Narrow Proofs May Be Maximally Long

ALBERT ATSERIAS, Universitat Politècnica de Catalunya
MASSIMO LAURIA, Tokyo Institute of Technology
JAKOB NORDSTRÖM, KTH Royal Institute of Technology

We prove that there are 3-CNF formulas over n variables that can be refuted in resolution in width w but require resolution proofs of size $n^{\Omega(w)}$. This shows that the simple counting argument that any formula refutable in width w must have a proof in size $n^{O(w)}$ is essentially tight. Moreover, our lower bound generalizes to polynomial calculus resolution and Sherali-Adams, implying that the corresponding size upper bounds in terms of degree and rank are tight as well. The lower bound does not extend all the way to Lasserre, however, since we show that there the formulas we study have proofs of constant rank and size polynomial in both n and w.

CCS Concepts: • **Theory of computation** → **Proof complexity**; • **Mathematics of computing** → Combinatoric problems;

Additional Key Words and Phrases: Proof complexity, resolution, width, polynomial calculus, polynomial calculus resolution, PCR, Sherali-Adams, SAR, degree

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1. INTRODUCTION

Proof complexity studies how hard it is to prove that propositional logic formulas are tautologies. While the original motivation for this line of research, as discussed in Cook and Reckhow [1979], was to prove superpolynomial lower bounds on proof size for increasingly stronger proof systems as a way towards establishing NP $\neq$ coNP (and hence P $\neq$ NP), it is probably fair to say that most current research in proof complexity is driven by other concerns.

One such concern is the connection to satisfiability solving (SAT solving). By a standard transformation, any propositional logic formula can be converted to another formula in conjunctive normal form (CNF) that has the same size up to constant factors

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Authors' addresses: A. Atserias, Dept. Ciències de la Computació Universitat Politècnica de Catalunya Barcelona, Catalonia, Spain; email: atserias@cs.upc.edu; M. Lauria, Tokyo Institute of Technology, Tokyo, Japan; email: lauria.massimo@gmail.com; J. Nordström, School of Computer Science and Communication, KTH Royal Institute of Technology, Stockholm, Sweden; email: jakobn@kth.se.
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and is unsatisfiable if and only if the original formula is a tautology. Any algorithm for solving SAT defines a proof system in the sense that the execution trace of the algorithm constitutes a polynomial-time verifiable witness of unsatisfiability (such a witness is often referred to as a *refutation* rather than a *proof*, and these two terms are sometimes used interchangeably). In fact, most modern-day SAT solvers can be seen to search for proofs in systems at fairly low levels in the proof complexity hierarchy, and upper and lower bounds for these proof systems hence give information about the potential and limitations of the corresponding SAT solvers. In this work, we focus on such proof systems.

1.1. Background

The dominant strategy in applied SAT solving today is so-called *conflict-driven clause learning (CDCL)* [Bayardo Jr. and Schrag 1997; Marques-Silva and Sakallah 1999; Moskewicz et al. 2001], which is ultimately based on the *resolution* proof system [Blake 1937]. The most studied complexity measure for resolution is *size* (also referred to as *length*), which gives lower bounds on the running time on CDCL solvers and for which (optimal) exponential lower bounds are known [Haken 1985; Urquhart 1987; Chvátal and Szemerédi 1988]. Another more recently studied measure is *space*,[1] which corresponds to memory usage and for which (again optimal) linear lower bounds have been proven [Alekhnovich et al. 2002; Ben-Sasson and Galesi 2003; Esteban and Torán 2001]. For all of these results, the concept of *width*, measured as the size of a largest clause in a resolution proof, has turned out to play a key role. Width was identified as a crucial resource already in Galil [1977], and strong lower bounds on proof width have been shown to imply lower bounds on proof size [Ben-Sasson and Wigderson 2001] and space [Atserias and Dalmau 2008].

Interestingly, although the relationships and tradeoffs between width and space in resolution are by now fairly well understood [Ben-Sasson 2009; Ben-Sasson and Nordström 2008], as are those between size and space [Ben-Sasson and Nordström 2008, 2011; Beame et al. 2012; Beck et al. 2013], very basic questions about the connections between size and width have remained open. Let us give two examples of such questions that we find particularly interesting.

For our first example, consider the main technical result in Ben-Sasson and Wigderson [2001], which when applied to CNF formulas of constant width says that if a formula over n variables has a resolution proof of size S, then it also has one of width $O(\sqrt{n \log S})$. This is established by exhibiting a general transformation that turns any small resolution proof into a (reasonably) narrow one. It was shown in Bonet and Galesi [2001] that the bound above is essentially tight in that polynomial-size resolution proofs cannot in general be squeezed down to width smaller than $\Omega(\sqrt{n})$, but one other aspect of this result has remained unclear. Namely, once the original small resolution proof has been transformed into a narrow proof, this new proof is no longer small, since the transformation in Ben-Sasson and Wigderson [2001] causes an exponential increase in size. It is not known whether such a blow-up is necessary, that is, whether there are tradeoffs between size and width, or whether the analysis in Ben-Sasson and Wigderson [2001] could be sharpened to show that short proofs can be made simultaneously narrow.[2]

[1] In this article, we are interested in the most well-studied measure of space for resolution, which counts the number of clauses in memory and is hence known as *clause space*, and in its natural generalization to (semi)algebraic proof systems where one instead counts monomials. For completeness, we want to mention that there is also a measure called *total space*, counting the total number of literals in memory (with repetitions), which has been studied in Alekhnovich et al. [2002] and Bonacina et al. [2014a, 2014b].

[2] Since this article first appeared, Thapen [2014] has shown that a size blow-up as in Ben-Sasson and Wigderson [2001] is indeed necessary. We discuss this result briefly in Section 1.4 below.

Our second example concerns what can be said in the other direction, that is, what conclusions can be drawn regarding proof size based on an upper bound on proof width. Clearly, an upper bound w on the refutation width for a formula over n variables implies a proof size of at most $n^{O(w)}$ simply by counting the number of possible distinct clauses of width w (as also noted in Ben-Sasson and Wigderson [2001]). But, again, it is not clear how tight this argument is. To the best of our knowledge, it has been open whether there exist formulas refutable in width $w = o(\sqrt{n})$ that require size $n^{\Omega(w)}$, that is, with the width complexity appearing in the exponent.[3]

From a theoretical point of view, the ubiquity of CDCL in SAT solving is somewhat puzzling, since resolution is quite a weak proof system. A different approach is to translate CNF formulas to multilinear polynomials and do Gröbner basis computations, which corresponds to *polynomial calculus resolution (PCR)* as defined in Clegg et al. [1996] and Alekhnovich et al. [2002]. (The resolution "R" in PCR stands for the fact that negated literals get their own formal variables when translating CNF formulas to polynomials. Such variables were missing in the original definition in Clegg et al. [1996] but adding them as in Alekhnovich et al. [2002] makes for a more natural and well-behaved proof system from a proof complexity point of view.) Intriguingly, although PCR is known to be exponentially stronger than resolution, implementations of search methods for this proof system such as PolyBoRi [Brickenstein and Dreyer 2009; Brickenstein et al. 2009] have a hard time competing with CDCL solvers.

Proof size and space in PCR are defined in analogy with resolution, but counting monomials rather than clauses, and the measure corresponding to width of clauses is (total) *degree* of polynomials. It is straightforward to show that PCR can simulate resolution efficiently with respect to all of these measures, meaning that the same worst-case upper bounds as in resolution apply to PCR. It was proven in Impagliazzo et al. [1999] that strong degree lower bounds imply strong size lower bounds, which is an exact analogue of the size–width relation for resolution in Ben-Sasson and Wigderson [2001] discussed above, and this size–degree relation has been employed to prove exponential lower bounds on size in a number of articles, with the most general setting perhaps provided in Alekhnovich and Razborov [2003] and Mikša and Nordström [2015]. The first lower bounds on space in PCR were obtained in Alekhnovich et al. [2002] but only worked for CNF formulas unbounded of width. In Filmus et al. [2015], the techniques in Alekhnovich et al. [2002] were adapted to prove space lower bounds also for formulas of constant width, and optimal (linear) lower bounds on space were finally obtained in Bonacina and Galesi [2015]. It is worth noting, however, that these bounds are *not* derived from degree lower bounds—it remains unknown whether an analogue of Atserias and Dalmau [2008] holds for PCR (although Filmus et al. [2013] has reported some progress on this and related open questions). Strong tradeoffs between size and space as well as between degree and space have been shown in Beck et al. [2013], but—again in analogy with resolution—the exact relations between size and degree remains unclear. The same blow-up as in Ben-Sasson and Wigderson [2001] occurs in Impagliazzo et al. [1999] when small size is converted to small degree, but it is not known whether this is necessary or just an

[3]Paul Beame has pointed out to us that there have been examples in the literature of formulas over n variables refutable in resolution in width $O(\sqrt{n})$ that require size $2^{\Omega(\sqrt{n})}$, namely the tiling formulas studied in Alekhnovich [2004] and Dantchev and Riis [2001]. Another example is provided by so-called Tseitin formulas over $\sqrt{n} \times \sqrt{n}$ grids with two copies of every edge. Such formulas are refutable in width $O(\sqrt{n})$ by simulating Gaussian elimination, but it can be shown by a standard restriction argument that any proof of unsatisfiability again requires size $2^{\Omega(\sqrt{n})}$, However, these examples require width $w = \Omega(\sqrt{n})$, and even in this range do not quite achieve size lower bounds of type $n^{\Omega(w)}$ but only $2^{\Omega(w)}$. Also, the lower bounds are known only for resolution and not for stronger proof systems.

artifact of the proof. Also, it was shown in Clegg et al. [1996] that a degree upper bound of d implies proof size at most $n^{O(d)}$, but it has been open whether this is tight or not.

Yet another way to achieve greater expressivity than in resolution is to translate clauses into linear inequalities and manipulate them using 0-1 linear programming. Perhaps the simplest and most well-known example of this approach is the *cutting planes* proof system introduced in Cook et al. [1987] based on ideas in Chvátal [1973] and Gomory [1963]. In this article, however, we will be interested in somewhat related but different *semialgebraic* methods operating on linear programming relaxations of the CNF translations, such as the *Sherali-Adams*, *Lovász-Schrijver*, and *Lasserre* hierarchies used for attacking NP-hard optimization problems.

The *Sherali-Adams (SA)* method [Sherali and Adams 1990] provides a hierarchy of linear programming relaxations of any given 0-1 integer program. The nth level of the hierarchy, where n is the number of 0-1 integer variables, wipes out the integrality gap and is thus exact but also leads to an exponential blow-up in problem size. The main point of the method, however, is that any linear function of the variables can be optimized over the kth level of the hierarchy in time $n^{O(k)}$, and, in particular, feasibility of the kth level relaxation can be checked in that time. In the context of proof complexity, what this means is that if the kth level relaxation of the integer programming formulation of a CNF formula in infeasible (the minimal such k is known as the *SA rank* of the integer program), then there is an $n^{O(k)}$-time algorithm that can detect this. Furthermore, since the kth level of the hierarchy is an explicitly defined linear program, its infeasibility can be certified as a positive linear combination of its defining inequalities. Such a certificate is a rank-k Sherali-Adams refutation of the corresponding CNF formula.

The *Lovász-Schrijver* approach [Lovász and Schrijver 1991] can be thought of as (and indeed it is formally equivalent to) an iterated version of the level-2 SA relaxation. The point is again that any linear function can be optimized over the linear program after k iterations in time $n^{O(k)}$. Lovász and Schrijver also introduced a method LS^+, which uses semidefinite programming instead of linear programming and which is significantly stronger in some notable cases of interest in combinatorial optimization.

The *Lasserre* method [Lasserre 2001], finally, is basically the Sherali-Adams method with semidefinite programming conditions at all levels of the hierarchy. Again, it stratifies into levels and the kth level can be solved in time $n^{O(k)}$. Moreover, Lasserre's method is the strongest of all three in the sense that, level by level, it provides the tightest of all three approximations of the integer linear program. We refer to Laurent [2001] and Chlamtáč and Tulsiani [2012] for a more detailed discussion of Sherali-Adams, Lovász-Schrijver, and Lasserre and a comparison of their relative strengths.

In view of the important algorithmic applications that these methods have (see, e.g., Parrilo [2000] and subsequent work), it is a natural question whether the upper bounds $n^{O(k)}$ for rank k are tight, just as for resolution and polynomial calculus resolution.

From the proof complexity side, some notable early articles investigating semialgebraic proof systems were published around the turn of the millennium [Pudlák 1999; Grigoriev and Vorobjov 2001; Grigoriev et al. 2002], but then this area of research seems to have gone mostly dormant. In the past few years, these proof systems have made an exciting reemergence in the context of hardness of approximation, revealing unexpected and intriguing connections between approximation and proof complexity. Some examples of this include the article [Schoenebeck 2008] essentially rediscovering results from Grigoriev [2001] and more recent articles such as Barak et al. [2012] and O'Donnell and Zhou [2013]. There have also been articles such as Beame et al. [2007] and Göös and Pitassi [2014] focusing on *semantic* versions of these proof systems, with less attention to the actual syntactic derivation rules used.

