Invariance properties in the root sensitivity of time-delay systems with double imaginary roots

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Abstract

If $i\omega \in i\mathbb{R}$ is an eigenvalue of a time-delay system for the delay τ_0 then $i\omega$ is also an eigenvalue for the delays $\tau_k := \tau_0 + k2\pi/\omega$, for any $k \in \mathbb{Z}$. We investigate the sensitivity, periodicity and invariance properties of the root $i\omega$ for the case that $i\omega$ is a double eigenvalue for some τ_k. It turns out that under natural conditions (the condition that the root exhibits the *completely regular splitting property* if the delay is perturbed), the presence of a double imaginary root $i\omega$ for some delay τ_0 implies that $i\omega$ is a simple root for the other delays τ_k, $k \neq 0$. Moreover, we show how to characterize the root locus around $i\omega$. The entire local root locus picture can be completely determined from the square root splitting of the double root. We separate the general picture into two cases depending on the sign of a single scalar constant; the imaginary part of the first coefficient in the square root expansion of the double eigenvalue.

Key words: Time-delay systems, sensitivity, perturbation analysis, imaginary axis, root locus, double roots, critical delays

1 Introduction

Consider functions $f : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ of the form $f(s, \tau) = p(s, e^{-s\tau})$ where p is a bivariate function $p : \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto p(x, y)$. We will also assume that p is sufficiently smooth and independent of τ, i.e., the dependence of f on τ is only via the exponential $e^{-s\tau}$. The roots of this type of function f are very important in the analysis of stability of time-delay systems, which is the context of this paper. For instance, the characteristic equation of the time-delay system with a single delay and constant coefficients,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau),\ A_0, A_1 \in \mathbb{C}^{n\times n}, \qquad (1)$$

is

$$0 = \det\left(-sI + A_0 + A_1 e^{-\tau s}\right) = p(s, e^{-\tau s}). \qquad (2)$$

See [9,8] for results on the stability of linear time-delay systems. Note that the characteristic equation of many different generalizations of (1) also correspond to the type $f(s, \tau) = p(s, e^{-s\tau})$. In particular, the characteristic equation of neutral time-delay systems and systems with multiple commensurate delays can be written as $f(s, \tau) = p(s, e^{-s\tau})$, as well as the characteristic equation of time-delay systems with multiple delays if perturbations in only one of the delays are considered.

When analyzing the stability of (1), the delays τ for which (2) has a purely imaginary root $s = i\omega \in i\mathbb{R}$ play a crucial role. In this work, these delays will be referred to as critical delays. They are important, since the critical delays and the sensitivity of the imaginary roots, i.e., the derivative with respect to the delay, can be used to produce a complete stability picture by keeping track of the number of roots entering and leaving the right half-plane. This type of reasoning is used in several works in the literature, e.g., [2,10], and many more. This is often combined with a theory crossing direction, as in e.g. [3,14].

It is widely known, and often exploited, that the presence of an imaginary root at $i\omega$ is periodic in the delay parameter with periodicity $\frac{2\pi}{\omega}$.

As a first result we will see that in general the same type of periodicity property does not hold for the presence of a double imaginary root. That is, if $i\omega$ is a double root for some delay τ_0 then for $\tau_k = \tau_0 + \frac{2\pi k}{\omega}$, $k \neq 0$, $i\omega$ is generally not a double root. This somewhat unexpected result motivates our study of properties of double imaginary roots and sequences of critical delays $\{\tau_k\}$ for which

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the time-delay system has a double imaginary root for one of the delays.

Stability conditions based on reasoning with imaginary roots are often some form of elimination of either the exponential $e^{-\tau s}$ or the scalar s in the characteristic equation. The resulting condition is typically expressed in terms of roots of a polynomial (as in e.g., [13,10,11]) or in terms of the eigenvalues of a generalized eigenvalue problem (as in e.g., [7,1,5,6]). See also [8, Section 4.3.2] and [9, Section 4.4] for more references to delay-dependent stability results expressed in terms of imaginary eigenvalues. These standard results on imaginary eigenvalues do not reveal properties of repeated imaginary eigenvalues.

