A FUNCTIONAL EXTREMAL CRITERION*

P. K. Jordanova (Shumen, Bulgaria) and E. I. Pancheva (Sofia, Bulgaria)  UDC 519.2

1. Introduction

In this paper, we are concerned with the following model:

(A) Let \( \mathcal{N} = \{(t_k, X_k): k \geq 1\} \) be a point process with time space \([0, \infty)\) and state space \([0, \infty)^d\), where \(\{t_k\}\) are distinct nonrandom time points. We assume them ordered and increasing to \(\infty\), i.e., \(t_1 < t_2 < \ldots\). So, the point process \(\mathcal{N}\) is simple in time; \(\{X_k\}\) are independent and identically distributed (i.i.d.) random vectors (r.v.'s) on a given probability space with values in \([0, \infty)^d\) and with common distribution function (d.f.) \(F\) nondefective at \(+\infty\).

Assume that almost all realizations of \(\mathcal{N}\) are Radon measures on \(S := [0, \infty) \times E\), where \(E := [0, \infty]^d \setminus \{0\}\), i.e.,

\[ \mathcal{N}(A) < \infty \quad \text{a.s.} \quad \forall \ \text{compact sets} \ A \in \mathcal{B}(S). \tag{1} \]

We consider two random processes associated with \(\mathcal{N}\), namely, the extremal process

\[ X(t) = \{\vee X_k: t_k \leq t\} \]

and the process \(Z\) with additive increments

\[ Z(t) = \left\{ \sum X_k: t_k \leq t \right\}. \]

Because of (1), the sum and the maximum are a.s. finite for every fixed \(t > 0\). Both processes are right continuous with increasing sample paths. Here and further on we use the notion increasing in the sense of nondecreasing.

We denote by \(\mathcal{M}\) the set of all increasing, right-continuous functions \(y: (0, \infty) \to [0, \infty]^d\). Then the set \(\mathcal{P}\) of all probability measures on \(\mathcal{M}\) is compact. Let \(\{P_n\}\) be a sequence of probability measures on \(\mathcal{M}\). We say \(\{P_n\}\) is weakly convergent to \(P \in \mathcal{P}\), briefly \(P_n \Rightarrow P\), if \(\int \varphi \, dP_n \to \int \varphi \, dP\) for bounded \(\varphi: \mathcal{M} \to R\) which are continuous in the weak topology of \(\mathcal{M}\). Now denote by \(\mathcal{P}_c\) and \(\mathcal{P}_s\) the subsets of \(\mathcal{P}\) corresponding to an extremal process (with independent max-increments) and to a sum process (with independent additive increments), respectively. In [1, Theorem 6.4] it is shown that the space \(\mathcal{P}_c\) with the topology of weak convergence is closed in \(\mathcal{P}\). The same is also true for \(\mathcal{P}_s\). So, the weak convergence of extremal processes \(Y_n \Rightarrow Y\) and of sum processes \(S_n \Rightarrow S\) is equivalent to the convergences in distribution \(Y_n(t) \overset{d}{\to} Y(t)\) and \(S_n(t) \overset{d}{\to} S(t)\) for each continuity point \(t\) of the limit process.

Further, for normalizing we use an unboundedly increasing in \(n\) sequence of mappings

\[ \zeta_n(t, x) = (\tau_n(t), u_n(x)), \]

continuous and strictly increasing in each coordinate. We call them time-space changes. Suppose \(\{\zeta_n\}\) is regular in the sense that there exists a pointwise limit of \(\zeta_n^{-1} \circ \zeta_{[ns]}\) for \(n \to \infty\) and \(s > 0\) which is again continuous and strictly increasing (cf. [6]). We assume the weak convergence

\[ Y_n(t) = \{\vee u_n^{-1}(X_k): t_k \leq \tau_n(t)\} \Longrightarrow Y(t), \quad n \to \infty, \tag{2} \]

to a nondegenerate extremal process \(Y\) with initial value \(Y(0) \overset{\text{a.s.}}{=} 0\).

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We also form the associated processes with additive increments

\[ S_n(t) := \left\{ \sum u_n^{-1}(X_k): t_k \leq \tau_n(t) \right\}. \]

Note that the space changes \( \{u_n\} \) preserve the max-operation, i.e., \( u_n^{-1}(\vee X_k) = \vee u_n^{-1}(X_k) \), but do not preserve (in general) the summing operation. Hence, \( Y_n(t) = u_n^{-1} \circ X \circ \tau_n(t) \) but \( S_n(t) \neq u_n^{-1} \circ Z \circ \tau_n(t) \) in general. If \( u_n \) preserves both operations \( \vee \) and \( \sum \), then \( u_n \) is just a scale change and the convergence \( S_n = u_n^{-1} \circ Z \circ \tau_n \Rightarrow S \) implies that \( S \) is a self-similar process (cf. [4]).