1.2. Our Results

The main contribution of this article is showing that the upper bounds on proof size in terms of width for resolution, degree for PCR, and rank for Sherali-Adams are essentially tight (up to constant factors in the exponent). Moreover, an interesting feature of our result is that we can actually use the same formula family to prove tightness simultaneously for all the proof systems. What this means is that we obtain upper bounds on size in resolution that tightly match lower bounds in the much stronger systems PCR and Sherali-Adams (which are, in turn, tight for these systems since resolution width is an upper bound on both PCR degree and Sherali-Adams rank). The formal statement of this result is as follows.

THEOREM 1.1. *Let $w = w(n)$ be such that $w = O(n^c)$ for some positive constant $c < 1/2$. Then there are 3-CNF formulas $F_{n,w}$ with $\mathrm{O}(wn)$ clauses over $\mathrm{O}(n)$ variables such that the following holds:*

(*1*) *$F_{n,w}$ has a resolution refutation in simultaneous size $n^{\mathrm{O}(w)}$, width $\mathrm{O}(w)$, and space $\mathrm{O}(w)$.*
(*2*) *Any refutation of $F_{n,w}$ in resolution, PCR, or Sherali-Adams must have size $n^{\Omega(w)}$.*

For resolution this actually shows something slightly stronger than that the counting upper bound on size in terms of width is tight. Namely, since the formulas in Theorem 1.1 have the same asymptotic upper bound on space as on width, it follows that even for formulas of space complexity $\mathrm{O}(w)$—which is a more stringent requirement than width complexity $\mathrm{O}(w)$—it is still impossible to obtain any size upper bound better than $n^{\mathrm{O}(w)}$ in general. It is also worth emphasizing that the formulas in the theorem are refutable in resolution width w but require polynomial calculus size $n^{\Omega(w)}$, which, in particular, answers in the negative a question raised in Clegg et al. [1996] and Beame and Pitassi [1998] whether a tighter simulation of resolution by polynomial calculus is possible.

Theorem 1.1 has an interesting consequence for the analysis of CDCL solver performance, which we state as a formal corollary. By way of background, it was shown in Atserias et al. [2011] that if a CNF formula F over n variables has a resolution refutation in width w, then with high probability any CDCL solver[4] will only need time $n^{\mathrm{O}(w)}$ to decide that F is indeed unsatisfiable. Perhaps this might not seem so impressive at first sight—after all, exhaustive search in bounded width runs within this time bound deterministically—but the point is that a CDCL solver is very far from doing exhaustive width search and does not care at all about the existence or non-existence of narrow refutations. An obvious question is whether this bound on the running time is tight. Theorem 1.1 shows that the answer is "yes," since no CDCL solver can run faster than the shortest resolution proof it can possibly find.[5]

COROLLARY 1.2. *There are formulas F over n variables refutable in resolution in width w for which any resolution-based CDCL solver cannot run faster than $n^{\Omega(w)}$, and hence the result in Atserias et al. [2011] is optimal up to constants in the exponent.*

[4]This result holds for a fairly general mathematical model of what a CDCL solver is, which agrees reasonably well with how state-of-the-art solvers are actually implemented in practice (though making this claim precise would require a detailed discussion of implementation details of CDCL solvers that is beyond the scope of this article).

[5]This is, of course, assuming that the solver does not implement features such as, for example, cardinality reasoning or extended resolution, since these fall outside of the standard CDCL framework and go beyond resolution-based reasoning.

Another interesting aspect of our lower bound for resolution is in the context of Berkholz's EXPTIME-completeness result for deciding resolution width [Berkholz 2012]. What Berkholz showed is that given a formula F over n variables and a parameter w, it cannot be decided in time less than $n^{(w-3)/12}$ whether F has a resolution refutation in width w or not. Optimizing the constants in Theorem 1.1, we can show that there are 4-CNF formulas refutable in width w for which no resolution refutation can be shorter than $n^{w/2-o(1)}$. It is worth noting that this bound is stronger than that in Berkholz [2012], although it of course applies only for the more restricted setting where the algorithm has to output a width-w resolution refutation rather than for the decision problem. Still, we believe this sheds interesting light on Berkholz's result.

1.3. Discussion of Proof Techniques

We conclude the overview by outlining the proof of the lower bound in Theorem 1.1 for resolution and how it differs from previously used methods. At a high level, our proof is a standard restriction argument, but it turns out to have some twists which we believe might be of interest and could be useful elsewhere. (In fact, in a sense this has already happened in that our article draws on ideas from Atserias et al. [2013], which used a similar approach in a different context.)

Before going into the details of our new restriction argument, let us revisit previous lower bounds on size in terms of width and see how they fall short of proving what we are after. On the one hand, the result in Ben-Sasson and Wigderson [2001] states that if a 3-CNF formula on n variables requires width w to refute in resolution, then it also requires size $2^{\Omega(w^2/n)}$. This lower bound is vacuous for w smaller than $\sqrt{n}$ and, in any case, can never be larger than $2^{\Omega(w)}$ since w is bounded by n. On the other hand, for formulas refutable in width w smaller than $\sqrt{n}$, a direct random restriction argument can sometimes still be applied to get meaningful lower bounds. The idea is that setting a random literal to true will kill off a $\frac{w}{2n}$-fraction of the wide clauses on average. After r rounds of such restrictions, the expected number of surviving wide clauses is at most $(1-\frac{w}{2n})^r S$, where S is the size of the refutation, and choosing $r = (2n/w)\log S$ brings the number of wide clauses down to zero. A contradiction is then derived by showing that the residual formula still requires width w to refute. Note, however, that we cannot apply the restriction for more than n rounds (or else there will be no residual formula to argue about), and so the best size lower bound this method can achieve is again $2^{\Omega(w)}$, which is smaller than the $n^{\Omega(w)}$ bound that we are after.

In some sense, the problem is that using restrictions in the style of Håstad's switching lemma [Håstad 1987] does not work in our setting. Instead, it turns out that a seemingly weaker argument inspired by Furst et al. [1984] is just what we need. Let us now describe this modified restriction argument and how it overcomes the problems discussed above.

We start with a carefully chosen family of formulas $F_{n,w}$ and an associated probability distribution over restrictions ρ_n. Then we assume that we have a resolution refutation π of $F_{n,w}$ in size $n^{o(w)}$ and analyze how a randomly chosen restriction ρ_n affects π. We get two cases:

(1) For clauses C in the refutation π that are noticeably wide, ρ_n is very likely to satisfy a literal in C and so the clause disappears.
(2) Clauses that are not so wide will not be satisfied by ρ_n, but since they are reasonably small they are very likely to be shortened by ρ to width strictly less than w.

Admittedly, the first case looks no different from the standard restriction argument, and the second case seems quite weak. But the point is that by considering also the second case, we can afford a significantly bigger bound for "wide" than before, thus

getting a bigger probability of success. This is the key to our argument. The rest is now standard: $F_{n,w}$ and ρ_n are chosen so $F_{n,w}$ restricted by ρ_n is a bounded-width version of a pigeonhole principle (PHP) formula with w pigeons that are supposed to fit into $w-1$ holes. Since π is short enough, by a counting argument there is some restriction ρ_n that eliminates all wide clauses to give a resolution refutation of the PHP formula in width significantly less than w. A separate argument shows that such a narrow resolution refutation cannot exist, and the lower bound on resolution refutation size follows.

The lower bounds for PCR and Sherali-Adams are quite similar. The restriction part of the argument is basically the same, but one has to work a bit harder to prove the final punchline that the restricted refutations have impossibly low degree and rank, respectively.

It should perhaps be stressed that while the final argument is quite straightforward and natural (at least for resolution), a crucial component in the proof is to find the right formulas $F_{n,w}$ and associated restrictions ρ_n to plug into the argument and to make a case analysis of the action of ρ_n as above. Both of these aspects use the techniques developed in Atserias et al. [2013] in an essential way.

1.4. Subsequent Developments

The two main questions driving the research behind this article are what upper bounds on proof width/degree/rank imply about proof size, and in the other direction whether upper bounds on size and width/degree/rank can be optimized simultaneously.

Regarding the latter question, it was recently shown by Thapen [2014] that the size blow-up in Ben-Sasson and Wigderson [2001] when width is reduced is indeed inherent. That is, there are formulas with resolution refutations of size S for which any refutation in width $\mathrm{O}(\sqrt{n \log S})$ must have exponential size. Indeed, Thapen's result is stronger than this in that it covers a wider range of parameters. It is also robust in the sense that even a smaller decrease in the width, not going all the way down to $\mathrm{O}(\sqrt{n \log S})$, still causes an exponential blow-up. We refer to the article [Thapen 2014] for more details.

It is worth noting, though, that Thapen's result does not work for formulas of constant width but requires clauses of logarithmic size. Also, and more importantly, the proof only works for resolution. The corresponding question for PCR whether the size blow-up in Impagliazzo et al. [1999] is still open.

Regarding the former question, in this article we show that there are formulas which can be refuted in resolution in width w (and hence essentially the same degree in PCR and rank in Sherali-Adams) but which require refutation of size $n^{\Omega(w)}$ in resolution, PCR, and Sherali-Adams. In the conference version [Atserias et al. 2014], we left as an open problem whether an analogous result could be proven also for the stronger Lasserre proof system. This has now been achieved in Lauria and Nordström [2015] using very similar techniques to those in the current article, but with much worse bounds on the constant hidden in the asymptotic notation in the exponent.

1.5. Outline of This Paper

The rest of this article is organized as follows. After the necessary preliminaries in Section 2, we state the main theorem for resolution and give a full proof in Section 3. We believe this can serve as a useful warm-up to the more complicated proofs for stronger proof systems that follow in Section 4. In Section 5 we show that our lower bounds do *not* extend all the way to Lasserre. We conclude in Section 6 with some final remarks and a discussion of open problems.

2. PRELIMINARIES

A *literal* over a Boolean variable x is either the variable x itself (a *positive literal*) or its negation $\overline{x}$ (a *negative literal*). A *clause* $C = a_1 \vee \cdots \vee a_k$ is a disjunction of literals. A *k-clause* is a clause that contains at most k literals. A *CNF formula* $F = C_1 \wedge \cdots \wedge C_m$ is a conjunction of clauses. A *k-CNF formula* is a CNF formula consisting of k-clauses. We think of clauses and CNF formulas as sets: The order of elements is irrelevant and there are no repetitions. We denote the logical true value as $\top$ and the logical false value as $\bot$. The empty clause (containing no literals) is also denoted $\bot$, since it is always false. For integers m and n, $m < n$, we use the standard notation $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$.

A *resolution derivation* of a clause C from a CNF formula F is a sequence of clauses $(C_1, \ldots, C_\tau)$ such that $C_\tau = C$ and for $1 \le t \le \tau$ the clause C_t is obtained by one of the following derivation rules:

—**Axiom:** C_t is a clause in F (an *axiom clause*);
—**Inference:** $C_t = A \vee B$, where $C_i = A \vee x$ and $C_j = B \vee \overline{x}$ for $1 \le i, j < t$;
—**Weakening:** $C_t \supseteq C_i$ for some $1 \le i < t$.

A *resolution refutation* of F is a derivation of the empty clause $\bot$ from F.

Every resolution derivation $\pi = (C_1, \ldots, C_\tau)$ can be associated with a directed acyclic graph (DAG) G_π with vertices labelled by clauses C_t in π and edges (C_i, C_j) if C_j is obtained by an inference or a weakening step and C_i is used as a premise in that step. The derivation π is said to be *treelike* if G_π is a tree. The *(clause) space* of π at time t is the number of clauses derived before or at time t that will be used after or at time t, that is, all clauses C_i, $i \le t$, in G_π having an outgoing edge to clauses C_j, $j \ge t$ (plus the clause C_t itself). We say that these are the clauses *in memory at time t*. The space of π is the maximal number of clauses in memory at any time t in the derivation. The *width* of π is the maximal number of literals in any clause C_t in π, and the *size* (or *length*) of $\pi = (C_1, \ldots, C_\tau)$ is τ. We remark that it is straightforward to show that all applications of the weakening rule can be eliminated from a resolution refutation without any increase in size, width, or space, and while maintaining treelikeness.

