SOLUTIONS TO EXERCISES

Chapter 1

Ex 1.2.1 $Re(e^{at}) = e^{\mu t}\cos(\omega t), Im(e^{at}) = e^{\mu t}\sin(\omega t)$

Ex 1.2.2 a) $\omega = 0, \mu < 0$. b) $\omega = 0, \mu > 0$. c) $\mu = 0, \omega \neq 0$. d) $\mu > 0, \omega \neq 0$. e) $\mu < 0, \omega \neq 0$.

Ex 1.2.3 $\mu = Re(a) \leq 0$

Ex 1.2.4 $u = u(\xi(x,t), \eta(x,t))$, $u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$,
$u_t = u_\xi \xi_t + u_\eta \eta_t = a u_\xi - a u_\eta$
$u_t + a u_x = 2a u_\xi = 0, \rightarrow u = F(\eta) = F(x - at)$

Ex 1.2.5 Use the result from Ex 1.2.4.
At $t = 0, x = x_0$ we have $u(x_0, 0) = u_0(x_0) = u_0(x - at) = constant$.

Chapter 2

Ex 2.1.1 a) $u(t) = Ce^{-t^2/2}$. b) $u(t) = e^{-t^2/2}$.

Ex 2.1.2 a) $u(t) = 1/(t + C)$. b) $u(t) = 1/(t + 1)$. c) $u(t) = 0$.

Ex 2.1.3 a) Characteristic equation: $\lambda^2 + 2\lambda - 3 = 0, \rightarrow \lambda_1 = -3, \lambda_2 = 1$
$u(t) = C_1 e^{-3t} + C_2 e^t$
b) $u(0) = C_1 + C_2 = 0, u'(0) = -3C_1 + C_2 = -1 \rightarrow u(t) = (e^{-3t} - e^t)/4$
c) $u(0) = C_1 + C_2 = 0, u(1) = C_1 e^{-3} + C_2 e = 1 \rightarrow u(t) = (e^t - e^{-3t})/(e - e^{-3})$
d) $u(\infty) = 0$ if $C_2 = 0$, $u(0) = C_1 = 1, \rightarrow u(t) = e^{-3t}$

Ex 2.1.4 Let $v = \dot{y}$ and we get the 2 ODEs
$\dot{y} = v,\ y(0) = y_0, \quad \dot{v} = -g - (c/m)v^2,\ v(0) = v_0$
If we write the system on vector form, let $u_1 = y$, $u_2 = v$
$\dot{u}_1 = u_2,\ u_1(0) = y_0, \quad \dot{u}_2 = -g - (c/m)u_2^2,\ u_2(0) = v_0$

Ex 2.1.5 We write the system on vector form: let $u_1 = x_1, u_2 = \dot{x}_1, u_3 = x_2, u_4 = \dot{x}_2$.
We get the system
$\dot{u}_1 = u_2, u_1(0) = 0$
$\dot{u}_2 = (-(d_{\nu_1} + d_{\nu_2})u_2 + d_{\nu_2}u_4 - (\kappa_1 + \kappa_2)u_1 + \kappa_2 u_3)/m_1, u_2(0) = 0$
$\dot{u}_3 = u_4, u_3(0) = 0$
$\dot{u}_4 = (d_{\nu_2}u_2 - d_{\nu_2}u_4 + \kappa_2 u_1 - \kappa_2 u_3 + \hat{F}_2 \sin(\omega t))/m_2, u_4(0) = 0$

Ex 2.1.6

$$\frac{1}{r}\frac{d}{dr}(r\frac{dc}{dr}) = \frac{d^2u}{dr^2} + \frac{1}{r}\frac{dc}{dr}$$

$$lim_{r\rightarrow 0}\frac{1}{r}\frac{dc}{dr} = \frac{0}{0} = lim_{r\rightarrow 0}\frac{c''}{1} = c''(0)$$

Hence, at $r = 0$, the ODE is $2c'' = kc^2$.