Our main result, proved in Sec. 2, concerns the convergence \( S_n \Rightarrow S \), if given (2). We call it a functional extremal criterion (for the convergence \( S_n \Rightarrow S \)), having in mind the extremal criterion in [5, §2.2.4.c].

**Theorem 1.** Let \( \mathcal{N} = \{(t_k, X_k), k \geq 1\} \) be the point process described in (A) and let \( \zeta_n(t, x) = (\tau_n(t), u_n(x)) \) be a regular norming sequence of time-space changes of \((0, \infty)^{d+1}\) such that the sequence of the associated extremal processes \( Y_n(t) = \{\vee u_n^{-1}(X_k): t_k \leq \tau_n(t)\} \) is weakly convergent to a nondegenerate extremal process \( Y(t) \). Assume that the d.f. \( F \) of \( Y(1) \) satisfies the condition

\[ I_F = \int_{\mathcal{A}} ||x|| d(\log F(x)) < \infty. \]

Then there exists a time-space change \( \zeta(t, x) = (\tau(t), h_n(x)) \) such that the sequence of the associated sum processes \( S_n(t) := \{\sum u_n^{-1}(X_k): t_k \leq \tau_n(t)\} \) is weakly convergent to an infinitely divisible process \( S(t) \) whose characteristic function is given by

\[ \mathbb{E} e^{i(\theta, S(t))} = \exp \left\{ \tau(t) \int_E \left\{ e^{i(\theta, h_n^{-1}(x))} - 1 \right\} d\Pi(x) \right\}, \]

where \( \Pi(dx) \) is the Lévy measure of Proposition 2, (iii).

2. Stepwise Proof of the Functional Extremal Criterion

We put \( t_{nk} = \tau_n^{-1}(t_k) \), \( X_{nk} = u_n^{-1}(X_k) \), and consider the point process \( \mathcal{N}_n := \{(t_{nk}, X_{nk}, k \geq 1)\} \) associated with \( S_n \) and \( Y_n \).

**Step 1.** Denote by \( k_n(t) \) the nonrandom counting function of \( \mathcal{N}_n \), i.e.,

\[ k_n(t) = \max\{k: t_k \leq \tau_n(t)\} = \sum I_{[0, t]}(t_{nk}). \]

Here \( I_A(\bullet) \) is the indicator of the set \( A \). By (2), we have the weak convergence

\[ P(Y_n(t) < x) = F^{k_n(t)}(u_n(x)) \xrightarrow{w} g(t, x), \quad n \to \infty, \]  

(3)

where \( g \) is the d.f. of the limit process \( Y \). Thus, for fixed \( t > 0 \) \( F \) belongs to the partial max-DA of \( g(t) := g(t, x) \). Hence \( Y \) (respectively, \( g \)) is max-ID.

Moreover, by Propositions 2.1 and 2.3 in [6], \( Y \) (respectively, \( g \)) is self-similar (briefly \( Y \in \text{SS} \)) and stochastically continuous. The condition \( Y(0) \overset{a.s.}{=} 0 \) guarantees that \( G(x) := g_1(x) \) does not have a defect at \( +\infty \). In our case, where \( \{X_k\} \) are i.i.d., one can determine more precisely the subclass of SS which \( g \) belongs to.

**Lemma 1.** The regularity of the time-space changes \( \zeta_n \) implies the regularity of the sequence \( k_n := k_n(1) \).