In PCR, one instead refutes an unsatisfiable formula F over variables $x_1, \ldots, x_n$ by reasoning in terms of polynomials in the ring $\mathbb{F}[x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n]$, where $\mathbb{F}$ is some fixed field and x_i, $\overline{x}_i$ are formally independent variables. It is natural to think of polynomials as being satisfied by an assignment when they evaluate to 0, so in PCR the truth values $\top$ and $\bot$ are represented by 0 and 1, respectively, and a clause $\bigvee_{i \in \mathcal{I}} x_i \vee \bigvee_{j \in \mathcal{J}} \overline{x}_j$ is translated into the one-term polynomial $\prod_{i \in \mathcal{I}} x_i \cdot \prod_{j \in \mathcal{J}} \overline{x}_j$. A *PCR derivation* of a polynomial R from a set of polynomials $\mathcal{S} = \{Q_1, \ldots, Q_m\}$ is a sequence $(P_1, \ldots, P_\tau)$ such that $P_\tau = R$ and for $1 \le t \le \tau$ the polynomial P_t is obtained by one of the following derivation rules:

—**Boolean axiom:** P_t is $x^2 - x$ for some variable x (or $\overline{x}$);
—**Complementarity axiom:** P_t is $1 - x - \overline{x}$ for some variable x;
—**Initial axiom:** P_t is one of the polynomials $Q_j \in \mathcal{S}$;
—**Linear combination:** $P_t = \alpha P_i + \beta P_j$ for $1 \le i, j < t$ and some $\alpha, \beta \in \mathbb{F}$;
—**Multiplication:** $P_t = x P_i$ for $1 \le i < t$ and some variable x.

A *PCR refutation* of F is a PCR derivation of 1 from the set of polynomials representing the clauses of F as explained above. Note that the Boolean axioms make sure that variables can only take values $\top = 0$ and $\bot = 1$, and the complementarity axioms enforce that x and $\overline{x}$ take opposite values.

The *degree* of a PCR derivation π is the maximum of the (total) degrees of the polynomials in π. The *size* of π is the sum of the sizes of the polynomials in π, where

the size of a polynomial is defined as its number of terms. Just to make terminology precise, in this article a *monomial* is a product of variables, a *term* is a monomial multiplied by a non-zero coefficient from the field $\mathbb{F}$, and a *polynomial* is a sum of terms with distinct monomials. The *space* measure can also be generalized from resolution, counting terms instead of clauses, but we will not really need it in this article.

Let us next discuss *semialgebraic* proof systems. All such proof systems encode a CNF formula as a set of polynomial inequalities over the reals. A clause $\bigvee_{i\in\mathcal{I}} x_i \vee \bigvee_{j\in\mathcal{J}} \overline{x}_j$ is represented by the inequality $\sum_{i\in\mathcal{I}} x_i + \sum_{j\in\mathcal{J}}(1-x_j) - 1 \geq 0$, where we identify $\top = 1$ and $\bot = 0$—note that this is the opposite of the convention for PCR. A CNF formula F is represented by the inequalities corresponding to its clauses. A *SA derivation* of an inequality $R \geq 0$ from a set of polynomial inequalities $\{Q_1 \geq 0, \ldots, Q_m \geq 0\}$ is a formula of the form

$$\sum_{t=1}^{\tau} \alpha_t \cdot \prod_{i\in\mathcal{I}_t} x_i \cdot \prod_{i\in\mathcal{J}_t}(1-x_i) \cdot P_t, \tag{2.1}$$

that, when expanded into a sum of terms, gives the polynomial R, where $\alpha_t \in \mathbb{R}^+$ and P_t is one of the original polynomials Q_j, or an *axiom* of the form $x_i^2 - x_i$ or $x_i - x_i^2$, or the constant 1. A *Lasserre derivation* of $R \geq 0$ is a formula of the form of Equation (2.1) that expands to R where, in addition, P_t can be a square Q^2 for any arbitrary polynomial Q. Note that Sherali-Adams and Lasserre are *static* proof systems in that they have "one-shot" derivations, in contrast to resolution and PCR that construct derivations dynamically step by step.

We can augment Sherali-Adams by *twin variables* $\overline{x}_i$ whose intended meaning is the negation of x_i, that is, $1 - x_i$.[6] We define a *Sherali-Adams resolution (SAR) derivation* to be an SA derivation as in Equation (2.1) except that the set of variables is $\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ and that P_t can also be a complementarity axiom $1 - x_i - \overline{x}_i$ or $-1 + x_i + \overline{x}_i$.

A *SA, SAR, or Lasserre refutation* of F is a derivation in the respective system of the inequality $-1 \geq 0$ from the inequalities $Q_1 \geq 0, \ldots, Q_m \geq 0$ that encode the clauses of F. The *rank* of the derivation is the maximum of the degrees among the polynomials to which the formulas $\prod_{i\in\mathcal{I}_t} x_i \cdot \prod_{i\in\mathcal{J}_t}(1-x_i) \cdot P_t$ in Equation (2.1) expand, and the *size* of the derivation is the sum of the sizes of those polynomials, where again the size of a polynomial is defined as its number of terms.

A *restriction* (or *partial assignment*) ρ is a partial mapping from variables to $\{\bot, \top\}$. We identify ρ with the set of literals it sets to true. The *domain* of ρ is denoted $\mathrm{dom}(\rho)$ and the size of ρ is $|\rho| = |\mathrm{dom}(\rho)|$. The restriction $C\restriction_\rho$ of a clause C by ρ is the trivial clause $\top$ if ρ sets some literal of C to true—such a clause can just be removed from any formula or derivation—and otherwise it is the clause resulting from deleting all literals in C set to false by ρ. The restriction $F\restriction_\rho$ of a CNF formula F is the conjunction of its restricted clauses, and a restricted resolution derivation $\pi\restriction_\rho$ is the sequence of the restrictions of the clauses in π. It is a basic fact that if π is a refutation of F, then $\pi\restriction_\rho$ is a refutation of $F\restriction_\rho$.

For PCR derivations and the polynomials therein, restrictions are defined similarly: a restricted term vanishes if one of its variables is set to $\top = 0$ and is otherwise obtained by deleting all variables set to $\bot = 1$, and a restricted polynomial is the sum of its restricted terms. Again, restrictions preserve PCR refutations. For SA and SAR, the definition is analogous except the roles of 0 and 1 are reversed.

[6]As briefly discussed above, this is how PCR was extended in Alekhnovich et al. [2002] from the original definition of polynomial calculus (PC) in Clegg et al. [1996].

3. UPPER AND LOWER BOUNDS IN RESOLUTION

In this section, we establish the special case of our main result for the resolution proof system. Although the lower bound part follows from the stronger results that we will prove in later sections, we believe it is instructive to develop the argument for resolution first. Let us start by stating a slightly more detailed version of Theorem 1.1, but restricted to resolution, which is what we will prove.

THEOREM 3.1. *Let $k = k(n)$ be any integer-valued function such that $k(n) \leq n/4\log n$. Then there is a family of 3-CNF formulas $\{F_{n,k}\}_{n\geq 1}$, where $F_{n,k}$ has $\mathrm{O}(n^2)$ variables and $\mathrm{O}(kn^2)$ clauses, such that:*

(1) *$F_{n,k}$ has a treelike resolution refutation in size $\mathrm{O}\big(k^k n^k\big)$, width $2k+1$, and space $2k+3$;*
(2) *any resolution refutation of $F_{n,k}$ has size $\Omega\big(n^{k-1}/(4k\log n)^k\big)$.*

Straightforward calculations show that if $k(n) = \mathrm{O}(n^c)$ for $c < 1$, then the upper bound is $n^{\mathrm{O}(k)}$ and the lower bound is $n^{\Omega(k)}$.

3.1. Definition of The Formula

The CNF formulas we use to establish Theorem 3.1 are *relativized* versions of the pigeonhole principle formulas encoding the contradictory statement that there is a way to choose k out of n pigeons and send them to $k-1$ pigeonholes so every pigeon gets its own hole. More formally, the formulas claim that there are (partial) functions $p : [k] \to [n]$ and $q : [n] \to [k-1]$ such that p is one to one and defined on $[k]$, and q is one to one and defined on the range of p. Let us first describe a straightforward CNF encoding of this claim with wide clauses that we denote $RPHP^{k,n}_{k-1}$. Once the general idea is clear, we transform this into a slightly more involved 3-CNF formula which is the formula we will work with.

The formula $RPHP^{k,n}_{k-1}$ is over variables $p_{u,v}$ that encode the function p, $q_{v,w}$ that encode the function q, and r_v that encode a superset of the range of p. It consists of the following collection of clauses:

$$p_{u,1} \vee p_{u,2} \vee \cdots \vee p_{u,n} \qquad u \in [k], \tag{3.1a}$$
$$\overline{p}_{u,v} \vee \overline{p}_{u',v} \qquad u, u' \in [k],\ u \neq u',\ v \in [n], \tag{3.1b}$$
$$\overline{p}_{u,v} \vee r_v \qquad u \in [k],\ v \in [n], \tag{3.1c}$$
$$\overline{r}_v \vee q_{v,1} \vee \cdots \vee q_{v,k-1} \qquad v \in [n], \tag{3.1d}$$
$$\overline{r}_v \vee \overline{r}_{v'} \vee \overline{q}_{v,w} \vee \overline{q}_{v',w} \qquad v, v' \in [n],\ v \neq v',\ w \in [k-1]. \tag{3.1e}$$

The clauses in Equations (3.1a)–(3.1b) say that p maps $[k]$ injectively into $[n]$; clauses (3.1c) encode the range of p; and clauses (3.1d)–(3.1e) force q to be defined and injective on this range.

Next, we convert $RPHP^{k,n}_{k-1}$ to a 3-CNF formula. This is done in the standard way by using extension variables to break up the wide clauses in Equations (3.1a) and (3.1d) and the 4-clauses in Equation (3.1e). For Equation (3.1a) we obtain the clauses

$$p_{u,1} \vee p_{u,2} \vee y_{u,2} \qquad u \in [k], \tag{3.2a}$$
$$\overline{y}_{u,v} \vee p_{u,v+1} \vee y_{u,v+1} \qquad u \in [k],\ v \in [2, n-3], \tag{3.2b}$$
$$\overline{y}_{u,n-2} \vee p_{u,n-1} \vee p_{u,n} \qquad u \in [k], \tag{3.2c}$$

splitting up Equation (3.1d) yields

$$\overline{r}_v \vee q_{v,1} \vee z_{v,1} \qquad v \in [n], \tag{3.2d}$$

$$\overline{z}_{v,w} \vee q_{v,w+1} \vee z_{v,w+1} \qquad v \in [n],\ w \in [k-4], \tag{3.2e}$$

$$\overline{z}_{v,k-3} \vee q_{v,k-2} \vee q_{v,k-1} \qquad v \in [n], \tag{3.2f}$$

and the rest of the clauses are

$$\overline{p}_{u,v} \vee \overline{p}_{u',v} \qquad u, u' \in [k],\ u \neq u',\ v \in [n], \tag{3.2g}$$

$$\overline{p}_{u,v} \vee r_v \qquad u \in [k],\ v \in [n], \tag{3.2h}$$

$$\overline{r}_v \vee \overline{r}_{v'} \vee r_{v,v'} \qquad v, v' \in [n],\ v \neq v', \tag{3.2i}$$

$$\overline{r}_{v,v'} \vee \overline{q}_{v,w} \vee \overline{q}_{v',w} \qquad v, v' \in [n],\ v \neq v',\ w \in [k-1]. \tag{3.2j}$$

The 3-CNF formula consisting of the clauses in Equations (3.2a)–(3.2j), which we will denote $ERPHP^{k,n}_{k-1}$, is the formula for which we will prove Theorem 3.1. It is easy to verify that this formula has $O(kn^2)$ clauses over $O(n^2)$ variables. We note that if we did not insist on bringing the clause size all the way down to 3, then we could get a 4-CNF formula with $O(kn^2)$ clauses over $O(kn)$ variables by not converting the 4-clauses in Equations (3.1e) into the 3-clauses (3.2i) and (3.2j). Our proof of Theorem 3.1 works for this formula as well after straightforward adjustments and gives a slightly better lower bound expressed in terms of the number of variables.

3.2. Proof of the Upper Bound

Let us first describe how we can refute the formula $ERPHP^{k,n}_{k-1}$ in resolution. In order to do so, we consider all sequences of the form $(v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k)$, where $v_u \in [n]$ and $w_u \in [k-1]$, and the corresponding clauses

$$\bigvee_{u\in[k]} \overline{p}_{u,v_u} \vee \bigvee_{u\in[k]} \overline{q}_{v_u,w_u}. \tag{3.3}$$

We derive all such clauses from the axiom clauses of $ERPHP^{k,n}_{k-1}$, and from these clauses we obtain a contradiction. All of these derivations are efficient, so the size of the whole refutation is dominated by the number of clauses in (3.3).