Ex 2.1.7 Differentiate the the algebraic system with respect to t

$$0 = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\frac{d\mathbf{y}}{dt}$$

$$\frac{d\mathbf{y}}{dt} = -(\frac{\partial \mathbf{g}}{\partial \mathbf{y}})^{-1}\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{y})$$

The initial values are $\mathbf{x}(0) = \mathbf{x}_0$ and a consistent $\mathbf{y}(0)$ value is obtained by solving the algebraic system $\mathbf{g}(\mathbf{x}_0, \mathbf{y}(0)) = 0$.

Ex 2.2.1 Characteristic equation $(-2-\lambda)(3-\lambda)+4 = 0, \rightarrow \lambda_1 = 2, \lambda_2 = -1$
Insert λ_1 into the system $(A - \lambda_1 I)\mathbf{x}_1 = 0 \rightarrow \mathbf{x}_1 = (1, 4)^T$ (times an arbitrary constant $\alpha_1 \neq 0$).
When λ_2 is inserted we obtain the second eigenvector $\mathbf{x}_2 = (1, 1)^T$ (times an arbitary constant $\alpha_2 \neq 0$).

Ex 2.2.2 According to Ex 2.2.1

$$S = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \rightarrow S^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}$$

$$e^{At} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} e^{2t} - 4e^{-t} & -e^{2t} + e^{-t} \\ 4e^{2t} - 4e^{-t} & -4e^{2t} + e^{-t} \end{pmatrix}$$

Ex 2.2.3 $\mathbf{u}(t) = e^{At}\mathbf{u}_0 = (e^{-t}, e^{-t})^T$

Ex 2.2.4 Characteristic equation $(-\lambda)(2-\lambda)+5 = 0 \rightarrow \lambda_1 = 1+2i, \lambda_2 = 1-2i$
First eigenvector $\mathbf{x}_1$ from $(A - \lambda_1 I)\mathbf{x}_1 = 0$

$$\begin{pmatrix} -1-2i & 1 \\ -5 & 1-2i \end{pmatrix}\mathbf{x}_1 = 0 \rightarrow \mathbf{x}_1 = (1, 1+2i)^T$$

Second eigenvector $\mathbf{x}_2$

$$\begin{pmatrix} -1+2i & 1 \\ -5 & 1+2i \end{pmatrix}\mathbf{x}_2 = 0 \rightarrow \mathbf{x}_2 = (1, 1-2i)^T$$

$$S = \begin{pmatrix} 1 & 1 \\ 1+2i & 1-2i \end{pmatrix}, \quad S^{-1} = -\frac{1}{4i}\begin{pmatrix} 1-2i & -1 \\ -1-2i & 1 \end{pmatrix}$$

Ex 2.2.5

$$\mathbf{u}(t) = e^{At}\mathbf{u}_0 = -\frac{1}{4i}\begin{pmatrix} 1 & 1 \\ 1+2i & 1-2i \end{pmatrix}\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix}\begin{pmatrix} 1-2i & -1 \\ -1-2i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1+2i & 1-2i \end{pmatrix}\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} e^{(1+2i)t} + e^{(1-2i)t} \\ (1+2i)e^{(1-2i)t} + (1-2i)e^{(1-2i)t} \end{pmatrix} = \begin{pmatrix} e^t \cos(2t) \\ e^t \cos(2t) - 2e^t \sin(2t) \end{pmatrix}$$

Observe that although both eigenvalues and eigenvectors are complex, the imaginary parts cancel so that the finally simplified expressions of the solution are real.

Ex 2.2.6 Since the matrix is triangular, we have the eigenvalues in the diagonal. Hence we have a triple zero eigenvalue and we have to investigate for this particular matrix how many linearly independent eigenvectors we have.

$$(B - \lambda I)\mathbf{c} = 0 \rightarrow \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hence there is only one eigenvector and the matrix is defect.