Some results of high-order analysis are given in [4] including results for multiple imaginary roots [4, Theorem 4]. The focus in this paper is on *invariance properties* of the imaginary root for the *other critical delays* when there is a repeated imaginary root. Invariance properties are not treated in [4]. A higher order analysis is adapted for the *direct method* in [12], where several involved cases are discussed. The invariance properties for double roots are however not treated.

Throughout this work we will implicitly assume that $i\omega \neq 0$ since otherwise, zero is a root for any delay, and the periodicity of the critical delays is not defined.

2 Main results

2.1 Root-path derivatives

Suppose that for $\tau = \tau_0$ the characteristic equation (2) has a double root at $i\omega$. The following theorem states $i\omega$ is a simple root for all other delays in the sequence $\mathcal{T} = \{\tau_0 + \frac{2\pi k}{\omega}\}_k$ under the condition that $f_\tau(i\omega, \tau_0) \neq 0$. Since the roots are simple, we can compute its sensitivity. It turns out that the sensitivity is purely imaginary telling us that the root path close to the $i\omega$ is vertical, and at one point, $s(\tau) = i\omega$. The behavior of the root path w.r.t. the imaginary axis can then be determined from a second order analysis.

Theorem 1 *Let $\mathcal{T} = \{\tau_k\}_{k\in\mathbb{Z}} := \{\tau_0 + k\frac{2\pi}{|\omega|}\}_{k\in\mathbb{Z}}$ be a set of delays for which $i\omega \in i\mathbb{R}$ is an eigenvalue. Let $s(\tau)$ be a continuous eigenvalue path defined in a neighborhood of τ_k for some $k \in \mathbb{Z}\backslash\{0\}$, i.e., $s(\tau_k) = i\omega$. Suppose that for the delay $\tau_0 \in \mathcal{T}$, $i\omega$ is a double (not triple) eigenvalue and $f_\tau(i\omega, \tau_0) \neq 0$. Then, $i\omega$ is a simple eigenvalue for the delay τ_k. Moreover,*

$$s'(\tau_k) = -i\frac{\omega|\omega|}{2\pi k} \tag{3}$$

and

$$s''(\tau_k) = 2i\frac{\omega^3}{(2\pi k)^2} + i\frac{\omega^5|\omega|}{(2\pi k)^3}\frac{f_{ss}(i\omega,\tau_0)}{f_\tau(i\omega,\tau_0)}. \tag{4}$$

PROOF. The characteristic equation of a time-delay system can be written as $f(s,\tau) = p(s, e^{-s\tau})$ where $p : (x,y) \mapsto p(x,y)$ is a bivariate function. It follows that

$$f_s(i\omega,\tau_k) = \underbrace{f_s(i\omega,\tau_0)}_{=0} + \frac{f_\tau(i\omega,\tau_k)}{i\omega}(\tau_k - \tau_0)$$

and $s = i\omega$ is simple for $\tau = \tau_k$ and $k \neq 0$. A root path $\tau \mapsto s(\tau)$ satisfies

$$f(s(\tau),\tau) = 0.$$

At $s = s_k$ for $k \neq 0$ we can differentiate this expression twice with respect to τ, yielding

$$\begin{aligned} &f_s(s,\tau)s' + f_\tau(s,\tau) = 0,\\ &[f_{ss}(s,\tau)s' + f_{s\tau}(s,\tau)]\, s' + f_{s\tau}(s,\tau)s' \\ &\qquad\qquad + f_s(s,\tau)s'' + f_{\tau\tau}(s,\tau) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} s'(\tau) &= -\tfrac{f_\tau(s,\tau)}{f_s(s,\tau)},\\ s''(\tau) &= -\tfrac{f_{\tau\tau}(s,\tau) - f_{ss}(s,\tau)s'^2 - 2f_{\tau s}(s,\tau)s'}{f_s(s,\tau)}. \end{aligned} \tag{5}$$