For each clause in Equation (3.3) we are in one of two cases: either $v_u = v_{u'}$ holds for some $u \neq u'$, or there must exist a pair $v_u \neq v_{u'}$ with $w_u = w_{u'}$ by the pigeonhole principle. In the former case, the clause (3.3) is just a weakening of the axiom (3.2g), namely $\overline{p}_{u,v} \vee \overline{p}_{u',v}$ with $v = v_u = v_{u'}$. In the latter case, we combine axioms $\overline{p}_{u,v_u} \vee r_{v_u}$ and $\overline{p}_{u',v_{u'}} \vee r_{v_{u'}}$ from Equation (3.2h), $\overline{r}_{v_u} \vee \overline{r}_{v_{u'}} \vee r_{v_u,v_{u'}}$ from Equation (3.2i), and $\overline{r}_{v_u,v_{u'}} \vee \overline{q}_{v_u,w} \vee \overline{q}_{v_{u'},w}$ from Equation (3.2j), where $w = w_u = w_{u'}$, to obtain the clause $\overline{p}_{u,v_u} \vee \overline{p}_{u',v_{u'}} \vee \overline{q}_{v_u,w} \vee \overline{q}_{v_{u'},w}$. It is easy to see that Equation (3.3) can be derived from this clause by weakening. Since a constant number of clauses is involved in this derivation, it requires only constant space, and it is straightforward to verify that it can in fact be carried out by a treelike derivation in space 3 (that is, keeping one clause in memory and resolving it with a sequence of axioms).

The rest of the refutation consists of derivations of all prefixes of clauses of the form of Equation (3.3) by backward induction. For the inductive step we assume that we are able to derive any prefix clause of size t in clause space $(2k-t)+3$ and show how to derive any prefix of size $t-1$ in clause space $(2k-t+1)+3$. The refutation ends when we reach the prefix clause of size 0 (that is, the empty clause) in clause space $2k+3$.

Suppose first that we can derive each clause of the form

$$\bigvee_{u\in[k]} \overline{p}_{u,v_u} \vee \bigvee_{u\in[k^*-1]} \overline{q}_{v_u,w_u} \vee \overline{q}_{v^*,w^*} = A \vee \overline{q}_{v^*,w^*} \tag{3.4}$$

for some $k^* \leq k$ in clause space s for some positive integers (writing $v^* = v_{k^*}$ and $w^* = w_{k^*}$ as a shorthand). We want to use the existence of such derivations to derive the clause A in space $s + 1$. To this end, start with the axiom $\overline{p}_{k^*,v^*} \vee r_{v^*}$ and note that the literal $\overline{p}_{k^*,v^*}$ also appears in the left-hand part of A in (3.4). We resolve this clause with the axiom $\overline{r}_{v^*} \vee q_{v^*,1} \vee z_{v^*,1}$ to get $\overline{p}_{u,v^*} \vee q_{v^*,1} \vee z_{v^*,1}$. Keeping the latter clause in memory, we invoke a subderivation in space s of the clause $A \vee \overline{q}_{v^*,1}$ and resolve to obtain $A \vee z_{v^*,1}$. Continuing, assume that we have derived $A \vee z_{v^*,w}$ for some $w \geq 1$. Then we resolve this clause with the axiom $\overline{z}_{v^*,w} \vee q_{v^*,w+1} \vee z_{v^*,w+1}$ to obtain $A \vee q_{v^*,w+1} \vee z_{v^*,w+1}$. Keeping the latter clause in memory, we derive $A \vee \overline{q}_{v^*,w+1}$ using no more space than $s+1$ all in all and then resolve to get $A \vee z_{v^*,w+1}$. When we reach the clause $A \vee z_{v^*,k-3}$ we resolve it with the axiom $\overline{z}_{v^*,k-3} \vee q_{v^*,k-2} \vee q_{v^*,k-1}$ and then with the inductively derived clauses $A \vee \overline{q}_{v^*,k-2}$ and $A \vee \overline{q}_{v^*,k-1}$ to obtain A. We point out again that the clause space of this derivation is $s + 1$.

After k steps of this backward induction we get to clauses of the form $\overline{p}_{1,v_1} \vee \overline{p}_{2,v_2} \vee \ldots \vee \overline{p}_{k,v_k}$. To derive the empty clause we do k more steps of backward induction, mimicking the procedure in the previous paragraph. Suppose that we have shown how to derive all clauses

$$\bigvee_{u \in [k^*-1]} \overline{p}_{u,v_u} \vee \overline{p}_{k^*,v^*} = A \vee \overline{p}_{k^*,v^*} \tag{3.5}$$

for $k^* \leq k$ and want to derive A. To do so, first resolve the axiom $p_{k^*,1} \vee p_{k^*,2} \vee y_{k^*,2}$ with the inductively derived clause $A \vee \overline{p}_{k^*,1}$ and then with $A \vee \overline{p}_{k^*,2}$ to get $A \vee y_{k^*,2}$. Suppose that we have shown how to derive $A \vee y_{k^*,v}$ in this way for $v \geq 2$. In order to obtain $A \vee y_{k^*,v+1}$ we resolve $\overline{y}_{k^*,v} \vee p_{k^*,v+1} \vee y_{k^*,v+1}$ with $A \vee y_{k^*,v}$ and then with $A \vee \overline{p}_{k^*,v+1}$. We iterate up to $A \vee y_{k^*,n-2}$ and, finally, resolve the axiom $\overline{y}_{k^*,n-2} \vee p_{k^*,n-1} \vee p_{k^*,n}$ with the clauses $A \vee y_{k^*,n-2}$, $A \vee \overline{p}_{k^*,n-1}$, and $A \vee \overline{p}_{k^*,n}$ to obtain A. After k steps of this second stage we reach the empty clause and the refutation is complete. As before, the clause space goes up by an additive one for every inductive step, so the clause space of the whole refutation is $2k + 3$.

To analyze the size of the resolution refutation obtained in this way, consider the prefix tree of the sequences $(v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k)$. Each vertex of this tree corresponds to one of the clauses A derived during the backward induction, with the empty clause at the root and clauses (3.3) at the leaves. The length of the derivation of each clause is linear in the number of children, and, in addition, we derived the leaves with a constant number of steps. Therefore we can charge a constant amount of steps per vertex. The size of the tree is $O(k^k n^k)$, and it follows that this is also the size of the refutation. Furthermore, the refutation is treelike since no intermediate clause is used more than once. One can also observe that the width of the refutation is $2k + 1$ and reaches this maximum at the induction step from sequences of length $2k$ to sequences of length $2k - 1$.

3.3. Proof of the Lower Bound for Resolution

As discussed in Section 1.3, we use a random restriction argument to prove our size lower bound for resolution refutations of the formula $ERPHP^{k,n}_{k-1}$. We define a distribution $\mathcal{D}$ on partial assignments ρ by picking a uniformly random subset $\mathcal{S}$ of k elements from $[n]$ and fixing an arbitrary bijection $\psi : [k] \to \mathcal{S}$ and then letting ρ assign values to variables as follows:

—$r_v = \top$ if $v \in \mathcal{S}$ (we say that v is *picked* in this case) and $r_v = \bot$ otherwise;
—$r_{v,v'} = r_v \wedge r_{v'}$ for all $v \neq v'$;
—$p_{u,\psi(u)} = \top$ and $p_{u,v} = \bot$ for all $u \in [k]$ and all $v \neq \psi(u)$;
—$y_{u,v}$ for all u and v are set arbitrarily so as to satisfy the clauses (3.2a)–(3.2c);
—$q_{v,w}$ and $z_{v,w}$ are left unset for all $v \in \mathcal{S}$ and all w;

—$q_{v,w} = b_v$ and $z_{v,w} = b_v$ for all $v \in [n] \setminus \mathcal{S}$ and all $w \in [k-1]$, where $b_v \in \{\bot, \top\}$ is chosen uniformly and independently at random for every $v \in [n] \setminus \mathcal{S}$.

Note that regardless of the choice of $\mathcal{S}$ the restricted formula is just a 3-CNF version of the pigeonhole principle formula over k pigeons and $k-1$ holes. We remark that the reason we can set $q_{v,w}$ and $z_{v,w}$ randomly and independently for different $v \notin \mathcal{S}$ is that the variables $r_{v,v'}$ are "guarding" the clauses (3.2j), which would otherwise be falsified with high probability. This will be an important technical point in our argument when we show that with high probability such restrictions remove or at least significantly shrink all wide clauses (which will then allows us to obtain impossibly narrow refutations of the 3-CNF PHP formulas).[7]

For $v \in [n]$, let us say that the variables $\{q_{v,1}, \ldots, q_{v,k-1}, z_{v,1}, \ldots, z_{v,k-1}\}$ *mention the pigeon* v. We say that a clause (or term) mentions v if it contains some variable in this set and define the *pigeon-width* to be the number of pigeons mentioned. The next lemma describes the effect of random restrictions ρ from $\mathcal{D}$ on clauses (or terms) depending on their pigeon-width. Namely, a sufficiently wide clause, that is, mentioning a lot of pigeons, is satisfied by the random restriction with high probability, whereas a narrower clause may not have its truth value fixed by the restriction but will with high probability contain few pigeons afterwards.

LEMMA 3.2. *Let k, ℓ, n be natural numbers such that $n \geq 16$ and $\ell \leq k \leq n/4\log n$. Let A be either a clause or term over the variables of $ERPHP^{k,n}_{k-1}$ and let ρ be a random restriction sampled from the distribution $\mathcal{D}$ as defined above. Then the pigeon-width of $A\restriction_\rho$ is less than ℓ with probability at least $1 - (4k\log n)^k/n^\ell$.*

PROOF. Let us assume that A is a clause—the proof for terms (which will be used for PCR and Sherali-Adams) is completely analogous. Let $v_1, \ldots, v_r$ be the pigeons mentioned in A sorted in some order and let $a_1, \ldots, a_r$ be a sequence of literals such that a_i witnesses that A mentions v_i.

If $r > 2k\log n$, then the probability that the clause A is not satisfied by the restriction is at most

$$\begin{aligned}
\Pr[\rho(a_i) \neq \top \text{ for all } i = 1, \ldots, r] &\leq \Pr[\rho(a_i) \neq \top \text{ for all } i = 1, \ldots, \lceil 2k\log n\rceil] \\
&= \prod_{i=1}^{\lceil 2k\log n\rceil} \Pr[\rho(a_i) \neq \top \,\big|\, \rho(a_j) \neq \top \text{ for } j < i] \\
&\leq \prod_{i=1}^{\lceil 2k\log n\rceil} \Pr\big[\rho(a_i) \neq \top \,\big|\, v_j \notin \mathcal{S} \text{ for } j < i\big] \qquad (3.6) \\
&\leq \prod_{i=1}^{\lceil 2k\log n\rceil} \left(\frac{1}{2} + \frac{k}{n-i+1}\right) \\
&< \left(\frac{5}{8}\right)^{2k\log n} < \frac{1}{n^k}.
\end{aligned}$$

To see this, note that the event $\rho(a_i) \neq \top$ occurs either if the pigeon v_i is picked or if the literal a_i is set to the wrong value. The first of these events is most probable if no pigeon so far was picked. The second event can occur only if the first event fails and, conditioned

[7]Indeed, proving resolution size lower bounds for relativized PHP formulas *without* "guard variables" in the clauses (3.2i) and (3.2j) (or for the wide version without the "guard variables" in the clauses (3.1e)) is an open problem as mentioned in Dantchev and Martin [2014].

on this, its probability is independent of the conditioning on the other pigeons so far. This explains why $\rho(a_j) \neq \top$ is replaced by $v_j \notin \mathcal{S}$ in the conditioning in the second inequality. For the next inequality, assuming that no pigeon $v_1, \ldots, v_{i-1}$ has been picked before v_i, the conditional probability of v_i being included in $\mathcal{S}$ is $k/(n - i + 1)$, and is less otherwise. If $v_i \notin \mathcal{S}$ instead, then a_i gets the wrong value with probability 1/2. The final inequalities hold because the ratio $k/(n - \lceil 2k \log n \rceil + 1)$ is at most $1/(2 \log n)$ when $k \leq n/4 \log n$, and therefore it is at most 1/8 for $n \geq 16$.