Ex 2.2.7 Here we cannot use the formula (2.23) since there is no matrix S^{-1}. Instead use (2.24)

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^3 = 0 \rightarrow e^{Bt} = I + tB + \frac{t^2}{2}B^2$$

Ex 2.2.8 Define the variable $\mathbf{v}(t)$ from $u(t) = e^{At}\mathbf{v}(t)$ From this transformation we have $\mathbf{u}(0) = \mathbf{v}(0)$ and $\mathbf{v}(t) = e^{-At}\mathbf{u}(t)$. - Differentiate $\rightarrow$ $\dot{\mathbf{u}}(t) = \dot{\mathbf{v}}(t)e^{At} + Ae^{At}\mathbf{v}(t)$. Insert into the differential equation (2.12) gives $\dot{\mathbf{v}}(t) = e^{-At}\mathbf{g}(t)$. Integrate

$$\mathbf{v}(t) - \mathbf{v}(0) = \int_0^t e^{-A\tau}\mathbf{g}(\tau)d\tau \rightarrow \mathbf{u}(t) = e^{At}\mathbf{u}(0) + \int_0^t e^{A(t-\tau)}\mathbf{g}(\tau)d\tau$$

Ex 2.2.9 The formulas (2.14) and (2.25) turn into

$$\mathbf{u}(t) = e^{A(t-t_0)}\mathbf{u}(t_0), \quad \mathbf{u}(t) = e^{A(t-t_0)}\mathbf{u}(t_0) + \int_{t_0}^t e^{A(t-t_0)}\mathbf{g}(\tau)d\tau$$

Ex 2.2.10 In the ODE the right hand side is linear in $\mathbf{u}$. The solution on the other hand is linear with respect to the initial vector. These statements are valid also if the matrix A depends on t, i.e. $A = A(t)$.

Chapter 3

Ex 3.3.1 Use Taylor's expansion formula $u(t_{k+1}) = u(t_k + h) = u(t_k) + hu'(t_k) + O(h^2) \rightarrow (u(t_{k+1}) - u(t_k))/h = u'(t_k) + O(h)$. Similar for the Euler backward formula

Ex 3.3.2 Use Taylor's expansion formula for $u(t_k + h)$ and $u(t_k - h)$.

Ex 3.3.3 Eulers explicit formula gives $u_k = u_{k-1} + h\lambda u_{k-1} \rightarrow u_k = (1 + h\lambda)u_{k-1} = (1 + h\lambda)^2 u_{k-2} = \ldots = (1 + h\lambda)^k u_0$

Ex 3.3.4

Ex 3.3.5 According to the error formula we have

$$u(t_k) - u_k^* = c_1 h + c_2 h^2 + ...(1)$$

$$u(t_k) - u_k^{**} = c_1 \frac{h}{2} + c_2(\frac{h}{2})^2 + ...(2)$$

Form the expression 2·(2)-(1)

$$u(t_k) = 2u_k^{**} - u_k^* - c_2\frac{h}{2}^2$$

Ex 3.3.6 According to (3.13), $u_k = 2u_{k-1} + h(f(t_{k-1} + f(t_{k-1}, u_{k-1}) + f(t_{k-\frac{1}{2}}, u_{k-\frac{1}{2}}) - (u_{k-1} + hf(t_{k-1}, u_{k-1}) = u_{k-1} + hf(t_{k-1} + h/2, u_{k-1} + hk_1/2)$, where $k_1 = f(t_{k-1}, u_{k-1})$.

Ex 3.3.7 An accurate value of $y(1) = 0.497615434$. A Matlabprogram generating a table for Heun's method similar to Table 3.1 is:

```
%Computation of the order of the Heun method
 y1res=[];
 N=8;
 for k=1:7
    N=2*N;
    h=1/N;
    y=[1 0]';t=0;
    for i=1:N
       ym=y+h*vdpolf(t,y);
       y=y+(h/2)*(vdpolf(t,y)+vdpolf(t+h,ym));
       t=t+h;
    end
    err(k)=y(1)-0.497615434;
    y1res=[y1res;[N,h,y(1),err(k)]];
 end
 y1res
 kvot=err(1:6)./err(2:7) %Result: 3.48 3.75 3.88  3.94  3.99  4.06
```

An accuracy of 4 decimals is achieved with $N = 16$ steps.
We also see that when the stepsize is halved the error is approximately decreased by a factor 4, which means that $error = O(h^2)$.