**Proof.** For \( t = 1 \), (3) reads as

\[ F^{k_n}(u_n(x)) \xrightarrow{w} G(x), \quad n \to \infty. \]  

(4)

Now we take \( s > 0 \) and observe the convergence in distribution for \( n \to \infty \)

\[ u_n^{-1} \circ X \circ \tau_{[ns]} = u_n^{-1} \circ u_{[ns]} \circ u_n^{-1} \circ X \circ \tau_{[ns]} \xrightarrow{d} U_s \circ Y, \]

where \( U_s(x) := \lim_{n \to \infty} u_n^{-1} \circ u_{[ns]}(x), \forall x \in \{G > 0\} \). Hence,

\[ P(u_n^{-1} \circ X \circ \tau_{[ns]}(1) < x) = F^{k_{[ns]}}(u_n(x)) \xrightarrow{w} P(Y(1) < U_s^{-1}(x)) = G(U_s^{-1}(x)). \]
On the other hand, $F^{k_n}_{[x]}(u_n(x)) = \left[F^{k_n}(u_n(x))\right]^{k_n_{[x]}/k_n}$. So there exists $\lim_{n \to \infty} (k_{[x]}/k_n) = k(s) \in (0, \infty)$. As is known, the last convergence is uniformly in $s$ and $k(s)$ is a power function of $s$. Say $s^\beta$.

The next examples show the consequences of the regularity of $k_n$ for the time process $\{t_n\}$.

**Example 1.** Let $t_k = e^k$, $k \in \{1, 2, \ldots, \}$, and take a time change $\tau_n(t) = (tn)^2$, then $t_{nk} = \tau^{-1}(t_k) = (1/n)e^{k/2}$ and $k_n = \sum_k t_{[k]}(t_{nk}) = 2\log n$ is not a regular sequence (but is slowly varying).

**Example 2.** Let $t_n = (n(n + 1))/2$ and $\tau_n(t)$ be the same as in Example 1. Now the sequence $\{t_n\}$ is regular, $t_{nk} = (1/n)^{k(k + 1)/2}$, and $k_n \sim 2n$ is regular, too.

Let us come back to (4) and observe that the limit d.f. $G(x)$ satisfies $G^k(x) = G(U_s^{-1}(x))$ for all $s > 0$, where $k(s) = s^\beta$.

Denote $L(t) := U \varphi(t)$. Now from $G^t(x) = G(L_t^{-1}(x)) \forall t > 0$ one can conclude that $G$ is max-stable with respect to the continuous one-parameter group $L = \{L_t: t > 0\}$. Note that $L$ bears the regularity of both sequences $\{k_n\}$ and $\{u_n\}$.

Analogously one can see that there exists $\lim_{n \to \infty} k_n(t)/k_n =: \tau(t)$ and, finally, we get

$$g(t, x) = G^{\tau(t)}(x).$$

The mapping $t \to \tau(t)$ is continuous and increasing (since $Y \in SS$), hence it is a time change and we can write

$$g(\tau^{-1}(t), x) = G^t(x) = G(L_t^{-1}(x)) = g(1, L_t^{-1}(x)).$$

This means that $Y \circ \tau^{-1}(t) = L_t \circ Y(1), \forall t > 0$. Thus the process $Y \circ \tau^{-1}$ has homogeneous max-increments. Note that $L_1 = 1d, \tau(1) = 1, \tau(t) \to 0, t \to 0, \tau(t) \to \infty, t \to \infty$.

**Step 2.** Choose $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < 1$ (later below we discuss this choice). Let $\Phi^*_\alpha(x_1, \ldots, x_d)$ be the d.f. on $[0, \infty)^d$ with Frechet univariate marginals

$$\Phi^*_\alpha(x_i) = e^{-x_i^{-\alpha_i}}, \quad i = 1, \ldots, d,$$

whose dependence (copula) function is the same as that of the limit d.f. $G(x)$. We determine the mapping

$$h_\alpha(x) = (h_{\alpha_1}(x_1), \ldots, h_{\alpha_d}(x_d))$$

for all $x$ in the support of $G$ (briefly Supp $G$) by $G(h^{-1}_\alpha(x)) = \Phi^*_\alpha(x)$ and use it to define the r.v. $Y^*_1 := h_\alpha \circ Y(1)$. It is distributed by $\Phi^*_\alpha$. Note that the mapping $h_\alpha$: Supp $G \to [0, \infty)^d$ is continuous and strictly increasing in each component.

Now the limit relation (4) implies that the d.f. $F'(x) := F \circ h^{-1}_\alpha(x)$ belongs to the max-DA($\Phi^*_\alpha$), i.e.,

$$[F'(T_n(x))]^{k_n} \buildrel {w} \over \to \Phi^*_\alpha(x), \quad n \to \infty,$$

with norming sequence $T_n(\cdot) = h_\alpha \circ u_n \circ \tau^{-1}(\cdot)$, which is regular at $\infty$.