We will now interpret the substitution $(s,\tau) \mapsto (s, e^{-\tau s}) = (x,y)$ as a coordinate transformation. Let $A(s,\tau)$ denote the matrix transforming the derivatives, i.e., $D_f(s,\tau) = A(s,\tau)D_p(s,e^{-\tau s})$ with $D_f := (f, f_s, f_\tau, f_{ss}, f_{s\tau}, f_{\tau\tau})^T$, $D_p := (p, p_x, p_y, p_{xx}, p_{xy}, p_{yy})^T$. The transformation $A(i\omega,\tau_k)$ can be explicitly computed from $f(s,\tau) = p(s,e^{-s\tau})$. Note that if $\tau_k \in \mathcal{T}$ then in the p-coordinate system, $D_p(i\omega, e^{-i\omega\tau_k}) = D_p(i\omega, e^{-i\omega\tau_0})$. Hence,

$$D_f(i\omega,\tau_k) = A(i\omega,\tau_k)A(i\omega,\tau_0)^{-1}D_f(i\omega,\tau_0). \tag{6}$$

The relations (3) and (4) follow from several algebraic manipulations of (6) and insertion into (5). The details of the algebraic manipulations are omitted due to space limitations.

Corollary 2 *Under the same conditions as in Proposition 1, let $s(\tau)$ be the continuous path close to τ_k. Then, the relation $s(\tau) = a(\tau) + ib(\tau)$ implicitly defines a function $a(b)$, where*

$$a(b) = -\frac{\omega|\omega|}{4\pi k}\left(\operatorname{Im}\frac{f_{ss}(i\omega,\tau_0)}{f_\tau(i\omega,\tau_0)}\right)(b-\omega)^2 + \mathcal{O}((b-\omega)^3). \tag{7}$$

PROOF. This results follows from (3)-(4) and the Taylor expansion of the root

$$s(\tau) = i\omega + s'(\tau_k)(\tau-\tau_k) + \frac{1}{2}s''(\tau_k)(\tau-\tau_k)^2 + \mathcal{O}(\tau-\tau_k)^3.$$

Remark 3 *If* $\operatorname{Im}\,(f_{ss}(s_0,\tau_0)/f_\tau(s_0,\tau_0))=0$ *we have a degenerate case. In this case, a second order analysis is inconclusive and does not reveal if roots enter or leave the right half-plane. An analysis using higher order derivatives would be necessary to analyze that situation.*

2.2 The double eigenvalue

In the previous section we saw that if $k\neq 0$ then $i\omega$ is a simple eigenvalue and we found formulae for the first terms in the Taylor expansion. When $k=0$, we have a square root splitting for the double eigenvalue, in the sense that the derivative of the root path at τ_0, i.e., $s'(\tau_0)$, is undefined, but the function $s(\tau)$ can be expanded in a Puiseux series around τ_0 where the first term is a square root. The following result gives a formula for the first coefficient in this expansion, and is a specialization of [4, Theorem 4].

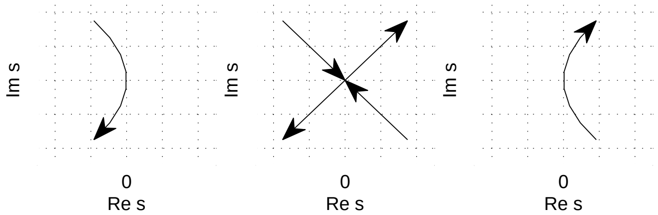
Theorem 4 *Under the same conditions as in Proposition 1, let* $s(\tau)$ *be a path for which* $s(\tau_0)=i\omega$ *is the double eigenvalue. Then,*

$$s(\tau)=i\omega\pm\left(-2\frac{f_\tau(i\omega,\tau_0)}{f_{ss}(i\omega,\tau_0)}(\tau-\tau_0)\right)^{1/2}+o(\sqrt{\tau-\tau_0}).$$