Ex 3.3.8 For Euler's explicit method the local error is given by (3.12). Taylor's expansion theorem gives:

$$l(t_k, h) = \frac{u(t_{k-1}) + hu'(t_{k-1}) + O(h^2) - u(t_{k-1})}{h} - f(t_{k-1}, u(t_{k-1})) = O(h),$$

since $u(t)$ satisfies the differential equation $u'(t) = f(t, u)$.

Ex 3.3.9 The ODE is $\ddot{u} + 0.4\dot{u} + 4.5u = 0, \rightarrow \lambda_1 = -0.2 + i\sqrt{4.46}, \lambda_2 = -0.2 + i\sqrt{4.46}$
Numerical instability if $|1 + h\lambda| > 1$, in our case $|0.98 + 0.1i\sqrt{4.46}| = 1.0025 > 1$, i.e. unstable.
Stable if $|1 - 0.2h + ih\sqrt{4.46}| = 1 \rightarrow h = 0.08$.

Ex 3.3.10

$$\frac{d\delta\mathbf{v}}{dt} = \frac{d\delta\mathbf{u}}{dt} \rightarrow \frac{d\delta\mathbf{v}}{dt} = J(\delta\mathbf{v} - J^{-1}\mathbf{c}) + \mathbf{c} = J\delta\mathbf{v}$$

Ex 3.3.11 The stability region for Heun's method, S_{HM} is obtained from

$$|1 + q + \frac{q^2}{2}| \leq 1$$

where $q = h\lambda$. Hence solve $1 + q + q^2/2 = e^{i\varphi}$, where φ goes from 0 to 4π, with Newton's method with respect to q.

```
%Stability region for the Heun method
 q0=0;resq=[0+i*1e-8];
 for fi=0:0.1:4*pi+0.1
     q1=q0;q0=q1+1;
     while abs(q0-q1)>1e-6
        q0=q1;
        f=1+q0+q0^2/2-exp(i*fi);
        fq=1+q0;
        q1=q0-f/fq;
     end
     resq=[resq q1];
 end
 plot(resq)
 title('Stability region for the Heun method')
 xlabel('Re(h\lambda)')
 ylabel('Im(h\lambda)')
 axis('equal')
```

Ex 3.3.12

Ex 3.3.13

Ex 3.4.1 The matrix is

$$\begin{pmatrix} 0 & & & \\ 1 & 1 & & \\ 1/2 & 1/4 & 1/4 & \\ & 1/2 & 1/2 & 0 \\ & 1/6 & 1/6 & 4/6 \end{pmatrix}$$

giving the following k- and u_k-values

$$k_1 = f(t_{k-1}, u_{k-1}), \quad k_2 = f(t_{k-1}+h, u_{k-1}+hk_1), \quad k_3 = f(t_{k-1}+\frac{h}{2}, u_{k-1}+\frac{h}{4}(k_1+k_2))$$

$$u_k = u_{k-1} + \frac{1}{2}(k_1 + k_2), \quad \hat{u}_k = u_{k-1} + \frac{h}{6}(k_1 + k_2 + 4k_3)$$

Applying this embedded RK-method to the model equation $\dot{u} = \lambda u$ gives

$$u_k = (1 + h\lambda + \frac{h^2\lambda^2}{2})u_{k-1}, \quad \hat{u}_k = (1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6})u_{k-1}$$

Since $u(t_k) = e^{h\lambda}u(t_{k-1})$ the local error in u_k is $O(h^3)$. Since the global error is one less than the local error u_k has second order accuracy. This is not really a proof, but a heuristic argument based on conclusions from the model equation.