If instead the number of pigeons mentioned by A is $r \leq 2k \log n$, then we want to bound the probability that there are at least ℓ pigeons mentioned in A that are chosen in $\mathcal{S}$ and hence survive. The number of choices $\mathcal{S}$ with exactly i pigeons mentioned in A is $\binom{r}{i}\binom{n-r}{k-i}$. Considering all possible intersections of size at least ℓ between the set $\mathcal{S}$ and the r pigeons mentioned in A, we obtain that the probability of ℓ surviving pigeons is at most

$$\begin{aligned}\sum_{i=\ell}^{k} \binom{r}{i}\binom{n-r}{k-i}\binom{n}{k}^{-1} &\leq k\binom{\lfloor 2k \log n \rfloor}{k}\binom{n}{k-\ell}\binom{n}{k}^{-1} \\ &\leq \frac{k(2k \log n)^k}{k!} \cdot \frac{n!}{(k-\ell)!(n-k+\ell)!} \cdot \frac{(n-k)!k!}{n!} \\ &\leq k(2k \log n)^k \cdot \frac{1}{(k-\ell)!} \cdot \frac{1}{(n-k)^\ell} < \frac{k(2k \log n)^k}{(n-k)^\ell}.\end{aligned} \tag{3.7}$$

To finish the computation we use that $n \geq 16$ and $k \leq n/4 \log n$ to get that $k \leq n/16$, and we observe that $k(16/15)^\ell \leq 2^k$ for every $1 \leq \ell \leq k$. We obtain that

$$\frac{k(2k \log n)^k}{(n-k)^\ell} \leq \frac{k(2k \log n)^k}{(15n/16)^\ell} = k(16/15)^\ell \cdot \frac{(2k \log n)^k}{n^\ell} \leq \frac{(4k \log n)^k}{n^\ell}. \tag{3.8}$$

This concludes the proof. □

We can use Lemma 3.2 to show that if we hit a sufficiently short resolution refutation of $ERPHP^{k,n}_{k-1}$ with a random restriction ρ sampled from the distribution $\mathcal{D}$, then in the restricted refutation all clauses are likely to have small pigeon-width. The reason this is useful is that the distribution $\mathcal{D}$ is constructed so that the restricted formula is just the standard pigeonhole principle formula over k pigeons and $k - 1$ holes or, rather, a 3-CNF version of it (up to renaming of variables). To spell this out explicitly, after renaming the k pigeons in $[n]$ chosen by ρ to $1, \ldots, k$, what remains is the following collection of clauses, which we will denote $EPHP^k_{k-1}$:

$$q_{v,1} \vee z_{v,1} \qquad v \in [k], \tag{3.9a}$$

$$\overline{z}_{v,w} \vee q_{v,w+1} \vee z_{v,w+1} \qquad v \in [k],\ w \in [k-4], \tag{3.9b}$$

$$\overline{z}_{v,k-3} \vee q_{v,k-2} \vee q_{v,k-1} \qquad v \in [k], \tag{3.9c}$$

$$\overline{q}_{v,w} \vee \overline{q}_{v',w} \qquad v, v' \in [k],\ v \neq v',\ w \in [k-1]. \tag{3.9d}$$

But the formula $EPHP^k_{k-1}$ can be shown to require almost maximal pigeon-width in resolution. This result is implicit in several previous articles on the proof complexity of pigeonhole principle formulas, but we write down the formal statement as a lemma and provide a proof for completeness.

LEMMA 3.3. *Every resolution refutation of the formula $EPHP^k_{k-1}$ consisting of the clauses* 3.9a–3.9d *has pigeon-width at least $k - 1$.*

PROOF. We use a game argument in the style of Pudlák [2000] and Atserias and Dalmau [2008] adapted to the notion of pigeon-width. The game is played between a *prosecutor* and a *defendant*. At each step of the game the prosecutor queries the

defendant for the value of a variable of $EPHP^k_{k-1}$ and stores the answer in his record. The prosecutor is also allowed to erase variable assignments from his record after any query, but if so the defendant can answer differently next time she is asked about an erased variable. The goal of the prosecutor is to force the defendant to falsify a clause from $EPHP^k_{k-1}$, while the goal of the defendant is to answer queries without falsifying any axiom clause in this formula.

To establish the lemma, it is sufficient to show that the prosecutor cannot win unless at some point he holds a record that mentions k pigeons. The reason for this is that if there exists a resolution refutation π of pigeon-width $\ell < k-1$, then the prosecutor can use such a refutation to construct a strategy that never mentions more than $\ell+1$ pigeons.

To build a winning strategy from a refutation π, the prosecutor walks backwards through the associated graph G_π from the final empty clause all the way to some axiom clause. The invariant maintained is that at each step the current assignment on record is the minimal falsifying assignment for the clause currently visited in G_π. At the beginning of the game the empty record corresponds to the empty clause in the refutation. If the current clause was obtained by resolution, then the prosecutor queries the resolved variable (which might temporarily increase the number of pigeons on record by 1), moves to the premise falsified by the answer, and then forgets all assignments not needed to falsify that clause. For a weakening step, the prosecutor just needs to forget variables. The prosecutor wins when the game reaches a source vertex in G_π (if not earlier), since by the invariant the corresponding axiom clause is falsified by the assignment on record at that point.

Switching to the lower-bound perspective, let us now briefly describe a defendant strategy that works against prosecutors mentioning less than k pigeons. The defendant privately keeps a partial matching of the pigeons mentioned in the current record of the prosecutor into holes, making sure that this mapping is compatible with the partial assignment in his record. If the prosecutor asks about a variable which mentions a pigeon already in the domain of the defendant's partial matching, then she answers consistently with her matching. If the prosecutor erases all variables mentioning a pigeon, the defendant removes that pigeon from the partial mapping, freeing up the corresponding hole for later reuse. If the prosecutor queries a variable that mentions a new pigeon, then we are in one of two cases: Either there is at least one free hole, or the record mentions $k-1$ pigeons. In the first case, the defendant assigns the new pigeon to some free hole and updates her partial matching accordingly. In the second case, the defendant has achieved her goal—although she is now forced to falsify a clause of $EPHP^k_{k-1}$ and loses, the prosecutor was able to win only by compiling a record that mentions k pigeons. □

Putting all the pieces together, we can now prove the lower bound in Theorem 3.1. Namely, let π be a resolution refutation of $ERPHP^{k,n}_{k-1}$ of size S. Hit π with a random restriction ρ distributed according to $\mathcal{D}$. Since resolution refutations are preserved under restrictions, $\pi\restriction_\rho$ is a refutation of $ERPHP^{k,n}_{k-1}\restriction_\rho$ which, as discussed above, is $EPHP^k_{k-1}$ after renaming of variables. By Lemma 3.3, this refutation must have pigeon-width at least $k-1$ with probability 1. On the other hand, using Lemma 3.2 with $\ell = k-1$ and taking a union bound over all clauses in π, the probability that this happens is at most $S \cdot (4k \log n)^k / n^{k-1}$ for large-enough n. We can hence conclude that $S \geq n^{k-1}/(4k \log n)^k$, and the proof of Theorem 3.1 is complete.

4. ALGEBRAIC AND SEMIALGEBRAIC PROOF SYSTEMS

Let us now show how the size lower bound for resolution in Section 3 can be generalized to PCR and SAR. The overall structure of the size lower bound proof is very similar to

that for resolution in that we first establish a lower bound on a parameter analogous to the pigeon-width in Section 3, which we call *pigeon-degree* for PCR and *pigeon-rank* for SAR and then plug this bound into the random restriction argument as in the proof of Lemma 3.2.

In this section, we also discuss how upper bounds for PCR and SAR analogous to those for resolution in Theorem 1.1 can be established. The upper bound in resolution more or less immediately carries over to PCR, in the sense that it is very easy to show that a resolution refutation can be simulated in PCR in essentially the same size and with PCR degree matching the resolution width. For SA and SAR, it requires a bit more work to construct such efficient simulations and we discuss it in some detail below. It should be noted that while PCR degree and SAR rank upper bounds $O(k)$ are sufficient to obtain refutation of size $n^{O(k)}$ in both proof systems, using explicit simulations like the ones discussed in this section gives better bounds.

4.1. Lower Bound on Degree for Polynomial Calculus Resolution

In a natural generalization of the terminology in Section 3, we say that not only the variables $q_{v,w}$ and $z_{v,w}$ of $EPHP^k_{k-1}$ but also their twins $\overline{q}_{v,w}$ and $\overline{z}_{v,w}$ *mention* the pigeon v. The *pigeon-degree* of a monomial is the number of pigeons that are mentioned by its variables, the pigeon-degree of a polynomial is the maximum pigeon-degree of its monomials, and the pigeon-degree of a PCR refutation of $EPHP^k_{k-1}$ is the maximum pigeon-degree of the polynomials in the refutation. The following lower bound for pigeon-degree of PCR refutations is the analogue of Lemma 3.3 for resolution.

LEMMA 4.1. *Every PCR refutation of $EPHP^k_{k-1}$ has pigeon-degree at least $\lceil \frac{k-1}{2} \rceil$.*

PROOF. We prove the lower bound by studying a different encoding $APHP^k_{k-1}$ of the pigeonhole principle for k pigeons and $k-1$ holes introduced by Razborov [1998], which we will described shortly. Given any PCR refutation of $EPHP^k_{k-1}$ as defined in Equation (3.9a)–(3.9d) in which all monomials mention at most d pigeons, we show how to transform it into a refutation of degree $d+1$ of $APHP^k_{k-1}$. Since $APHP^k_{k-1}$ requires degree strictly larger than $\lceil \frac{k-1}{2} \rceil$ by Theorem 3.9 in Impagliazzo et al. [1999], we can conclude that $d \geq \lceil \frac{k-1}{2} \rceil$ and the lemma follows.

The alternative formulation $APHP^k_{k-1}$ is defined on variables $x_{v,w}$ for $v \in [k]$ and $w \in [k-1]$, where $x_{v,w} = 1$ means that pigeon v sits in hole w. We stress that this interpretation of the variables is the opposite of the one we use for $EPHP^k_{k-1}$. Also, $APHP^k_{k-1}$ is not a (translation of a) CNF formula but consists of the following polynomials:

$$1 - \sum_{w \in [k-1]} x_{v,w} \qquad v \in [k], \tag{4.1a}$$

$$x_{v,w} x_{v',w} \qquad w \in [k-1],\ v, v' \in [k],\ v \neq v', \tag{4.1b}$$

$$x_{v,w} x_{v,w'} \qquad v \in [k],\ w, w' \in [k-1],\ w \neq w'. \tag{4.1c}$$

To obtain a degree-$(d+1)$ refutation for $APHP^k_{k-1}$, the first step is to apply a substitution δ to the variables in the refutation in pigeon-degree d of $EPHP^k_{k-1}$. For q-variables we define $\delta(q_{v,w}) = 1 - x_{v,w}$ and $\delta(\overline{q}_{v,w}) = x_{v,w}$, and for z-variables we let $\delta(z_{v,w}) = 1 - \sum_{j>w} x_{v,j}$ and $\delta(\overline{z}_{v,w}) = 1 - \sum_{j \leq w} x_{v,j}$. This substitution transforms the refutation of $EPHP^k_{k-1}$ into a sequence of polynomials over the variables in $APHP^k_{k-1}$. This is not yet a valid refutation, however, and in order to deal with this we need to show how to derive each substituted polynomial in the sequence. How to do so depends on what rule was used to derive the polynomial before the substitution.

For inference steps, if we derived xP from P, then $\delta(xP) = \delta(x)\delta(P)$ can be derived from $\delta(P)$ by a sequence of multiplications and linear combinations, and if the polynomial was derived via a linear combination, then the same derivation step is valid for the substituted polynomials.

If P is an application of the Boolean axiom $x^2 - x$ to a q-variable or z-variable, then $\delta(P)$ can be derived from Boolean axioms combined with polynomials (4.1c). Applications of complementarity axioms are either vacuous (for q-variables) or reduce to Equation (4.1a) (for z-variables).

Finally, we need to show how to derive $\delta(P)$ if P is obtained from one of the clauses in Equations (3.9a)–(3.9d). We describe how to do this for $P = \overline{z}_{v,w} q_{v,w+1} z_{v,w+1}$ as in Equation (3.9b); the other cases are very similar. We have

$$\begin{aligned}\delta(\overline{z}_{v,w} q_{v,w+1} z_{v,w+1}) &= \left(1 - \sum_{j \le w} x_{v,j}\right)(1 - x_{v,(w+1)})\left(1 - \sum_{j > w+1} x_{v,j}\right) \\ &= 1 - \sum_{j \in [k-1]} x_{v,j} + \sum_{\substack{j \neq j',\ j \le w+1,\\ j' \ge w+1}} x_{v,j} x_{v,j'} - \sum_{\substack{j < w+1,\\ j' > w+1}} x_{v,j} x_{v,(w+1)} x_{v,j'},\end{aligned} \tag{4.2}$$

where $1 - \sum_{j \in [k-1]} x_{v,j}$ is 4.1a and all monomials of degree two or three can be derived from Equation (4.1c). Thus, $\delta(\overline{z}_{v,w} q_{v,w+1} z_{v,w+1})$ can be derived from $APHP^k_{k-1}$.

