

Multiscale methods for the wave equation

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We consider the wave equation in a medium with a rapidly varying speed of propagation. We construct a multiscale scheme based on the heterogeneous multiscale method, which can compute the correct coarse behavior of wave pulses traveling in the medium, at a computational cost essentially independent of the size of the small scale variations. This is verified by theoretical results and numerical examples.

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1 Introduction

We consider wave propagation in heterogeneous media modeled by the scalar wave equation

$$u_{tt} = \nabla \cdot A^\varepsilon(\mathbf{x})\nabla u, \quad \mathbf{x} \in \mathbb{R}^d, t > 0, \quad (1)$$

with initial data $u(0, \mathbf{x}) = f(\mathbf{x})$, $u_t(0, \mathbf{x}) = g(\mathbf{x})$. The coefficient matrix $A^\varepsilon(\mathbf{x}) \in \mathbb{R}^{d \times d}$ is positive definite uniformly in \mathbf{x} and is highly oscillatory with a wave length on the scale $\mathcal{O}(\varepsilon)$. We will be studying the case when g and f are smooth and $\varepsilon \ll 1$. This case is difficult to treat with standard finite difference methods because the ε -scale must be resolved. At least the order of $N \sim \varepsilon^{-(1+d)}$ points is needed to include all the details of the problem in space and time.

The heterogeneous multiscale method (HMM) is a framework for treating this type of computationally challenging problems. The ε -microscale is only resolved locally but the correct macroscale can still be computed. The purpose of this presentation is to analyze the analytically well known case of hyperbolic homogenization in order to increase the understanding of multiscale approximation techniques. For references see the original HMM paper [1] as well as [2], [3] and, for a related framework for multiscale computations, [4].

2 Heterogeneous multiscale method

In HMM one does not attempt to resolve all details of the problem (1). Instead one focus on a macroscale problem: Let \hat{u} be the *coarse* part of u , e.g. a local average, and assume it satisfies a PDE, with an effective flux \hat{F} , of the form:

$$\hat{u}_{tt} = \nabla \cdot \hat{F}(\mathbf{x}, \nabla \hat{u}). \quad (2)$$

In HMM, the effective flux \hat{F} is unknown and determined by solving problems on the micro-scale (defined in more detail below). The inspiration is homogenization theory, where it can be shown that if A^ε is ε -periodic, then $u \rightarrow \hat{u}$ as $\varepsilon \rightarrow 0$. The limit function \hat{u} will satisfy (2) with $\hat{F} = \bar{A}\nabla\hat{u}$, where \bar{A} is a constant matrix.

The HMM algorithm we use here is based on a central finite difference scheme fitted in the framework described in [1]. It is similar to the schemes in [3], for parabolic equations, and in [5] for the one dimensional advection equation.

A more detailed description of the HMM algorithm follows:

1. Discretize (2) with a centered difference scheme on a Cartesian grid. In 2D we have (see Figure 1.1):

$$U_{i,j}^{n+1} = 2U_{i,j}^n - U_{i,j}^{n-1} + \frac{(\Delta T)^2}{\Delta X} [F_{i+1/2,j} - F_{i-1/2,j}] + \frac{(\Delta T)^2}{\Delta Y} [G_{i,j+1/2} - G_{i,j-1/2}],$$

where $F_{i,j}$ and $G_{i,j}$ are the discrete x - and y -components of the effective flux $\hat{F}(\mathbf{x})$ evaluated at $x_{i,j}$.

2. To compute the macro flux $\hat{F}(\mathbf{x})$ on half points as seen in Figure 1.1, solve a *micro problem* parametrised by values $U_{i,j}^n$ around \mathbf{x} . The micro problem consists of solving (1) over $I_\delta = [x - \delta/2, x + \delta/2] \times [y - \delta/2, y + \delta/2]$, $\delta \sim \varepsilon$, with linear initial data $u_0(\mathbf{x}) = \sigma^{(1)}x + \sigma^{(2)}y$, together with periodic boundary conditions for $u^\varepsilon - u_0$. The coefficients $\sigma^{(1)}$ and $\sigma^{(2)}$ is the normal of a plane, approximating the macro solution over I_δ , more precisely: $\sigma^{(1)} = (U_{i+1,j} - U_{i,j})/(\Delta x)$ and $\sigma^{(2)} = (U_{i,j+1} - U_{i,j-1} + U_{i+1,j+1} - U_{i+1,j-1})/(4\Delta y)$ for the flux components $F_{i+1/2,j}$ and $G_{i+1/2,j}$. Other fluxes are computed analogously.