**Step 3.** Combine Step 1 and Step 2. So, we started with the limit extremal process $Y(t)$ distributed by $g(t, x)$ and came to the time-space changed extremal process $Y^*_t := h_\alpha \circ Y \circ \tau^{-1}(t), t > 0$, with d.f. ($\Phi^*_\alpha$)$^t$. It is self-similar with respect to the continuous one-parameter group

$$U_\alpha = \{U_t(x) = (\sqrt[n]{U_{x_1}}, \ldots, \sqrt[n]{U_{x_d}}): t > 0\}.$$

More precisely, the extremal process $Y^*_t = U_t \circ Y^*(1)$ is stochastically continuous, has homogeneous max-increments, and $Y^*(0) \buildrel {}\over = 0$. Thus it is a Lévy process in the max-framework.

In what follows, we denote the vector $(\sqrt[n]{U_{x_1}}, \ldots, \sqrt[n]{U_{x_d}})$ simply by $\sqrt[n]{U}$. Then we can write

$$Y^*_t = \sqrt[n]{U}Y^*(1).$$

**Proposition 1.** Let (5) hold, i.e., $F' \in$ max-DA($\Phi^*_\alpha$) with respect to a regular norming sequence $\{T_n\}$ of space changes and a regular subsequence $\{k_n\}$ of $\{n\}$. Then there exists a space change $T$ such that

$$[F' \circ T(\sqrt[n]{k_n} x)]^{k_n} \buildrel {w} \over \to \Phi^*_\alpha(x), \quad n \to \infty.$$

(7a)
Now we can see that $T$ processes with additive increments. Consider now the process with additive increments (or equivalently, normalizations preserve both the $\text{(A) to a model (B), where the sequence of extremal processes needs scale normalization to converge. And scale}

Let us summarize what we have achieved within the three steps: we have transformed continuously our initial model

where $\nu_\alpha$ is the exponent measure of $\Phi^*_\alpha$ satisfying

$$s^{-1} \nu_\alpha(A^e_{x}) = \nu_\alpha(A^e_{\sqrt{s}x}),$$

(7b)

Statement (7a) of Proposition 1 is equivalent to (cf. [8, Proposition 5.17]):

$$1 - F' \circ T is regularly varying at $\infty$, i.e., for $A^e_x := [0, \infty]\setminus [0, x)$ and $e = (1, \ldots, 1) \in \mathbb{R}^d$

$$\frac{1 - F' \circ T(se)}{1 - F' \circ T(e)} \rightarrow \frac{\nu_\alpha(A^e_x)}{\nu_\alpha(A^e_e)} := \lambda(x), \quad s \rightarrow \infty,$$

Let us summarize what we have achieved within the three steps: we have transformed continuously our initial model (A) to a model (B), where the sequence of extremal processes needs scale normalization to converge. And scale normalizations preserve both the $\vee$ and $\sum$ operations. In the next step, we pursue convergence of the associated processes with additive increments.

**Step 4.** Model (B): Denote $t^*_k = \tau(t_k)$, $\sigma_n(t) = \tau \circ \tau_n \circ \tau^{-1}(t)$, and

$X^*_k = T^{-1} \circ h_\alpha(X_k), \quad k = 1, 2, \ldots,$

with common d.f. $F^* := F' \circ T$. In this model, we use the point process $\mathcal{N}^* = \{(t^*_k, X^*_k); k \in \{1, 2, \ldots\}\}$ and the norming sequence $\eta_n(t, x) = (\sigma_n(t), \sqrt{k_n} x)$ to generate the asymptotically homogeneous point process

$$\mathcal{N}_n^* = \left\{(t^*_nk = \sigma_n^{-1}(t_k), X^*_nk = \frac{1}{\sqrt{k_n}} X^*_k; \quad k \geq 1\right\}.$$

Consider now the process with additive increments

$$S_n^*(t) = \left\{\sum X^*_nk: \quad t^*_nk \leq t \right\} = \frac{k^*_n(t)}{\sqrt{k_n}} \sum_{k=1}^{k^*_n(t)} X^*_k$$

and the extremal process

$$Y_n^*(t) = \left\{\vee X^*_nk: \quad t^*_nk \leq t \right\} = \frac{k^*_n(t)}{\sqrt{k_n}} \vee_{k=1}^{k^*_n(t)} X^*_k$$
associated with the same point process \( N^*_n \) and with the same counting function
\[
k^*_n(t) = \sum_k I_{[0,t]}(t^*_nk) \sim k_n t.
\]
The last asymptotic relation is a consequence of \( k^*_n(t) = k_n(\tau^{-1}(t)) \) and \( k_n(t) \sim k_n \tau(t) \) established in the first step. Further, by (7a),
\[
Y^* \implies Y^*.
\]
In model (B), a lot of results are well known. We gather them in the next proposition. Let \( \mathcal{B}(E) \) denote the Borel \( \sigma \)-algebra of subsets of \( E \).