This shows how we can apply the substitution δ to a refutation of $EPHP^k_{k-1}$ to obtain a refutation of $APHP^k_{k-1}$. The substitution exchanges variables indexed by the pigeon v for degree-1 polynomials that mention just v, and therefore each monomial of this refutation mentions at most d pigeons as well. We then postprocess this refutation of $APHP^k_{k-1}$ as follows: Every time a polynomial P of degree $d+1$ is derived, we remove all its monomials of degree larger than d and we use the resulting polynomial in place of P in the rest of the proof. We can eliminate monomials of degree larger than d because they mention at most d pigeons, and so either they are divisible by an axiom (4.1c) or they contain a squared variable. After this postprocessing phase the refutation has a polynomial of degree at most d for each polynomial in the original refutation, plus there are small derivations of degree $d+1$ for the cancellations. Thus the (total) degree is at most $d+1$ and the lemma follows. □

4.2. Size and Rank Upper Bounds for Sherali-Adams Refutations

Let us next switch focus to upper bounds and show that SAR can simulate resolution refutations efficiently in term of size and rank. We remark that a similar simulation is given in Dantchev et al. [2009], but since that article uses a slightly different definition of Sherali-Adams, we give a full description of the simulation here for completeness.

We start by introducing notation for two polynomial forms which we will use to represent clauses. For any pair of sets of propositional variables Y, Z, $Y \cap Z = \emptyset$, we let

$$S(Y, Z) = \sum_{y \in Y} y + \sum_{z \in Z} \overline{z} \tag{4.3}$$

and

$$M(Y, Z) = \prod_{y \in Y} \overline{y} \prod_{z \in Z} z. \tag{4.4}$$

Consider a clause $C = \bigvee_{y \in V_C^+} y \vee \bigvee_{z \in V_C^-} \overline{z}$ where V_C^+ and V_C^- are the sets of variables appearing positively and negatively in C, respectively. Then we define

$$S(C) = S\big(V_C^+, V_C^-\big) \tag{4.5}$$

and

$$M(C) = M\big(V_C^+, V_C^-\big). \tag{4.6}$$

Observe that for any assignment of the variables to $\top = 1$ or $\bot = 0$ it holds that $S(C) - 1 \geq 0$ and $-M(C) \geq 0$ if and only if C is satisfied. The former, additive inequality is how clauses are translated to inequalities as discussed in Section 2, but for our simulation of resolution by SAR we will need to work with the latter, multiplicative version.

The following three lemmas show how to efficiently simulate the steps in a resolution derivation.

LEMMA 4.2 (SIMULATION OF AXIOM). *For a clause C of width w the inequality $-M(C) \geq 0$ has a derivation in SAR of rank $w + 1$ and size $O(w^2)$ from the inequality $S(C) - 1 \geq 0$.*

PROOF. If C is the empty clause, then the claim is obvious since in that case $-M(C) = S(C) - 1$. Let C be non-empty and assume for simplicity that it has a positive literal x. Then C has the form

$$x \vee \bigvee_{y \in Y} y \vee \bigvee_{z \in Z} \overline{z} \tag{4.7}$$

with $|Y| + |Z| < w$. By multiplying $S(C) - 1 \geq 0$ by $M(Y, Z)$ we obtain the polynomial inequality

$$xM(Y, Z) + \sum_{y \in Y} yM(Y, Z) + \sum_{z \in Z} \overline{z}M(Y, Z) - M(Y, Z) \geq 0. \tag{4.8}$$

For each $y \in Y$ we can derive

$$\begin{aligned}(1 - y - \overline{y}) \cdot yM(Y \setminus \{y\}, Z) \;+\; (y^2 - y) \cdot M(Y \setminus \{y\}, Z) = \\ -\; y \cdot \overline{y} \cdot M(Y \setminus \{y\}, Z) = -y \cdot M(Y, Z).\end{aligned} \tag{4.9}$$

In essentially the same way, we can derive

$$-\overline{z}M(Y, Z) \geq 0 \tag{4.10}$$

for each $z \in Z$. The inequality

$$-M(C) = -\overline{x}M(Y, Z) \geq 0 \tag{4.11}$$

is now the sum of inequality (4.8), all inequalities of the form (4.9) and (4.10) for all $y \in Y$ and $z \in Z$, and of the inequality

$$(1 - x - \overline{x})M(Y, Z) \geq 0. \tag{4.12}$$

This SAR derivation has size $O(w^2)$ and rank w. □

LEMMA 4.3 (SIMULATION OF WEAKENING). *For clauses $A \subseteq B$ of width at most w the inequality*

$$M(A) - M(B) \geq 0$$

has a derivation in SAR of rank w and size $O(w^2)$.

PROOF. Consider the decomposition $M(B) = M(A) \cdot M(B \setminus A)$, and let us write the monomial $M(B \setminus A)$ as $\prod_{i=1}^{\ell} a_i$, where a_i are literal variables in SAR and $\ell = |B \setminus A|$. By a telescoping sum we derive

$$\sum_{j=1}^{\ell} M(A)(1 - a_j) \prod_{i=1}^{j-1} a_i = M(A) \left(1 - \prod_{i=1}^{\ell} a_i\right) = M(A) - M(B) \tag{4.13}$$

which establishes the lemma. □

LEMMA 4.4 (SIMULATION OF RESOLUTION STEP). *Let A and B be clauses in which the variable x does not appear and let w be the width of $A \vee B$. Then the inequality*

$$M(A \vee x) + M(B \vee \overline{x}) - M(A \vee B) \geq 0$$

has a derivation in SAR of rank $w + 1$ and size $O(w^2)$.

PROOF. Using Lemma 4.3 twice we derive the two inequalities $M(A \vee x) - M(A \vee B \vee x) \geq 0$ and $M(B \vee \overline{x}) - M(A \vee B \vee \overline{x}) \geq 0$. Then we derive $(x + \overline{x} - 1)M(A \vee B) \geq 0$ from the axiom $x + \overline{x} - 1 \geq 0$. This is the same as

$$M(A \vee B \vee \overline{x}) + M(A \vee B \vee x) - M(A \vee B) \geq 0. \tag{4.14}$$

The inequality that we want to prove is the sum of these three inequalities just derived. This SAR derivation has size $O(w^2)$ and rank $w + 1$. □

Remark 4.5. In Lemmas 4.2, 4.3, and 4.4 we gave the SAR simulations of the steps of a resolution refutation. To get a simulation in SA it is sufficient to substitute $(1 - x_1), \ldots, (1 - x_n)$ for the variables $\overline{x}_1, \ldots, \overline{x}_n$. After the substitution we obtain a valid SA proof of the corresponding inequalities of the same rank but potentially of larger size. Notice that the proofs of the inequalites in Lemmas 4.2, 4.3, and 4.4 have the form of Equation (2.1), with $O(w)$ axioms, each of them multiplied by a degree $w + O(1)$ polynomial. Hence, the size of each of these proofs is at most $O(w2^w)$.

Now we can show how resolution refutations can be efficiently simulated in the SA and SAR proof systems.

LEMMA 4.6. *If a CNF formula F has a resolution refutation of width w and length L, then it has an SA refutation of rank $w + 1$ and size $\mathrm{O}(w2^w L)$ and an SAR refutation of rank $w + 1$ and size $\mathrm{O}(w^2 L)$.*

PROOF. Let $\pi = (C_1, C_2, \ldots, C_L)$ be a resolution refutation of F where all clauses have width at most w. Let us focus first on the SAR simulation. For each clause C_i in the refutation we derive an inequality as follows:

(1) If C_i is an axiom clause, then we derive $-M(C_i) \geq 0$.
(2) If C_i is obtained by weakening from C_j, then we derive $M(C_j) - M(C_i) \geq 0$.
(3) If C_i is obtained by resolving C_j and C_k, then we derive $M(C_j) + M(C_k) - M(C_i) \geq 0$.

All of these inequalities have SAR derivations of rank $w + 1$ and size $\mathrm{O}(w^2)$ by Lemmas 4.2, 4.3, and 4.4 (where we recall that the encoding of an axiom clause C in the SAR proof system is $S(C) - 1 \geq 0$, as required by Lemma 4.2).

Now we have a sequence of inequalities $Q_1 \geq 0, Q_2 \geq 0, \ldots, Q_L \geq 0$, where the inequality $Q_i \geq 0$ corresponds to the clause C_i as explained above. Observe that any positive combination $\sum_{i=1}^{L} \alpha_i Q_i \geq 0$ has a SAR derivation of rank $w + 1$ and size $O(L \cdot w^2)$. In order to conclude the proof of the lemma, we just need to argue that there are positive weights α_i such that $\sum_i \alpha_i Q_i = -1$.

The intuition is that if C_i is obtained by weakening from C_j, then adding $Q_i = M(C_j) - M(C_i)$ will cancel the term $-M(C_j)$ in Q_j representing C_j, and if C_i is inferred by resolution from C_j and C_k, then adding $Q_i = M(C_j) + M(C_k) - M(C_i)$ will cancel the terms $-M(C_j)$ and $-M(C_k)$ representing C_j and C_k in Q_j and Q_k, respectively. In the end, all monomials representing clauses are cancelled and the only term remaining is -1. However, if a clause is used in several different applications of the resolution or weakening rules we need to set the weights so it is cancelled the correct number of times.

To do so, consider the DAG of the resolution refutation oriented from the initial clauses towards the empty clause. We assign a weight to each clause C_i in this DAG inductively: The empty clause C_L gets weight 1, and if all immediate successors of a clause have already been assigned weights, then the clause gets the sum of the weights of its immediate successors as the weight for itself. The value of α_i is then the weight assigned to the clause C_i in this way. To verify that $\sum_i \alpha_i Q_i = -1$, notice that every polynomial $M(C_i)$ has negative coefficient in the inequality $Q_i \geq 0$ and positive one in every $Q_j \geq 0$ where C_i appears as a premise in the derivation of C_j. By construction, the coefficient of each $M(C_i)$ in the final sum is zero unless $i = L$. Since $\alpha_L = 1$, the final sum is equal to $-M(\emptyset, \emptyset)$ which is -1.

We can obtain a simulation in the SA proof system instead by substituting $(1 - x_i)$ for every negative variable $\overline{x}$ in the SAR simulation described above. Then we can reason as in Remark 4.5 to see that the the size and rank bounds claimed for SA hold. The lemma follows. □

4.3. Lower Bound on Rank for Sherali-Adams Resolution

The *pigeon-rank* of a Sherali-Adams resolution refutation of $EPHP^k_{k-1}$ of the form described in Equation (2.1) is the maximum pigeon-degree of the polynomials to which the formulas $\prod_{i \in \mathcal{I}_t} x_i \cdot \prod_{j \in \mathcal{J}_t} (1 - x_j) \cdot P_t$ expand.

In order to prove a lower bound on pigeon-rank it is useful to generalize this concept to a more abstract notion of rank for SA proofs. Let V be a set of variables and let H be a non-empty downward-closed family of subsets of V, that is, such that if Y belongs to H and $X \subseteq Y$, then X also belongs to H. We say that a polynomial (or polynomial inequality) is *H-bounded*, or has *H-bounded rank*, if H contains the variable set of every monomial in it. We say that an SA derivation as in (2.1) has H-bounded rank if the polynomial to which each formula $\prod_{i \in \mathcal{I}_t} x_i \cdot \prod_{j \in \mathcal{J}_t} (1 - x_j) \cdot P_t$ expands is H-bounded. Observe that if an SA derivation has rank r, then it has H-bounded rank where H is the family of all subsets of at most r variables. Similarly, if an SA refutation of $EPHP^k_{k-1}$ has pigeon-rank r, then it has H-bounded rank where H is the family of all subsets of variables that mention at most r pigeons.

Let $\mathcal{P}$ be a set of polynomial inequalities over the variable set V. We say that $\mathcal{P}$ admits an *H-consistent family of distributions* if there exists a collection of probability distributions $\{\Pi_X\}_{X \in H}$ over assignments $\{0,1\}^X$ as X ranges over H that satisfy the following properties:

H1. For every variable set $X \in H$ and every polynomial inequality $Q \geq 0$ in $\mathcal{P}$ that has all its variables in X, it holds that all assignments in the support of Π_X satisfy $Q \geq 0$.

H2. For every pair of variable sets $X, Y \in H$ such that $X \subseteq Y$ and for every assignment $\mu \in \{0,1\}^X$ it holds that

$$\Pi_X(\mu) = \sum_{\substack{\eta \in \{0,1\}^Y \\ \eta \supseteq \mu}} \Pi_Y(\eta), \tag{4.15}$$

where η ranges over all assigments to Y that are consistent with μ.

In the definition above and elsewhere, $\Pi_X(\mu)$ denotes the probability assigned to μ by the distribution Π_X. We will use such H-consistent families of distributions to establish the Sherali-Adams rank lower bound that we need. Before stating the formal lemma that we will appeal to, let us try to provide some intuition.