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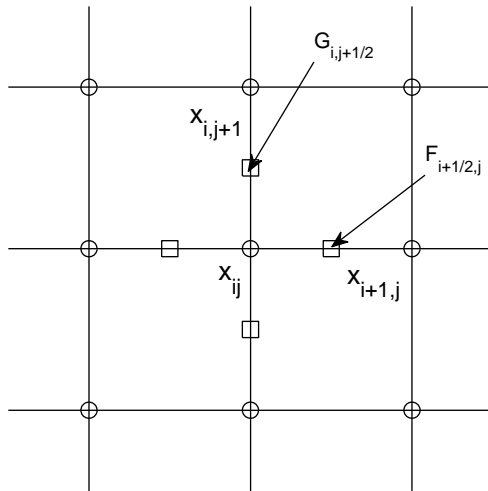


Fig. 1.1: Discretization on the macro level. The grid points appear as circles and the small squares indicate where the fluxes are computed.

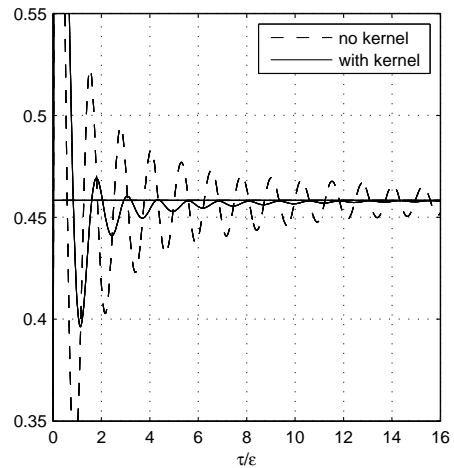


Fig. 1.2: Convergence of micro-scale flux as function of time scale τ/ϵ .

- Evaluate the macro scale flux $\hat{F}(\mathbf{x})$ as a time and space average of the $A^\epsilon \nabla u$ over the box I_δ and in time from 0 to $\tau \sim \epsilon$.

$$\hat{F}(\mathbf{x}) \approx \tilde{F}(\mathbf{x}) := \frac{1}{|I_\delta| \tau} \int_0^\tau \int_{I_\delta(\mathbf{x})} K(\mathbf{x}) A^\epsilon(\mathbf{x}) \nabla u(t, \mathbf{x}) \, dx dt$$

The volume of the box is $|I_\delta| = \delta^2$. Special consideration has to be taken when choosing τ . It should not be too big with respect to δ and A^ϵ . If waves from the boundary contaminate the sampling, chosen inside I_δ , the convergence will be damaged or completely destroyed. We have proved that if $u_0(\mathbf{x})$ is linear and $A^\epsilon(\mathbf{x}) = A(\mathbf{x}/\epsilon)$, then $\tilde{F}(\mathbf{x}) = \bar{A} \nabla u_0 + \mathcal{O}(\frac{\epsilon}{\delta} + \frac{\epsilon}{\tau})$, where \bar{A} is the homogenized A^ϵ operator.

3 Numerical results

In Figure 1.2 we see the average flux \tilde{F} as a function of the upper limit τ/ϵ . We can see that it has an oscillating behavior. To improve the convergence speed, we use an integrating kernel $K(x) = \frac{1}{3} (1 - x^2)$, with support over $[-1, 1]$, as described in [6]. The solid horizontal line at $y \approx 0.46$ corresponds to the homogenized coefficient.

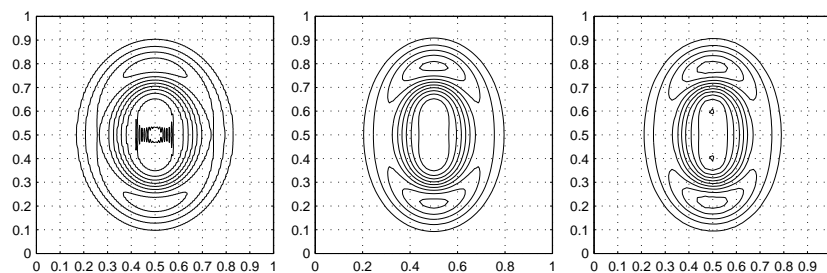


Fig. 1 Comparison between fully resolved solution (left), the solution to a homogenized \bar{A} operator (middle) and HMM solution (right). The coefficient used is $A^\epsilon(\mathbf{x}) = (1.1 + \sin(2\pi \frac{x}{\epsilon})) \mathbf{I}$ with initial data $\sim e^{-100((x-0.5)^2 + (y-0.5)^2)}$, $\epsilon = 10^{-3}$. The figure is a snapshot at $t = 0.25$. On the macro scale we discretize with $\Delta T = 1/128$, $\Delta X = \Delta Y = 1/32$; and on the micro scale with $\Delta t = \epsilon/64$, $\Delta x = \Delta y = \epsilon/16$. Note that the solution from an arithmetic average of A^ϵ over I_δ , i.e. $A_{\text{avg}} = 1.1 \mathbf{I}$, is completely wrong in this case, with equal wave propagation speed in all directions; it misses the anisotropic character of the exact solution.

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