Ex 3.4.2 Apply the RK-method to the model problem $\dot{u} = \lambda u$.

$$k_1 = \lambda u_{k-1}$$

$$k_2 = \lambda(u_{k-1} + (h/2)\lambda u_{k-1})$$

$$k_3 = \lambda(u_{k-1} + (h/2)\lambda(u_{k-1} + (h/2)\lambda u_{k-1}))$$

$$k_4 = \lambda(u_{k-1} + h\lambda(u_{k-1} + (h/2)\lambda(u_{k-1} + (h/2)\lambda u_{k-1})))$$

Inserting these expression into (3.35) gives the stability condition (3.37).

Ex 3.4.3 See the solution to Ex 3.3.11. Exchange the to the following three rows:

```
 for fi=0:0.1:8*pi+0.1

        f=1+q0+q0^2/2+q0^3/6+q0^4/24-exp(i*fi);
        fq=1+q0+q0^2/2+q0^3/6;
```

Ex 3.4.4
```
%Solution to Exemple 3.4.4, particle motion
%u(1)=x,u(2)=dx/dt,u(3)=y,u(4)=dy/dt
%the right hand side of the ODE-system in fpar.m
%in the graph y is plotted as a function of x
clear,clf,hold off
global c m g
c=0.01;m=1;g=10;v0=10;y0=2;
for alfa=[20,60]
    u0=[0,v0*cos(pi/180*alfa),y0,v0*sin(pi/180*alfa)]';
    t0=0;
    u=u0;t=t0;h=0.1;
    result=[u0'];time=[t0];
    while u(3)>0
        k1=fparticle(t,u);
        k2=fparticle(t+h/2,u+h*k1/2);
        k3=fparticle(t+h/2,u+h*k2/2);
        k4=fparticle(t+h,u+h*k3);
        u=u+h*(k1+2*k2+2*k3+k4)/6;
        t=t+h;
        result=[result;u'];time=[time;t];
    end
    plot(result(:,1),result(:,3))
    title('Particle motion with air resistance')
    xlabel('x [m]')
    ylabel('y [m]')
    hold on
end
grid
```

where the function fparticle is defined as

```
function rhs=fparticle(t,u)
global c m g
rhs=[u(2);
     -(c/m)*sqrt(u(2)*u(2)+u(4)*u(4))*u(2);
```

```
u(4);
-g-(c/m)*sqrt(u(2)*u(2)+u(4)*u(4))*u(4)];
```

Ex 3.4.5 **Adams-Bashforth** 1st order: $u_k = u_{k-1} + hf_{k-1}$, Euler's explicit method
2nd order: $u_k = u_{k-1} + h(\frac{3}{2}f_{k-1} - \frac{1}{2}f_{k-2})$

Adams-Moulton 1st order: $u_k = u_{k-1} + hf_k$, Euler's implicit method
2nd order: $u_k = u_{k-1} + h(\frac{3}{2}f_k - \frac{1}{2}f_{k-1})$

Gear's method 1st order: $u_k = u_{k-1} + hf_k$, Euler's implicit method
2nd order: $\frac{3}{2}u_k = 2u_{k-1} - \frac{1}{2}u_{k-2} + hf_k$

Ex 3.4.6 Insert the model equation into BDF-2:

$$u_k - u_{k-1} + \frac{1}{2}(u_k - 2u_{k-1} + u_{k-2}) = hf_k, \rightarrow (\frac{3}{2} - h\lambda)u_k - 2u_{k-1} + \frac{1}{2}u_{k-2} = 0$$

The characteristic equation has the general solution $u_k = A\mu_1^k + B\mu_2 k, \quad (*)$ where

$$\mu_1 = \frac{2 + \sqrt{1 + 2h\lambda}}{3 - 2h\lambda}, \quad \mu_2 = \frac{2 - \sqrt{1 + 2h\lambda}}{3 - 2h\lambda}$$