If the set of polynomial inequalities $\mathcal{P}$ were satisfiable, then it would not be hard to come up with a family of probability distributions with properties H1 and H2: We could just fix a global probability distribution over all satisfying assignments and then let Π_X be the corresponding marginal distribution on any set of variables X. For an unsatisfiable set $\mathcal{P}$ there is no such globally consistent family, but if we can find an H-consistent family of distributions for $\mathcal{P}$, then $\mathcal{P}$ will still "look satisfiable" to any derivation that does not go "outside of H." Whenever we look at a specific inequality $Q \geq 0$ in $\mathcal{P}$, property H1 yields a "marginal distribution" that satisfies the inequality. Furthermore, property H2 ensures that such "marginal distributions" over different sets look locally consistent. The following lemma makes this precise.

LEMMA 4.7. *Let H be a downward-closed family of sets of variables and let $\mathcal{P}$ be a set of H-bounded polynomial inequalities. If $\mathcal{P}$ has an SA refutation of H-bounded rank, then $\mathcal{P}$ does not admit an H-consistent family of distributions.*

PROOF. Let us think of each X in H as a new formal variable. For each monomial M, let X_M denote the set of variables in M. If R is an H-bounded polynomial, then let us write $\widehat{R}$ to denote the linear form on the variables H obtained from R by replacing each term $c \cdot M$ by $c \cdot X_M$ and collecting all terms of the same variable into a single term by adding their coefficients (which could result in cancellations of terms). Note that $\widehat{R}$ can also be thought of as the multilinearization of R, namely, the polynomial obtained from R by removing all higher powers in the monomials to get $\widehat{M} = X_M$ instead of M. We write 1_Y to denote the assignment $\{x \mapsto 1 : x \in Y\}$ to a set of variables Y, and for a monomial M (multilinear or not), we define $1_M = 1_{X_M}$.

Let $\mathcal{P} = \{Q_1 \geq 0, \ldots, Q_m \geq 0\}$ be a set of polynomial inequalities and suppose that there exists an SA refutation of $\mathcal{P}$ of the form (2.1) that has H-bounded rank. Let us write R_t for the polynomial to which the formula $\prod_{i \in \mathcal{I}_t} x_i \cdot \prod_{j \in \mathcal{J}_t}(1 - x_j) \cdot P_t$ expands for $1 \leq t \leq \tau$. The assumption that the refutation has H-bounded rank means that every monomial in the polynomial R_t is H-bounded.

Assume for contradiction that $\mathcal{P}$ admits an H-consistent family $\{\Pi_X\}_{X \in H}$. Let $h : H \to \mathbb{R}$ be the real-valued assignment defined by

$$h(X) = \Pi_X(1_X), \tag{4.16}$$

that is, the probability of the all-ones assignment to the variables in X according to the distribution Π_X, and extend h to all linear forms on the variables $X \in H$ linearly; that is, if $L = \sum_i c_i X_i$ is such a linear form with coefficients c_i and variables X_i, then $h(L) = \sum_i c_i \cdot h(X_i)$.

We claim that h satisfies $h(\widehat{R}_t) \geq 0$ for every $1 \leq t \leq \tau$. By linearity, it then further follows that $h(\sum_{t=1}^{\tau} \alpha_t \widehat{R}_t) = \sum_{t=1}^{\tau} \alpha_t \cdot h(\widehat{R}_t) \geq 0$, where all $\alpha_t \geq 0$, which is a contradiction since $\sum_{t=1}^{\tau} \alpha_t R_t = -1$ and hence also $\sum_{t=1}^{\tau} \alpha_t \widehat{R}_t = -1$.

Let us prove that the assignment h as defined in Equation (4.16) satisfies every inequality $\widehat{R}_t \geq 0$ for $1 \leq t \leq \tau$. We do so by establishing a stronger claim: if X_t is the set of variables in R_t and $\mathbb{E}_{X_t}$ denotes expectation under the distribution Π_{X_t}, then the following holds:

A1. The assignment $h : H \to \mathbb{R}$ satisifies $h(\widehat{R}_t) = \mathbb{E}_{X_t}[R_t]$.
A2. Every assignment in the support of Π_{X_t} satisfies the inequality $R_t \geq 0$.

To see that A1 holds, we evaluate each monomial M in R_t separately to get

$$h(\widehat{M}) = h(X_M) = \Pi_{X_M}(1_M) = \sum_{\substack{\eta \in \{0,1\}^{X_t} \\ \eta \supseteq 1_M}} \Pi_{X_t}(\eta) = \mathbb{E}_{X_t}[M]. \tag{4.17}$$

The first and second equalities in (4.17) hold by definition, the third one follows from property H2 of H-consistent families of distributions, and the final equality is true since a monomial M evaluates to 1 under an assignment $\eta \in \{0,1\}^{X_t}$ if and only if η is compatible with 1_M. Adding over all terms, we get $h(\widehat{R}_t) = \mathbb{E}_{X_t}[R_t]$ by applying linearity of h on the left and linearity of expectation on the right.

To verify the claim in A2, consider an assignment η in the support of Π_{X_t}. Substituting the values assigned by η to the variables of R_t, we deduce that

$$\eta(R_t) = \eta\left(\prod_{i \in \mathcal{I}_t} x_i \cdot \prod_{j \in \mathcal{J}_t} (1 - x_j) \cdot P_t\right) = \prod_{i \in \mathcal{I}_t} \eta(x_i) \cdot \prod_{j \in \mathcal{J}_t} (1 - \eta(x_j)) \cdot \eta(P_t) \geq 0. \tag{4.18}$$

To see this, it suffices to observe that all factors in the final expression in 4.18 are nonnegative. First, regardless of what the assignment η is, we clearly have $\eta(x) \in \{0,1\}$ for any variable x in its domain and hence $\eta(x_i) \geq 0$ and $1 - \eta(x_j) \geq 0$. Second, from property H1 we know that if P_t is one of the polynomials Q_i in $\mathcal{P}$, then $\eta(P_t) \geq 0$ since η is in the support of Π_{X_t}. Third, if P_t is one of the axioms $x_i^2 - x_i$ or $x_i - x_i^2$ then $\eta(P_t) = 0$ since the range of η is $\{0,1\}$, and if P_t is the axiom 1, then of course $\eta(P_t) = 1 \geq 0$. This concludes the proof of the lemma. □

Dantchev et al. [2009] proved a rank lower bound on SAR refutations of PHP^k_{k-1}. Let us show how this result can be extended to a pigeon-rank lower bound for $EPHP^k_{k-1}$.

LEMMA 4.8. *Every SAR refutation of $EPHP^k_{k-1}$ has pigeon-rank at least k.*

PROOF. First, note that by replacing each variable $\overline{x}$ by $1 - x$ we transform an SAR proof into an SA proof of the same pigeon-rank. Thus, by Lemma 4.7 it will suffice to build an H-consistent family of distributions where H is the family of sets of variables that mention up to $k-1$ pigeons.

Intuitively, it is clear what the distributions should be: since there is room for up to $k-1$ pigeons in the pigeonholes, we can just choose any one-to-one mapping uniformly at random and set the Boolean variables accordingly. Formally, for every set X of variables that mention at most $k-1$ pigeons we define the distribution Π_X as follows:

(1) Let A be the set of at most $k-1$ pigeons that are mentioned by the variables in X.
(2) Let φ be a uniformly chosen one-to-one map $\varphi : A \to [k-1]$.
(3) For $q_{v,w} \in X$ set $q_{v,w} = 1$ if $\varphi(v) = w$ and $q_{v,w} = 0$ otherwise.
(4) For $z_{v,w} \in X$ set $z_{v,w} = 1$ if $\varphi(v) > w$ and $z_{v,w} = 0$ otherwise.

Let us verify that a family of distributions defined in this way satisfy properties H1 and H2.

That property H1 is satisfied is immediate by construction. If C is a clause in $EPHP^k_{k-1}$ with all variables contained in X, then all assignments in the support of Π_X satisfy C since they encode one-to-one mappings (with the extension variables $z_{v,w}$ set appropriately).

Property H2 is also straightforward to verify. Fix any sets X and Y such that $X \subseteq Y$ and that mention up to $k-1$ pigeons and any assignment $\mu \in \{0,1\}^X$. Let A and B be the sets of at most $k-1$ pigeons that are mentioned in X and Y, respectively, and note that $A \subseteq B$. Let us write $a = |A|$ and $b = |B|$. By construction, the assignments

$\eta \in \{0,1\}^Y$ in the support of Π_Y are in bijective correspondence with the one-to-one mappings $\psi : B \to [k-1]$ and the same holds for μ in the support of Π_X vis-a-vis $\varphi : A \to [k-1]$. Moreover, each one-to-one mapping $\varphi : A \to [k-1]$ can be chosen in $(k-1)(k-2)\cdots(k-a) = \binom{k-1}{a}a!$ ways, and for a fixed φ the number of one-to-one mappings $\psi : B \to [k-1]$ that extend φ is $(k-a-1)(k-a-2)\cdots(k-b) = \binom{k-1-a}{b-a}(b-a)!$. Since all involved distributions are uniform over their support, for $\mu \in \{0,1\}^X$ in the support of Π_X we have

$$\sum_{\substack{\eta\in\{0,1\}^Y \\ \eta\supseteq\mu}} \Pi_Y(\eta) = \sum_{\substack{\eta\,:\,\Pi_Y(\eta)>0 \\ \eta\supseteq\mu}} \frac{1}{\binom{k-1}{b}b!} = \frac{\binom{k-1-a}{b-a}(b-a)!}{\binom{k-1}{b}b!} = \frac{1}{\binom{k-1}{a}a!} = \Pi_X(\mu) \tag{4.19}$$

and for μ outside the support of Π_X the whole summation in (4.19) is zero. This finishes the proof of the lemma. □

4.4. Size Bounds for PCR and SAR Refutations

Given the lower bounds on pigeon-degree and pigeon-rank for refuting $EPHP^k_{k-1}$ in Lemmas 4.1 and 4.8, respectively, the size lower bounds on refutations of $ERPHP^{k,n}_{k-1}$ in polynomial calculus resolution and Sherali-Adams resolution are straightforward adaptions of the lower bound for resolution in Theorem 3.1. We write down the details here for completeness, starting with the PCR bounds.

THEOREM 4.9. *Let $k = k(n)$ be any integer-valued function such that $k(n) \le n/4\log n$. Then $ERPHP^{k,n}_{k-1}$ can be refuted in PCR in size $O(k^{k+1}n^k)$, and any PCR refutation requires size $\Omega(n^{\lceil (k-1)/2\rceil}/(4k\log n)^k)$.*

PROOF. Fix any PCR refutation of $ERPHP^{k,n}_{k-1}$ and let $\mathcal{M}$ be the set of monomials appearing in it. We hit the refutation with a random restriction ρ distributed according to $\mathcal{D}$. Since restrictions preserve PCR derivations we obtain a refutation of $ERPHP^{k,n}_{k-1}\restriction_\rho$, which, as before, is $EPHP^k_{k-1}$ after renaming of variables.

Assume that $|\mathcal{M}| < n^{\lceil (k-1)/2\rceil}/(4k\log n)^k$. Applying Lemma 3.2 with $\ell = \left\lceil \frac{k-1}{2} \right\rceil$ and taking a union bound over the monomials in $\mathcal{M}$, we conclude that there must be at least one restriction ρ in the support of $\mathcal{D}$ such that the pigeon-degree of $\pi\restriction_\rho$ is at most $\left\lceil \frac{k-1}{2} \right\rceil - 1$ if n is large enough. This contradicts Lemma 4.1, and hence $|\mathcal{M}|$ must be at least $n^{\lceil (k-1)/2\rceil}/(4k\log n)^k$.

To obtain the upper bound we start with the resolution refutation in Theorem 3.1. It is not hard to see that any resolution refutation of size S and width w translates into a PCR refutation of size wS and degree $w+1$. The additional factor w in the size is due to the fact that while resolution can arbitrary weaken a clause in one step, the way multiplication is defined in PCR means that we need one multiplication step per literal to simulate the same weakening. □

The proof of the bounds for Sherali-Adams is very similar.

THEOREM 4.10. *Let $k = k(n)$ be any integer-valued function such that $k(n) \le n/4\log n$. Then $ERPHP^{k,n}_{k-1}$ can be refuted in SAR in size $O(k^{k+2}n^k)$, and any SAR refutation requires size $\Omega(n^k/(4k\log n)^k)$.*

PROOF. Fix any SAR refutation of $ERPHP^{k,n}_{k-1}$ and let $\mathcal{M}$ be the set of monomials appearing in it. Hit the refutation with a random restriction ρ distributed according

to $\mathcal{D}$. Since restrictions preserve soundness of SAR proofs, this yields a refutation of $ERPHP^{k,n}_{k-1}\restriction_\rho$, which is $EPHP^k_{k-1}$.

Suppose now that $|\mathcal{M}| < n^k/(4k\log n)^k$. Using Lemma 3.2 with $\ell = k$ and a union bound argument for $\mathcal{M}$, we conclude that there exists at least one restriction ρ in the support of $\mathcal{D}$ such that the pigeon-rank of $\pi\restriction_\rho$ is at most $k-1$, assuming that n large enough. But this contradicts Lemma 4.8, and hence the lower bound in the theorem follows.

