}^{(S)}(x))$ has $\operatorname{supp} {H_1}^{(S)} = (0, \infty)$ and has a jump of height 1/2 in $l_H = m_H = 0$;

(2) ${H_2}^{(S)} = \Phi({\tau_2}^{(S)}(x))$ has $\operatorname{supp} {H_2}^{(S)} = (-\infty, 0)$ and has a jump of height 1/2 in $r_H = m_H = 0$;

(3) ${H_3}^{(S)} = \Phi({\tau_3}^{(S)}(x))$ has $\operatorname{supp} {H_3}^{(S)} = (-\infty, \infty)$ and is continuous on its support;

(4) ${H_4}^{(S)} = \Phi({\tau_4}^{(S)}(x))$ is the two jump distribution with jumps of height 1/2 both in l_H and r_H.

The results from Theorem 3 coincide with Smirnov's results if we take $G_n(x) = a_n x + b_n$, $a_n > 0$, $b_n \in \mathbf{R}$, to be regular. In case we use general monotone regular norming sequences there are differences. Namely,

(i) H_1 might have a jump at r_H;

(ii) H_2 might have a jump at l_H;

(iii) H_3 might have jumps at r_H, l_H and m_H.

5. Example for convergence of central upper order statistics. Let $X \sim F(x) = \sqrt{x}$, $x \in (0,1)$. For (X_i) independent identically distributed copies of X, let $X_{n,n} < \cdots < X_{k,n} < \cdots < X_{1,n}$ be the corresponding upper order statistics. Assume that $k_n/n \to \theta \in (0,1)$; hence $k_n \to \infty$, $n - k_n \to \infty$ as $n \to \infty$. We ask for a regular norming sequence $\{G_n\} \subset \mathrm{GMA}$ such that

$$F_{k_n,n}(G_n(x)) \xrightarrow[n]{w} \Phi(\tau(x)),$$

where

$$\tau(x) = \lim_{n\to\infty} \sqrt{n}\,\frac{\theta - \overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}}.$$

It is well known that the kth central upper order statistic $U_{k_n,n}$ of a sample uniformly distributed on $(0,1)$, properly normalized, converges weakly to a standard normal random variable $N(0,1)$ (see Theorem 2.1 in [1]). More precisely

$$\sqrt{n}\,\frac{k_n/n-(1-U_{k_n},n)}{\sqrt{(k_n/n)(1-k_n/n)}}\xrightarrow[n]{d}N(0,1),\ \text{i.e.,}$$
$$\sqrt{n}\,\frac{U_{k_n,n}-(n-k_n)/n}{\sqrt{(k_n/n)(1-k_n/n)}}\xrightarrow[n]{d}N(0,1).$$

Equivalently (with $k_n/n\sim p_n$, $(n-k_n)/n\sim 1-p_n=q_n$),

$$\mathbf{P}\{U_{k_n,n}<L_n(x)\}\xrightarrow[n]{w}\Phi(x),\qquad x\in(-\infty,\infty),$$

where $L_n(x)=\sqrt{p_nq_n/n}\cdot x+q_n$ is a linear transformation.

For $x\in(0,\infty)$ choose the function $\tau(x)=\sqrt{x}$. It is continuous and strictly increasing with $\tau^{-1}(x)=x^2$. For $x\in(0,1)$, $\tau^{-1}(x)=F^{-1}(x)$. Then

$$\mathbf{P}\{U_{k_n,n}<L_n(x)\}=\mathbf{P}\{\tau^{-1}(U_{k_n,n})<\tau^{-1}\{L_n(x)\}\}\xrightarrow[n]{w}\Phi(x),$$

and thus, by the quantile transformation (Lemma 4.1.9 in [10])

$$\mathbf{P}\{X_{k_n,n}<G_n(x)\}\xrightarrow[n]{w}\Phi(\tau(x))=H(x),\qquad x\in(0,\infty),$$

where

$$G_n(x)=\tau^{-1}\circ L_n\circ\tau(x)=\left(\sqrt{\frac{p_nq_n}{n}}\cdot\sqrt{x}+q_n\right)^2$$

and

$$\lim_{n\to\infty}\sqrt{n}\,\frac{\theta-\overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}}=\lim_{n\to\infty}\sqrt{n}\,\frac{p_n-(1-\sqrt{G_n(x)})}{\sqrt{p_nq_n}}=\tau(x).$$