**Proposition 2.** The following statements are equivalent:

(i) \( Y^*_n \Rightarrow Y^* \) and the limit process is max-stable with respect to the multiplicative group \( U_n \) defined in (6);
(ii) \( N^*_n \Rightarrow \pi \) and the limit point process \( \pi \) is a homogeneous Poisson point process whose structural measure \( \mu \) does not charge instant spaces and
\[
\mu([0,t] \times A) = t \nu_\alpha(A) \quad \text{for} \quad A \in \mathcal{B}(E);
\]
(iii) \( S^*_n \Rightarrow S^* \) and the limit process is \( \alpha \)-stable. Its Lévy measure \( \Pi \) satisfies
\[
\Pi(A) = \nu_\alpha(A), \quad A \in \mathcal{B}(E), \quad \Pi(\{0\}) = 0.
\]

**Proof.** The equivalence (i) \( \Leftrightarrow \) (ii) is a special case of Proposition 3.21 in [8]. Recall that every max-ID extremal process (with d.f. \( g \)) is associated with a Poisson point process (with structural measure \( \mu \)) and the connection between them is given by
\[
g(t, x) = e^{-\mu([0,t] \times A^c_x)}
\]
(cf. [1]). Let \( (T_k, Y^*_k), k = 1, 2, \ldots, \) be the points of \( \pi \). Then
- \( Y^*(t) = \bigcup Y^*_k : T_k \leq t \) is max-ID \( \Leftrightarrow \pi \) is Poisson;
- \( Y^*(t) \) is stochastically continuous \( \Leftrightarrow \mu \) does not charge instant spaces, i.e., \( \mu(\{t\} \times A) = 0, \ A \in \mathcal{B}(E) \);
- \( Y^*(t) \) has homogeneous max-increments \( \Leftrightarrow \mu([s,t] \times A) = (t-s) \nu_\alpha(A), \quad 0 \leq s < t < \infty \).

Note that \( \nu_\alpha \) is a Radon measure on \( E \), i.e., finite on compact subsets far away from zero.

On (i) \( \Rightarrow \) (iii). The process
\[
S^*(t) = \left\{ \sum_k Y^*_k : T_k \leq t \right\}
\]
is associated with the time-homogeneous Poisson point process \( \pi \) on \( S \), whose structural measure does not charge instants. Hence it is stochastically continuous. Further it has nonnegative independent increments. Thus for \( \theta \in (0, \infty)^d \) its characteristic function \( \varphi_\pi(\theta) := \mathbb{E} e^{i(\theta, S^*(t))} \) has the form
\[
\varphi_\pi(\theta) = \exp \left\{ t \int_E \left\{ e^{i(\theta, x)} - 1 \right\} \Pi(dx) \right\}, \quad (9)
\]
where the \( \sigma \)-finite Lévy measure \( \Pi \) has the properties
\[
\int_{A^c_x} \|x\| \Pi(dx) < \infty, \quad \Pi(\{0\}) = 0.
\]
Further, \( \Pi \) is determined by the limit relation
\[
k_n[1 - F^*(\sqrt{k_n} x)] \longrightarrow \Pi(A^*_x), \quad \forall x > 0, \quad n \to \infty.
\]
By (7a) and since \( \mathcal{B}(E) \) is generated by sets of the form \( A^*_x, x > 0, \)
\[
\Pi(A) = \nu_\alpha(A), \quad \forall A \in \mathcal{B}(E).
\]
The last limit relation together with the regularity of the tail \((1 - F^*)\), expressed in (7b), is equivalent to the weak convergence

\[
S_n^*(1) \sim \frac{1}{\sqrt{k_n}} \sum_{k=1}^{k_n} X_k^* \overset{d}{\to} S^*(1), \quad n \to \infty, \tag{10}
\]

where \(S^*(1)\) is a one-sided \(\alpha\)-stable r.v. (see, e.g., [9]) with \(\alpha_i \in (0, 1), i = 1, \ldots, d\). From here and the asymptotic \(k_n(t) \sim k_n t\) we get

\[
S_n^*(t) \overset{d}{\to} S^*(t), \quad \forall t > 0, \quad n \to \infty. \tag{11}
\]

So \(S^*(t), t > 0\), is one-sided \(\alpha\)-stable process with \(\alpha_i \in (0, 1)\) and \(S^*(0) = 0\) a.s. In fact,

\[
S^*(t) \overset{d}{\sim} \sqrt{t} S^*(1).
\]

The inverse implication \((iii) \Rightarrow (i)\) is obvious.