The difference equation $(*)$ is stable if $|\mu_1| \leq 1$ and $|\mu_2| \leq 1$. At $h\lambda = 0$, $\mu_1 = 1$ and $\mu_2 = 1/3$, hence solve the equation $\mu_1(h\lambda) = e^{i\varphi}$ with respect to $h\lambda$ for $\varphi = 0.0, 0.2, \ldots$ and check that $|\mu_2(h\lambda)| \leq 1$.
Compare with the solution to 3.3.11. Exchange the following lines

```
for fi=0:0.1:2*pi+0.1;

        f=((3-2*q0)*exp(i*fi)-2)*((3-2*q0)*exp(i*fi)-2)-1-2*q0;
        fp=2*(-2*exp(i*fi))*((3-2*q0)*exp(i*fi)-2)-2;

end
```

Ex 3.4.7 Use the same parameter values as in Example 3.3.

```
%Solution to the ESCEP-problem
global k1 k2 k3 k4
k1=10;k2=0.1;k3=1;k4=10;
E0=0.1;S0=1;
u0=[E0,S0,0,0]';
t0=0;
result=[u0'];time=[t0];h=0.01;
u=u0;t=t0;korr=1;
while norm(korr)>1e-5 %one step with implicit Euler
    F=u-h*fenzym(t,u)-u0;
    Fprime=eye(4,4)-h*jacenzym(t,u);
    korr=-Fprime\F;
    u=u+korr;
end
```

```
t=t+h;
result=[result;u'];time=[time;t];
for k=2:1000
    u0=result(k-1,:)';u1=result(k,:)';
    korr=1;%u2=u1;
    while norm(korr)>1e-5
        F=1.5*u-h*fenzym(t,u)-2*u1+0.5*u0;
        Fprime=1.5*eye(4,4)-h*jacenzym(t,u);
        korr=-Fprime\F;
        u=u+korr;
    end
    t=t+h;
    result=[result;u'];time=[time;t];
end
semilogx(time,result)
```

where the right hand side function `fenzym} of the ODE-system is`

```
function rhs=fenzym(t,u);
global k1 k2 k3 k4
r1=k1*u(1)*u(2);
r2=k2*u(3);
r3=k3*u(3);
r4=k4*u(1)*u(4);
rhs=[-r1+r2+r3-r4;
     -r1+r2;
     r1-r2-r3+r4;
     r3-r4];
```

and the jacobian of the ODE-system `jacenzym} is`

```
function rhs=jacenzym(t,u);
global k1 k2 k3 k4
rhs=[-k1*u(2)-k4*u(4) k1*u(1) k2+k3 k4*u(1);
     -k1*u(2) -k1*u(1) k2 0;
      k1*u(2)-k4*u(4) -k1*u(1) -k2-k3 -k4*u(1);
      -k4*u(4) 0 k3 -k4*u(1)];
```

Chapter 4

Ex 4.2.1 Use the grid **G**. With the technique described in appendix A.3 the following difference formula can be derived:

$$\frac{d^2u}{dx^2}(x_i) = \frac{2}{h_{i-1}(h_i+h_{i-1})}u_{i-1} - \frac{2}{h_{i-1}h_i}u_i + \frac{2}{h_i(h_i+h_{i-1})}u_{i+1} + hot$$

Insert into the model problem $-u'' = f(x), \quad u(0)=0, u(1)=1$ and we obtain a linear system of N equations $A\mathbf{u} = \mathbf{b}$ where

$$A = tridiag(\frac{2}{h_{i-1}(h_i+h_{i-1})}, -\frac{2}{h_{i-1}h_i}, \frac{2}{h_i(h_i+h_{i-1})}), \quad \mathbf{b} = (f(x_1), ...f(x_N))^T$$