We obtain the upper bound by using the simulation in Lemma 4.6 on the resolution refutation in Theorem 3.1. □

5. AN UPPER BOUND FOR RELATIVIZED PHP FORMULAS IN LASSERRE

In this section, we show that our lower bound in Theorem 1.1 does not generalize to Lasserre. Indeed, the formulas $ERPHP^{k,n}_{k-1}$ (and also $RPHP^{k,n}_{k-1}$) have Lasserre refutations in constant rank and hence polynomial size. To establish this we will use the easily verified identity

$$\sum_{\substack{i,j\in[n]\\ i\neq j}} (1-z_i-z_j)z_j + (n-2)\sum_{j\in[n]}\left(z_j^2-z_j\right) + \left(1-\sum_{i\in[n]} z_i\right)^2 = 1-\sum_{i\in[n]} z_i \tag{5.1}$$

a couple of times. A direct application of 5.1 shows that the inequality $1-\sum_{i\in[n]} z_i \geq 0$ has a rank-2 Lasserre derivation from the set of all inequalities of the form $1-z_i-z_j \geq 0$ for $i, j \in [n]$, $i \neq j$. We remark that this fact is a direct consequence of Lemma 1.5 in Lovász and Schrijver [1991]. Let us first use this to get a rank-2 Lasserre refutation of the standard pigeonhole principle PHP^k_{k-1} encoded as the set of clauses

$$x_{u,1} \vee x_{u,2} \vee \cdots \vee x_{u,k-1} \qquad u \in [k], \tag{5.2a}$$

$$\overline{x}_{u,w} \vee \overline{x}_{v,w} \qquad u, v \in [k],\ u \neq v,\ w \in [k-1]. \tag{5.2b}$$

The proof we give next is essentially from Grigoriev et al. [2002].

Proposition 5.1 (Grigoriev et al. [2002]). *The formulas PHP^k_{k-1} have Lasserre refutations of rank 2.*

Proof. Combining all hole axioms $1-x_{u,w}-x_{v,w} \geq 0$ in Equation (5.2b) for a fixed hole $w \in [k-1]$ and using Equation (5.1) we can get the inequality $1-\sum_{u\in[k]} x_{u,w} \geq 0$. Adding these inequality over all holes $w \in [k-1]$ we obtain

$$k-1-\sum_{u\in[k]}\sum_{w\in[k-1]} x_{u,w} \geq 0. \tag{5.3}$$

Adding together instead all the pigeon axioms $\sum_{w\in[k-1]} x_{u,w} - 1 \geq 0$ in Equation (5.2a), we get

$$\sum_{u\in[k]}\sum_{w\in[k-1]} x_{u,w} - k \geq 0. \tag{5.4}$$

Summing Equations (5.3) and (5.4) yields $-1 \geq 0$. □

Two more applications of Equation (5.1) will help us get rank-9 Lasserre refutations of $RPHP^{k,n}_{k-1}$ (and $ERPHP^{k,n}_{k-1}$) by reduction to PHP^k_{k-1}. The main idea of the proof is to substitute variables in the derivation in Proposition 5.1 with polynomials defined over the variables of $RPHP^{k,n}_{k-1}$.

PROPOSITION 5.2. *The formulas* $RPHP^{k,n}_{k-1}$ *and* $ERPHP^{k,n}_{k-1}$ *have Lasserre refutations of rank* 7 *and size polynomial in* n.

PROOF. The bound on size will follow from the bound on rank since the number of monomials that are produced by a constant rank refutation is bounded by a fixed polynomial in the number of variables. Next, let us observe that it suffices to get a Lasserre refutation of $RPHP^{k,n}_{k-1}$ since, once we have one, it is easy to convert it to a refutation of $ERPHP^{k,n}_{k-1}$ of the same rank. This follows from the observation that the encoding of a wide clause $C = a_1 \vee \ldots \vee a_w$ is the sum of the encodings of the corresponding 3-clauses $a_1 \vee a_2 \vee z_2$, $\overline{z}_2 \vee a_3 \vee z_3, \ldots, \overline{z}_{w-2} \vee a_{w-1} \vee a_w$. Thus, once we have a refutation of $RPHP^{k,n}_{k-1}$ we can get a valid refutation of $ERPHP^{k,n}_{k-1}$ of the same rank by substituting the sum of the corresponding short axioms in $ERPHP^{k,n}_{k-1}$ for any long axiom in $RPHP^{k,n}_{k-1}$.

Therefore, for the rest of the proof we focus on $RPHP^{k,n}_{k-1}$. Let $\mathcal{P}$ be the set of polynomial inequalities that encode it and let us define the shorthand

$$x_{u,w} = \sum_{\ell \in [n]} p_{u,\ell}\, r_\ell\, q_{\ell,w}. \tag{5.5}$$

We want to use the proof of the pigeonhole principle in Proposition 5.1 together with the substitution (5.5) for $x_{u,w}$. In order to do so, we need to show how to derive the substituted axioms used in that proof.

The inequalities $x^2_{u,w} - x_{u,w} \geq 0$ can be obtained by summing

$$\sum_{\substack{\ell,m \in [n] \\ \ell \neq m}} p_{u,\ell}\, r_\ell\, q_{\ell,w}\, p_{u,m} r_m q_{m,w} \geq 0 \tag{5.6}$$

and

$$\sum_{\ell \in [n]} \left(p^2_{u,\ell} - p_{u,\ell}\right) r^2_\ell\, q^2_{\ell,w} + \left(r^2_\ell - r_\ell\right) p_{u,\ell}\, q^2_{\ell,w} + \left(q^2_{\ell,w} - q_{\ell,w}\right) p_{u,\ell}\, r_\ell \geq 0, \tag{5.7}$$

and these inequalities have direct rank-6 derivations not even using $\mathcal{P}$. To derive the inequalities $\sum_{w \in [k-1]} x_{u,w} - 1 \geq 0$ for $u \in [k]$ we can sum up

$$\sum_{\ell \in [n]} \left(\sum_{w \in [k-1]} q_{\ell,w} - r_\ell \right) p_{u,\ell}\, r_\ell \geq 0, \tag{5.8}$$

$$\sum_{\ell \in [n]} \left(r^2_\ell - r_\ell\right) p_{u,\ell} + \sum_{\ell \in [n]} \left(r_\ell - p_{u,\ell}\right) p_{u,\ell} + \sum_{\ell \in [n]} \left(p^2_{u,\ell} - p_{u,\ell}\right) \geq 0, \tag{5.9}$$

and

$$\sum_{\ell \in [n]} p_{u,\ell} - 1 \geq 0, \tag{5.10}$$

which can all be derived directly from $\mathcal{P}$ in rank 3. The inequality $1 - x_{u,w} - x_{v,w} \geq 0$ is the sum of

$$\sum_{\ell \in [n]} (1 - p_{u,\ell} - p_{v,\ell}) r_\ell\, q_{\ell,w} \geq 0 \tag{5.11}$$

and

$$1 - \sum_{\ell \in [n]} r_\ell q_{\ell,w} \geq 0, \tag{5.12}$$

where (5.11) has a direct rank-3 derivation from $\mathcal{P}$. For Equation (5.12), we need to do some more work. Fix indices $\ell, m \in [n]$ with $\ell \neq m$ and observe that

$$\begin{aligned}\left(1 - r_\ell q_{\ell,w} - r_m q_{m,w}\right) r_\ell q_{\ell,w} \;=\; & \left(3 - r_\ell - r_m - q_{\ell,w} - q_{m,w}\right) q_{\ell,w}\, r_m\, q_{m,w} \\ & + \left(q_{\ell,w}^2 - q_{\ell,w}\right) r_m\, q_{m,w} + \left(q_{m,w}^2 - q_{m,w}\right) r_m\, q_{\ell,w} \\ & + \left(r_m^2 - r_m\right) q_{\ell,w} q_{m,w} + \left(r_\ell - r_\ell^2\right) q_{\ell,w} + \left(q_{\ell,w} - q_{\ell,w}^2\right) r_\ell^2. \end{aligned} \tag{5.13}$$

Note that the first term on the right-hand side of this equation is the polynomial translation of axiom (3.1e). Writing z_ℓ for $r_\ell\, q_{\ell,w}$, this shows that the inequality $(1 - z_\ell - z_m) z_\ell \geq 0$ has a rank-4 derivation from $\mathcal{P}$. Combined with the fact that $z_\ell^2 - z_\ell = (r_\ell^2 - r_\ell) q_{\ell,w}^2 + (q_{\ell,w}^2 - q_{\ell,w}) r_\ell$, Equation (5.1) gives a rank-4 derivation of $1 - \sum_{\ell \in [n]} z_\ell \geq 0$. This is precisely Equation (5.12).

Now we mimic the refutation of PHP_{k-1}^k in Proposition 5.1. For a fixed $w \in [k-1]$ we can use the derivations of $1 - x_{u,w} - x_{v,w} \geq 0$ and $x_{v,w}^2 - x_{v,w} \geq 0$ in combination with (5.1) to obtain the inequality $1 - \sum_{u \in [k]} x_{u,w} \geq 0$ by a rank-7 derivation. Adding all such inequalities for $w \in [k-1]$ gives

$$k - 1 - \sum_{w \in [k-1]} \sum_{u \in [k]} x_{u,w} \geq 0. \tag{5.14}$$

On the other hand, adding $\sum_{w \in [k-1]} x_{u,w} - 1 \geq 0$ over all $u \in [k]$ yields

$$\sum_{u \in [k]} \sum_{w \in [k-1]} x_{u,w} - k \geq 0 \tag{5.15}$$

in rank 3 (the rank of the derivation of $\sum_{w \in [k-1]} x_{u,w} - 1 \geq 0$), and a final addition allows us to derive $-1 \geq 0$, never going above rank 7. □

6. CONCLUDING REMARKS

In this article, we exhibit a family of 3-CNF formulas over n variables that can be refuted in resolution in width w but require refutations of size $n^{\Omega(w)}$. Furthermore, this lower bound can be extended to PCR and Sherali-Adams, but not to Lasserre, where we give an upper bound that is polynomial in both n and w for the formula we study. This shows that the seemingly naive counting upper bounds on proof size in terms of width for resolution, degree for PCR, and rank for Sherali-Adams are actually all tight up to small constant factors in the exponent. As noted in Section 1.4, subsequent work in Lauria and Nordström [2015] has shown that a different but very much related formula family achieves the same conclusions also for Lasserre, albeit for much weaker settings of parameters.

Regarding open problems, perhaps the most obvious one concerns the tightness of our result. Our formulas have roughly $N = n^2$ variables and are refutable in width roughly $w = 2k$, and our size lower bounds are on the order of $n^k = N^{w/4}$. However, the direct counting argument for width w gives an upper bound of about N^w. Could this gap in the exponent be closed? If so, then this would have to be for a different formula family since ours has an upper bound of roughly $n^k = N^{w/4}$. One point worth noting is that one can shave a factor 2 off the gap in the exponent by considering the 4-CNF formulas obtained if the 4-clauses in Equation (3.1e) are *not* converted to 3-CNF. In this case, the same upper and lower bounds still hold, but the number of variables is on the

order of $N = kn$, which means that we get a lower bound of the form $N^{w/2}$ if we focus on widths w that are bounded by a constant. In all the remarks above, the upper bounds refer to resolution, and the lower bounds refer to resolution, PCR, or Sherali-Adams. For Lasserre, however, the new results in Lauria and Nordström [2015] are currently far from being this tight.

A natural formula for which it would be interesting to prove similar size lower bounds as in this article is the so-called *clique formula* claiming that there is a k-clique in some fixed n-vertex graph chosen so this claim is false. It has been conjectured (e.g., in Beyersdorff et al. [2012]) that such formulas require resolution refutation size $n^{\Omega(k)}$ for the right kind of graphs, and this has been proven for the restricted case of treelike resolution [Beyersdorff et al. 2013]. If such a lower bound could be established for general resolution, then it would have interesting consequences for parameterized proof complexity.

Finally, in the earlier version of this article [Atserias et al. 2014], we asked about the necessity, or not, of the exponential blow-up in size incurred by the transformation that takes short resolution refutations into narrow ones in Ben-Sasson and Wigderson [2001]. As discussed in Section 1.4, Neil Thapen addressed this question after the first version of this article had been published and showed that the blow-up cannot be avoided [Thapen 2014]. However, it should be noted that Thapen's result concerns only formulas of logarithmic initial width; for CNF formulas of constant initial width (that is, k-CNFs with constant k), the question remains open, although it does not seem impossible that an extension of Thapen's technique could be made to work for this case. A more fundamental challenge would be to prove also the necessity of an exponential blow-up in the transformation in Impagliazzo et al. [1999] that takes small-size polynomial calculus (PC or PCR) refutations into small-degree ones.

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