**Remarks.** 1. Now the choice of \(\alpha\) with \(\alpha_i \in (0, 1)\) is plausible: in this case \(\vee X_k^*\) and \(\sum X_k^*\) need the same scale normalization \(\sqrt{k_n}\).

2. It is no surprise that the spectral measure \(\Pi\) of \(S^*(1)\) and the exponent measure \(\nu_\alpha\) of \(Y^*(1)\) coincide on \(\mathcal{B}(E)\). By construction, \(Y^*(1)\) is the largest jump of \(S^*\) in \([0, 1]\) and \(\Pi(A_\alpha)\) is just the expected value of the number of jumps in \([0, 1]\) larger than \(x\) (cf. [5, XI]). More interesting is that the dependence structure of the process \(S^*(t)\) for all \(t > 0\) is determined by the dependence structure of the maximal jump of \(S^*\) in \([0, 1]\). Indeed, in the integral expression of the exponent measure

\[
\nu_\alpha(A_\alpha) = \int_{S_\alpha^t} \max_{1 \leq e \leq d} \left( \frac{s_e}{x_i} \right)^{\alpha_i} Q(ds),
\]

the dependence structure of the r.v. \(Y^*(1) = (Y_1^*, \ldots, Y_d^*)\) is borne by \(Q\). Here \(S_\alpha^t\) is the intersection of \(E\) and the unit sphere in \(\mathbb{R}^d\), and \(Q\) is a finite Borel measure on \(S_\alpha^t\) (cf., e.g., [8]). In the case of full dependence, i.e., if \(P(Y_1^* = \cdots = Y_d^*) = 1\), \(\nu_\alpha\) is concentrated on the orbit \(\{\sqrt{s} \in E : s > 0\}\), respectively, \(Q\) is concentrated at the point \(e/\|e\|\). Hence \(\Pi(A_\alpha) = \nu_\alpha(A_\alpha) = \sqrt{d} \sum_{i=1}^d x_i^{-\alpha_i}\). In the case of independent marginals,

\[
P(Y_1^* < x_1, \ldots, Y_d^* < x_d) = \exp\left\{-\sum_{i=1}^d x_i^{-\alpha_i}\right\}.
\]

So \(\Pi(A_\alpha) = \nu_\alpha(A_\alpha) = \sum_{i=1}^d x_i^{-\alpha_i}\). Consequently, \(Q\) is discrete and concentrated on \(e_i, i = 1, \ldots, d\), where \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) with 1 in the \(i\)th coordinate.

How \(Q\) reflects the dependence structure of the associated vector in \(\mathbb{R}^d\) is in general a hard problem (cf. [3]).

Now we come back to our starting problem: to determine the limit behavior of the sum process in the initial model (A)

\[
S_n(t) = \sum_{k=1}^{k_n(t)} u_n^{-1}(X_k), \quad n \to \infty.
\]

By the multivariate Central Criterion of Convergence (CCC), this sum is convergent if and only if

\begin{itemize}
  \item [(C1)] \(k_n(t)[1 - P(u_n^{-1}(X_k) < x)]\) converges weakly to a nonnegative, nonincreasing, and right-continuous function;
  \item [(C2)] \(k_n(t) P(u_n^{-1}(X_k) | \{u_n^{-1}(X_k) < v\})\) converges to a finite vector, say \(a(t, v)\).
\end{itemize}

Below, we shall check these conditions in our case.