2.3 Combination of results

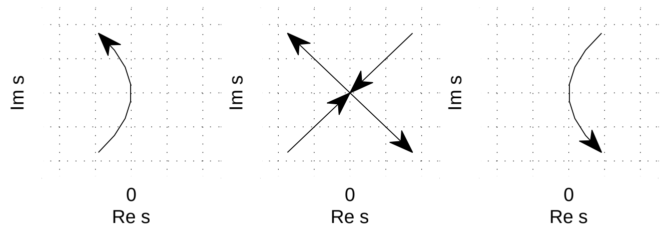
Note that expressions for the coefficients of the expansions in the square root splitting (Theorem 4) and the expression (7) in Corollary 2) both contain the expression $f_{ss}(i\omega,\tau_0)/f_\tau(i\omega,\tau_0)$. Hence, the local root behaviour of all $\tau_k\in\mathcal{T}$ can be determined by the function f (and the derivatives) at $\tau=\tau_0$. This allows us to categorize the local behaviour of the roots into two separate cases. Without loss of generality we assume that $\omega>0$ for this categorization.

1) If $\operatorname{Im} f_{ss}(i\omega,\tau_0)/f_\tau(i\omega,\tau_0)>0$, then for critical delays $\tau_k\in\mathcal{T}$ greater than τ_0 (positive k) the root path touches the imaginary axis from above and in the left half-plane. For delays $\tau_k\in\mathcal{T}$ less than τ_0 (negative k) the imaginary axis is touched from the left and upward.



(a) $k>0$ (b) $k=0$ (c) $k<0$

2) Analogously, if $\operatorname{Im} f_{ss}(i\omega,\tau_0)/f_\tau(i\omega,\tau_0)<0$, the root path for critical delays $\tau_k\in\mathcal{T}$ less than τ_0 touch the imaginary axis in the left half-plane and for delays greater than τ_0 touch the imaginary axis from above and in the right half-plane.



(d) $k<0$ (e) $k=0$ (f) $k>0$

3 Example

Let

$$A_0=\begin{pmatrix}0 & 1\\ -9\pi^2 & 2\end{pmatrix}\text{ and }A_1=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}.$$

For this example $s=i\omega=3\pi i$ is a double (not triple) root for $\tau=\tau_0$ since $f(3\pi i,1)=0$ and $f_s(3\pi i,1)=0$ but $f_{ss}(3\pi i,1)=-2+6\pi i\neq 0$. Moreover, $f_\tau(3\pi i,1)=18\pi^2$. Note that

$$\operatorname{Im}\frac{f_{ss}(3\pi i,1)}{f_\tau(3\pi i,1)}=\operatorname{Im}\frac{-2+6\pi i}{18\pi^2}=\frac{1}{3\pi}>0.$$

The first case in the behavior described in Section 2.3 can be observed in Figure 1. We see that for $k=-1$, i.e., $\tau=1/3$ (which is the only negative k for which the delay is positive) the root path is in the right half plane whereas for $k>0$ all root paths lie in the left half-plane. In Figure 1 we have also plotted the truncated expansions for the roots touching the imaginary axis by using Corollary 2 and Theorem 4.

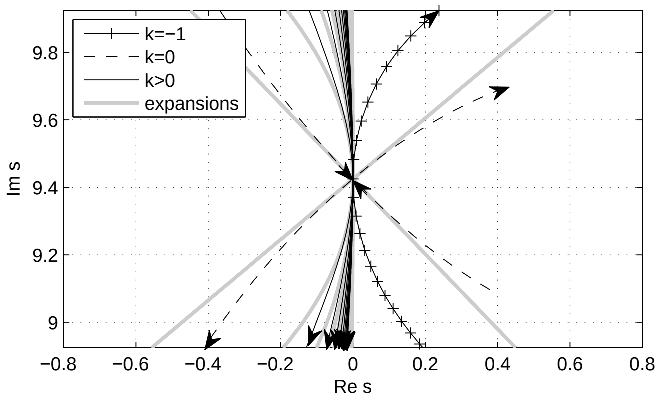


Fig. 1. The root locus close to the imaginary eigenvalue $s=3\pi i$ for $k=-1,0,1,\ldots$ and the expansions corresponding to Theorem 4 and Corollary 2. This corresponds to case 1.