We check that $\{G_n\}$ is a regular norming sequence. Indeed,

$$G_n{}^{-1}(y)=\left(\frac{\sqrt{y}-q_n}{\sqrt{p_nq_n/n}}\right)^2=\frac{(\sqrt{y}-q_n)^2}{p_nq_n}\,n.$$

Consequently, for $m\sim[nt]$, $t\in(0,1)$

$$\begin{aligned}G_m{}^{-1}\circ G_n(x)&=\frac{(\sqrt{G_n(x)-q_m})^2}{p_mq_m}\cdot m\\&=\frac{(\sqrt{p_nq_nx/n}+(q_n-q_m))^2}{p_mq_m}\cdot m\sim\frac{p_n}{p_m}\cdot\frac{q_n}{q_m}\cdot\frac{m}{n}\cdot x.\end{aligned}$$

Since $p_n\sim k_n/n$ one can see that $(q_n-q_m)\to 0$, $p_n/p_m\to 1$, $q_n/q_m\to 1$, as $n,m\to\infty$. Thus $G_{[nt]}{}^{-1}\circ G_n(x)\to t\cdot x=:g_t(x)\in$ GMA. Moreover, $g_t(x)=l^{-1}(l(x)+\log t)$ with $l(x)=\log x$, $x>0$. Consequently,

$$\tau(x)=\begin{cases}e^{l(x)/2}=\sqrt{x}, & x\geqq 0,\\ -\infty, & x<0\end{cases}$$

and

$$H(x)=\begin{cases}\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\sqrt{x}}e^{-u^2/2}\,du=\Phi(\sqrt{x}), & x\geqq 0,\\ 0, & x<0\end{cases}$$

has a jump of height 1/2 at the left end point of $\operatorname{supp}H=(0,\infty)$.

Appendix.

Proof of Theorem 1. The proof is the same as in the classical framework. For methodological reasons, we give here, a brief sketch.

Let relation (1) hold. Notice that relation (2) is equivalent to

$$\frac{k_n - n\overline{F}(G_n(x))}{\sqrt{k_n(1-k_n/n)}} \xrightarrow[n]{w} \tau(x)$$

in view of the assumptions. Define

$$B_n = \mathrm{card}\,\{\text{exceedances over level } G_n(x) \text{ by } X_1, X_2, \dots, X_n\} = \sum_{i=1}^{n} I\{X_i \geq G_n(x)\}.$$

The probability of exceedance (success) is $p_n = \overline{F}(G_n(x))$. Then $F_{k_n,n}(G_n(x)) = \mathbf{P}\{X_{k_n,n} < G_n(x)\} = \mathbf{P}\{B_n < k_n\}$. Since B_n is a Bernoulli random variable, using the central limit theorem we get

$$\begin{aligned} F_{k_n,n}(G_n(x)) &= \mathbf{P}\biggl\{\frac{B_n - np_n}{\sqrt{n}p_n(1-p_n)} < \frac{k_n - np_n}{\sqrt{np_n(1-p_n)}}\biggr\} \\ &\sim \Phi\biggl(\frac{k_n - np_n}{\sqrt{np_n(1-p_n)}}\biggr) \longrightarrow \Phi(\tau(x)). \end{aligned}$$

Due to the continuity of Φ, we have

$$\frac{k_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow[n]{w} \tau(x).$$

It is easy to see that $k_n \sim np_n$, $n - k_n \sim n(1-p_n)$ (see, e.g., [8]). Therefore,

$$\frac{k_n - n\overline{F}(G_n(x))}{\sqrt{k_n(1-k_n/n)}} \xrightarrow{w} \tau(x),$$

and hence the equivalent condition (2) holds.

On the other hand, let (2) hold. Without any additional restrictions, for $k_n \to \infty$, $n - k_n \to \infty$, we may use Lemma 2 in [9], “translated” for upper order statistics. It states that for

$$\tau_n(x) := \sqrt{n}\,\frac{\theta - \overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}} \xrightarrow{w} \tau(x), \qquad n \to \infty,$$

the difference

$$R_{k_n,n}(x) = F_{k_n,n}(G_n(x)) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau_n(x)} e^{-t^2/2}\, dt$$

uniformly converges to 0. Now,

$$\begin{aligned} \Bigl|F_{k_n,n}(G_n(x)) - \Phi(\tau(x))\Bigr| &\leqq \Bigl|F_{k_n,n}(G_n(x)) - \Phi(\tau_n(x))\Bigr| \\ &\quad + \Bigl|\Phi(\tau_n(x)) - \Phi(\tau(x))\Bigr| \to 0, \end{aligned}$$

i.e., (1) holds. Theorem 1 is proved.