The limit relation \((iii)\) of Proposition 2 implies

\[
P(S_n^*(t) < x) = P\left\{ \sum_{k=1}^{k_n(t)} \frac{1}{\sqrt{k_n}} T^{-1} \circ h_\alpha(X_k) < x \right\} \sim P\left\{ \sum_{k=1}^{k_n(t)} h_\alpha \circ u_n^{-1}(X_k) < x \right\} \overset{w}{\to} P(S^*(t) < x), \quad n \to \infty.
\]

Here we have used the relation \(T(\sqrt{k_n} x) \sim T_n(x) \sim h_\alpha \circ u_n \circ h_\alpha^{-1}(x)\). Hence necessarily we have for \(t > 0\) and \(x > q := \inf \text{Supp } G\)

\[
k_n(t)[1 - P(u_n^{-1}(X_k) < x)] \rightarrow \tau(t)\Pi(A_\alpha(x)), \quad n \to \infty. \tag{12a}
\]
Furthermore, the sequence of the truncated by \( v > q \) mean of \( u_n^{-1}(X_k) \) is formally convergent, i.e., for \( A_v = \{ y \in E: y < v \} \)

\[
k_n(t)E\{u_n^{-1}(X_k)I(u_n^{-1}(X_k) < v)\} = k_n(t)\int_{A_v} x dP(u_n^{-1}(X_k) < x)
\]

\[
= -\int_{A_v} x d\{k_n(t)(1 - P(u_n^{-1}(X_k) < x))\} \xrightarrow{w} \tau(t) \int_{A_{q,v}} x d\Pi(h_\alpha(x)), \quad A_{q,v} := A_v \cap \{ x > q \}.
\] (12b)

Consequently,

\[
a(t, v) = \tau(t) \int_{A_{q,v}} x d\Pi(h_\alpha(x)).
\] (13)

Here we have used that \( P(u_n^{-1}(X_k) < x) \to 0 \) for \( x < q \) and \( k_n(t) \sim \tau(t)k_n \). Observe that \( a(1, v) := a(v) \) is zero if \( v = q \).

Caution: With abuse of notation we denote the Lévy measure and its d.f. by the same letter. So, \( (\gamma, \Pi) \) well defined, i.e., \( a(v) < \infty \).

Note that \( (\Pi(A_\alpha(x)) = -\log G(x) \). The d.f. \( \Psi(x) \) of the measure \( \Psi = \Pi \circ h_\alpha \) is defined by \( \Psi(A_\alpha(x)) = \Psi(\infty) - \Psi(x) \), i.e., \( \Psi(\infty) = \log G(x) \). Thus, it possesses the following properties:

(1) it is nondecreasing in each component;

(2) \( \Psi(\infty) = 0 \);

if, additionally,

(3) \( \int_{A_{q,v}} \|x\| d\Psi(x) \) is finite,

then \( \Psi(A) \) is the Lévy measure of an infinitely divisible random vector whose characteristic function has the form \( (9) \). Thus, questions (a) and (b) are positively answered if

\[
I_G = \int_{A_{q,v}} \|x\| d(\log G(x)) < \infty.
\] (14)

Obviously, no max-stable d.f. \( G(x) \) satisfies condition (14). (Recall in \( \mathbb{R}^1 \) each continuous and strictly increasing d.f. is max-stable with respect to a certain one-parameter norming group (cf. [7]).)

**Example 3.** \( G(x) = e^{-x^{-\alpha}} \) for \( x > 0 \) and \( \alpha > 0 \) is a univariate max-stable d.f. with respect to the group \( \mathcal{L} = \{ L_\alpha(x) = x^\alpha: t > 0 \} \), since \( G^1(x) = G(L_\alpha^{-1}(x)) \), \( \forall t > 0 \).

(a) Let \( 0 < \alpha < 1 \). In this case, \( I_G = a(v) = (v^\alpha - 1)/(1 - \alpha) < \infty \).

(b) Let \( \alpha \geq 1 \). Here \( I_G \) is infinite, so the corresponding measure \( \Psi(A) \) is not a Lévy measure of a distribution of the kind \( (9) \).

Now let us come back to conditions (12) with Lévy measure \( \Psi := \Pi \circ h_\alpha \) and finite \( a(v) \). They are equivalent to the convergence \( S_n(t) \xrightarrow{d} S(t) \), \( t > 0 \), \( n \to \infty \). The limit process \( \{ S(t): t > 0 \} \) has nonnegative and independent increments and is nonhomogeneous in the general case. The characteristic function of \( S(t) \) is expressed by

\[
Ee^{i\theta S(t)} = \exp \left\{ \tau(t) \int_{[q, \infty]} \left( e^{i\theta x} - 1 - i\theta \pi \right) d\Pi(h_\alpha(x)) \right\} = \exp \left\{ \tau(t) \int_{E} \left( e^{i\theta h_\alpha^{-1}(x)} - 1 \right) d\Pi(x) \right\}.
\] (15)

Note the shift parameter here is zero, because the limit of the truncated means in \( C_2 \) is just \( \tau(t) \int_{A_{q,v}} x d\Pi(h_\alpha(x)) \). Recall, in the general case, that the shift parameter \( \gamma(t) \) is \( a(t, v) - \int_{A_{q,v}} x d\pi(t) \) and does not depend on \( v \).