Ex 4.2.2 It is appropriate to construct a model problem satisfying the BVP given. The ansatz $u(x) = acos(x)$ satifies the BC $u'(0) = 0$. By choosing $a = 1/(sin(1) + cos(1))$ the right BC is satisfied. This implies that $u(x) = cos(x)/(sin(1) + cos(1))$. Finally, since $u''(x) = -acos(x)$, we obtain $f(x) = cos(x)/(sin(1) + cos(1))$. A Matlabprogram computing the error in the midpoint $x = 0.5$ of the interval could be:

```
%Solution of the boundary value problem (4.32)
a=0;b=1;
miderror=[];
for N=[9,17,33,65,129]
   h=(b-a)/(N-1);
   x=[a:h:b]';
   A=zeros(N,N);
   for i=1:N-1
      A(i,i)=2;
      A(i,i+1)=-1;
      A(i+1,i)=-1;
   end
   A(1,1)=1;A(N,N)=1-h;
   f=h*h*cos(x)/(sin(1)+cos(1));
   f(1)=0;f(N)=-h;
   u=A\f;
   uexakt=cos(x)/(sin(1)+cos(1));
   miderror=[miderror;[h u((N-1)/2)-uexakt((N-1)/2)]];
end
miderror
%answer:  h     error
%         1/8   0.0426
%         1/16  0.0224
%         1/32  0.0115
%         1/64  0.0058
%         1/128 0.0029
```

The table produced by the program shows that the accuracy is of first order.

4.2.3

4.2.4

$$u(x_i+h)-2u(x_i)+u(x_i-h) = u(x_i)+hu'(x_i)+\frac{h^2}{2}u''(x_i)+\frac{h^3}{6}u'''(x_i)+O(h^4)-2u(x_i)+$$

$$u(x_i) - hu'(x_i)) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(x_i) + O(h^4) = h^2u''(x_i) + O(h^4)$$

4.2.5 Use the mean value of the values at left and right side of $x = 0$.

$$\frac{u(x_0) + u(x_1)}{2} = \frac{u(0 - h/2) + u(0 + h/2)}{2} = u(0) + O(h^2)$$

4.2.6

4.3.3 Let $v(x)$ be the straight line through the points $(0, \alpha)$ and $(1, \beta)$. Define $w(x) = u(x) - v(x)$. Then $w(0) = 0$ and $w(1) = 0$. Also $w''(x) = u''(x)$, since $v''(x) = 0$. Hence $w(x)$ satisfies the homogeneous BVP $-w''(x) = f(x), w(0) = 0, W(1) = 0$ just as the model problem. Hence the ansatz is (4.70) with basis functions satisfying (4.71) leading to the linear system of equations (4.77) giving $\tilde{w}(x) = \sum_{j=1}^{N} c_j \varphi_j(x)$. The ansatz solution is $\tilde{u}(x) = \tilde{w}(x) + v(x)$.
Another method is to start with the ansatz

$$\tilde{u}(x) = \alpha\varphi_0 + \sum_{j=1}^{N} c_j\varphi_j + \beta\varphi_{N+1}$$

where $\varphi_0(0) = 1, \varphi_j(0) = 0, j = 1, 2, .., N+1$ and $\varphi_{N+1}(1) = 1, \varphi_j(1) = 0, j = 0, 1, .., N$. These BC's are met by e.g. the "roof" functions. For the coefficients c_j we obtain the same linear system of equations as (4.77).

Chapter 5

5.2.1 a) parabolic, b) elliptic, c) hyperbolic, d) parabolic

5.2.2 Let $u = ve^{\alpha x}$. Then $u_x = v_x e^{\alpha x} + \alpha v e^{\alpha x}$ and

$$u_{xx} = v_{xx}e^{\alpha x} + 2\alpha v_x e^{\alpha x} + \alpha^2 v e^{\alpha x}$$

. Insert into the PDE, divide by $e^{\alpha x}$: $v_t + v_x + \alpha v = v_{xx} + 2\alpha v_x + \alpha^2 v$ Let $\alpha = 1/2$ and we get $v_t = v_{xx} + (\alpha^2 - \alpha)v$. The BCs and the IC are transformed according to $v(0, t) = 1$, $v(1, t) = 0$, $v(x, 0) = u_0(x)e^{-\alpha x}$.