Proof of Lemma 1. Let $\lim_{n\to\infty} {G_{[nt]}}^{-1} \circ G_n(x) = g_t(x) \in \text{GMA}$ and $s \in (0,1]$, $s \cdot t \in (0,1]$. We consider ${G_{[nts]}}^{-1} \circ G_n(x) = {G_{[nts]}}^{-1} \circ G_{[nt]} \circ {G_{[nt]}}^{-1} \circ G_n(x)$. Because of the continuity and strictly increasing property of the G_n's, we can regroup the mappings and

$$ {G_{[nts]}}^{-1} \circ G_n(x) = ({G_{[nts]}}^{-1} \circ G_{[nt]}) \circ ({G_{[nt]}}^{-1} \circ G_n(x)) \to g_t \circ g_s(x). $$

On the other hand, by definition ${G_{[nts]}}^{-1} \circ G_n(x) \to g_{t\cdot s}(x)$, therefore $g_t \circ g_s(x) = g_{t\cdot s}(x)$. Lemma 1 is proved.

Proof of Theorem 2. Since there exists $\lim_{n\to\infty} {G_{[nt]}}^{-1} \circ G_n(x) = g_t(x)$ and $F_{k_{[nt]},[nt]}(G_{[nt]}(x)) \xrightarrow[n]{w} H(x)$, using the result of Theorem 1, we have

$$ \begin{aligned} F_{k_{[nt]},[nt]}(G_n(x)) &= F_{k_{[nt]},[nt]}(G_{[nt]}({G_{[nt]}}^{-1} \circ G_n(x))) \xrightarrow{w} H(g_t(x)) \\ &= \Phi \circ \tau(g_t(x)). \end{aligned} $$

Notice that

$$ \sqrt{[nt]}\, \frac{\theta - \overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}} = \sqrt{\frac{[nt]}{n}}\, \sqrt{n}\, \frac{\theta - \overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}} \underset{n\to\infty}{\longrightarrow} \sqrt{t} \cdot \tau(x). \tag{11} $$

Let $B_{n'} = \sum_{i=1}^{[nt]} I\{X_i \ge G_n(x)\}$ count down the exceedances over level $G_n(x)$ by $X_1, X_2, \dots, X_{[nt]}$. Thus $B_{n'}$ is a Bernoulli random variable with the success probability $p_n = \overline{F}(G_n(x))$, $\mathbf{E}\, B_{n'} = [nt]p_n$, $\mathbf{D}B_{n'} = [nt]p_n(1-p_n)$. Then

$$ \begin{aligned} F_{k_{[nt]},[nt]}(G_n(x)) &= \mathbf{P}\{X_{k_{[nt]},[nt]} < G_n(x)\} = \mathbf{P}\{B_{n'} < k_{[nt]}\} \\ &= \mathbf{P}\left(\frac{B_{n'} - [nt]p_n}{\sqrt{[nt]p_n(1-p_n)}} < \frac{k_{[nt]} - [nt]p_n}{\sqrt{[nt]p_n(1-p_n)}} \right) \sim \Phi\left(\frac{k_{[nt]} - [nt]p_n}{\sqrt{[nt]p_n(1-p_n)}} \right). \end{aligned} $$

On the other hand, using (11) we obtain

$$ \begin{aligned} \frac{k_{[nt]} - [nt]p_n}{\sqrt{[nt]p_n}(1-p_n)} &= \frac{[nt]}{\sqrt{[nt]}} \cdot \frac{k_{[nt]}/[nt] - p_n}{\sqrt{p_n}(1-p_n)} \\ &\sim \sqrt{[nt]}\, \frac{\theta - \overline{F}(G_n(x))}{\sqrt{\theta(1-\theta)}} \longrightarrow \sqrt{t} \cdot \tau(x). \end{aligned} $$

Hence, $F_{k_{[nt]},[nt]}(G_n(x)) \xrightarrow{w} \Phi(\sqrt{t} \cdot \tau(x))$.

Smirnov [9] stated Theorem 5 in which the second order condition is a sufficient condition for the uniqueness of the limit distribution. Applying this result, without any additional conditions, it is clear that in our case $H(g_t(x)) = \Phi(\sqrt{t} \cdot \tau(x))$, hence $\sqrt{t} \cdot \tau(x) = \Phi^{-1}(H(g_t(x))) = \tau(g_t(x))$; i.e., the functional equation (5) is satisfied. Theorem 2 is proved.

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