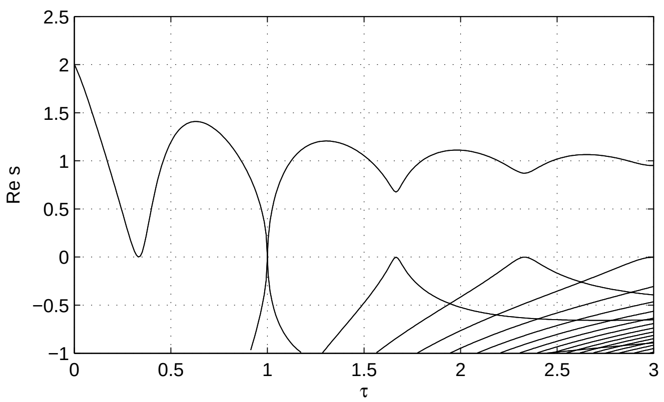


Fig. 2. The real part vs τ. The touching points to the right of the $\tau_0 = 1$ are from below. This corresponds to case 1.

In Figure 2 we see that the parabola corresponding to the critical delay to the left of the double eigenvalue is from above, i.e., the path lies in the right half-plane. Conversely, all the critical delays (touching points) to the right of the double eigenvalue are touching Re $s = 0$ from below, which means that they lie in the left half-plane.

The above system shows the first case. The second situation occurs for the system

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b_3 & -b_2 & -b_1 \end{pmatrix},$$

where

$$a_1 = \frac{2}{5}\frac{(65\pi + 32)}{8 + 5\pi},\ a_2 = \frac{9\pi^2(13 + 5\pi)}{8 + 5\pi},$$
$$a_3 = \frac{324}{5}\frac{\pi^2(5\pi + 4)}{8 + 5\pi},\ b_1 = \frac{260\pi + 128 + 225\pi^2}{10(8 + 5\pi)},$$
$$b_2 = \frac{45\pi^2}{10(8 + 5\pi)} \text{ and } b_3 = \frac{81\pi^2(40\pi + 32 + 25\pi^2)}{10(8 + 5\pi)}.$$

This system is constructed such that $0 = f(3\pi i, 1) = f_s(3\pi i, 1)$ and Im $f_{ss}(3\pi i, 1)/f_\tau(3\pi i, 1) \approx -0.0667 < 0$.

4 Conclusions

It is well known that for a simple imaginary roots, the root tendency, i.e., the sign of the derivative of the root path sign(Re $s'(\tau)$), is independent of k for $\tau_k \in \mathcal{T} = \{\tau_0 + \frac{2\pi k}{\omega}\}_k$. $\tau_k \in \mathcal{T}$ determines the root tendency for all $\tau \in \mathcal{T}$. In this work we have considered the case where the time-delay system has a double imaginary root for $\tau_0 \in \mathcal{T}$, and shown that the multiplicity is not the same for all $\tau \in \mathcal{T}$. Thus, the multiplicity is not invariant with respect to k. However, a consequence of the results in this paper is that an invariance property similar to the case of simple roots still holds: the crossing behavior of all $\tau \in \mathcal{T}$ is *completely* determined from the crossing directions at $\tau = \tau_0$.

More precisely, we demonstrate that the local behavior of the root path $s(\tau)$ around any associated critical delay $\tau \in \mathcal{T}$ can be completely characterized by the sign of Im $f_{ss}(i\omega, \tau_0)/f_\tau(i\omega, \tau_0)$. In the technical derivation we have used that the coefficients in the Taylor expansion of $s(\tau)$ around $\tau = \tau_k \in \mathcal{T}$, can be expressed in terms of the first coefficient of the Puiseux series around $\tau = \tau_0$.

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