5.2.3

$$u = u(x(r, \varphi), y(r, \varphi))$$

$$u_r = u_x x_r + u_y u_r = u_x cos(\varphi) + u_y sin(\varphi), \quad u_\varphi = u_x x_\varphi + u_y y_\varphi = u_x(-rsin(\varphi)) + u_y(rcos(\varphi))$$

$$u_{rr} = u_{xx}x_r^2 + u_x x_{rr} + u_{yy}u_r^2 + u_y x_{rr} = u_{xx}(cos(\varphi))^2 + u_{yy}(sin(\varphi))^2$$

$$u_{\varphi\varphi} = u_{xx}x_\varphi^2 + u_x x_{\varphi\varphi} + u_{yy}u_y^2 + u_y y_{\varphi\varphi} = u_{xx}r^2(sin(\varphi))^2 + u_x(-rcos(\varphi)) + u_{yy}r^2(cos(\varphi))^2 + u_y(-rsin(\varphi))$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi) = u_{xx} + u_{yy}}$$

5.2.4

$$u_x = \frac{1}{2\sqrt{\pi\kappa t}}(-\frac{2x}{4\kappa t})e^{-x^2/4\kappa t}$$

$$u_{xx} = \frac{1}{2\sqrt{\pi\kappa t}}(-\frac{1}{2\kappa t})e^{-x^2/4\kappa t} + \frac{1}{2\sqrt{\pi\kappa t}}(\frac{-x}{2\kappa t})^2 e^{-x^2/4\kappa t}$$

$$u_t = -\frac{1}{4t\sqrt{\pi\kappa t}}e^{-x^2/4\kappa t} + \frac{1}{2\sqrt{\pi\kappa t}t}(\frac{x^2}{4\kappa t^2})e^{-x^2/4\kappa t}$$

From which we see that $u_t = \kappa u_{xx}$.

5.2.5

$$u_{xx} = \frac{1}{2}(u_0''(x-ct) + u_0''(x+ct)), \quad u_{tt} = \frac{1}{2}(c^2u_0''(x-ct) + c^2u_0''(x+ct))$$

$$u(x,0) = u_0(x), u_t(x,0) = 0$$

5.2.6

$$u_{xx} = -i^2\pi^2 sin(i\pi x)sin(j\pi y), u_{yy} = -j^2\pi^2 sin(i\pi x)sin(j\pi y) \rightarrow u_{xx}+u_{yy} = -\pi^2(i^2+j^2)u$$

5.2.7 a) $div(\nabla u) = (u_x)_x + (u_y)_y + u_z)_z = \Delta u$
b) $div(curl\mathbf{u}) = ((u_3)_y - (u_2)_z)_x + ((u_1)_z - (u_3)_x)_y + ((u_2)_x - (u_1)_y)_z = 0$
c) $curl(\nabla u) = (u_{zy} - u_{yz}, u_{xz} - u_{zx}, u_{yx} - u_{xy})^T = 0$
d) $div(\rho\mathbf{u}) = (\rho u_1)_x + ... = \rho_x u_1 + \rho(u_1)_x + ... = \rho div\mathbf{u} + \nabla\rho \cdot \mathbf{u}$
e) Start with the right hand side

5.3.2 The first two equations in 1D

$$\rho_t + (\rho u)x = 0, \quad (\rho u)_t + (\rho u^2)_x + p_x = 0$$

5.3.4

$$c_t + uc_x = Dc_{xx} + kc, \quad \rho C(T_t + uT_x) = \kappa T_{xx} + \Delta H kc$$

5.3.5

$$T_t = \alpha(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + Q(r,\varphi)$$

5.3.6

Chapter 6

6.1.1 The problem is parabolic: $\rho C v T_z = \kappa(T_{zz} + \frac{1}{r}(rT_r)_r)$

6.1.2 We obtain a system of two ODEs:

$$vc_z = Dc_{zz} - Ae^{-E/RT}c, \quad \rho C v T_z = \kappa T_{zz} + \Delta H A e^{[}-E/RTc, \quad c(0) = c_0, T(0) = T_0, c_z(L) = 0, T_c(L) =$$

If T is constant $vc_x = Dc_{xx} - kc, \quad c(0) = c_0, c_x(L) = 0$, where k is the rate constant.

6.3.1