From (15) one can see that the Lévy measure \( \Psi(t) \) of \( S(t) \) admits the factorization \( d\Psi_t(h_\alpha^{-1}(x)) = \tau(t) d\Pi(x) \).

In this way, we complete the proof of our main theorem, formulated in Sec. 1.

**Example 4.** Consider the point process \( N = \{ (t_k, X_k): k \in \{1, 2, \ldots \} \} \), where \( t_k = k(k + 1)/2 \) and \( X_k \) are i.i.d. r.v.’s with d.f.

\[
F(x) = e^{-(\log x)^{-\alpha}}, \quad x \in [1, \infty), \quad 0 < \alpha < 1.
\]
The distribution function $F$ is max-stable with respect to the norming group $\{L_t(x) = x^{1/\sqrt{7}}, t > 0\}$. Indeed,

$$F^t(x) = \exp\{-t(\log x)^{-\alpha}\} = \exp\{-t(\log x^{1/\sqrt{7}})^{-\alpha}\} = F(x^{1/\sqrt{7}}).$$

The sequence of the following time-space changes

$$\zeta_n(t, x) = (\tau_n(t), u_n(x)) = (n^2 t, (x + 1)^{1/\sqrt{7}})$$

satisfies the conditions of the Theorem 1, namely,

(i) it is regular:

$$u_{[ns]}^{-1} \circ u_n(x) \longrightarrow (x + 1)^{1/\sqrt{7}} - 1 =: L_s(x), \quad \forall s > 0, \quad n \rightarrow \infty$$

$$\tau_{[sn]}^{-1} \circ \tau_n(t) \longrightarrow ts^{-2} = \tau_s(t), \quad \forall s > 0, \quad n \rightarrow \infty;$$

(ii) the random variables $X_{nk} = u_n^{-1}(X_k) = X_k^{1/\sqrt{7}} - 1$ are asymptotically negligible, i.e.,

$$P(X_k^{1/\sqrt{7}} - 1 > x) = 1 - \sqrt{F(x + 1)} \longrightarrow 0, \quad \forall x > 0, \quad n \rightarrow \infty;$$

(iii) the sequence of extremal processes $Y_n(t) = \sqrt{k_n(t)} u_n^{-1}(X_k)$ is weakly convergent for $n \rightarrow \infty$.

Indeed, since

$$k_n(t) = \sum_k I \left\{ \frac{k(k + 1)}{2n^2} \in [0, t] \right\} \sim n\sqrt{t} \quad \text{for} \quad n \rightarrow \infty,$$

we have

$$P(Y_n(t) < x) \sim F^{\sqrt{7}}(u_n(x)) = F^{\sqrt{7}}(x + 1) = g(t, x),$$

$F^n(u_n(x)) = F(x)$ since $F \in MS$ with respect to $\{u_n\}$. Put $G(x) := g(1, x)$. Then $P(Y_n(t) < x) \rightarrow G^{\sqrt{7}}(x), n \rightarrow \infty$, and the limit d.f. $F$ is MS with respect to the one-parameter group $\{L_t: t > 0\}$ defined in (i), i.e., $G^t(x) = G(L_t^{-1}(x)), \forall t > 0, \forall x > 0$.

Furthermore, $G(x)$ satisfies (14), since $\Psi(x) = \log G(x) = -(\log(x + 1))^{-\alpha}$ and

$$\int_0^v x \, d(\log G(x)) = \int_0^v x \, (\log(x + 1))^{-\alpha} \, dx < \infty.$$

One can observe that the process $Y^*(t)$ has d.f. $\Phi^t$, where

$$Y^*(t) := h_0 \circ Y \circ \tau^{-1}(t) = \log(Y(t^2) + 1)$$

with $\tau(t) = \sqrt{t}$ and $h_0(x) = \log(x + 1), \forall x > 0$. Indeed,

$$P(Y^*(t) < x) = P(Y(t^2) < h_0^{-1}(x)) = F^t(e^x) = e^{-x^{-\alpha}}.$$
REFERENCES


University of Shumen,
Shumen,
Bulgaria
e-mail: pavlina.kj@abv.bg

Institute of Mathematics and Informatics,
Sofia,
Bulgaria
e-mail: pancheva@math.bas.bg