Beyond Satisfaction

Towards an Understanding of Real-World Efficient Computation

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KTH EECS

SRA ICT TNG Research Challenge Day
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This Is Me...

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. . . And This Is My Research Challenge

$(x_{1,1} \lor x_{1,2} \lor x_{1,3} \lor x_{1,4} \lor x_{1,5} \lor x_{1,6} \lor x_{1,7}) \land (x_{2,1} \lor x_{2,2} \lor x_{2,3} \lor x_{2,4} \lor x_{2,5} \lor x_{2,6} \lor x_{2,7}) \land (x_{3,1} \lor x_{3,2} \lor x_{3,3} \lor x_{3,4} \lor x_{3,5} \lor x_{3,6} \lor x_{3,7}) \land (x_{4,1} \lor x_{4,2} \lor x_{4,3} \lor x_{4,4} \lor x_{4,5} \lor x_{4,6} \lor x_{4,7}) \land (x_{5,1} \lor x_{5,2} \lor x_{5,3} \lor x_{5,4} \lor x_{5,5} \lor x_{5,6} \lor x_{5,7}) \land (x_{6,1} \lor x_{6,2} \lor x_{6,3} \lor x_{6,4} \lor x_{6,5} \lor x_{6,6} \lor x_{6,7}) \land (x_{7,1} \lor x_{7,2} \lor x_{7,3} \lor x_{7,4} \lor x_{7,5} \lor x_{7,6} \lor x_{7,7}) \land (x_{8,1} \lor x_{8,2} \lor x_{8,3} \lor x_{8,4} \lor x_{8,5} \lor x_{8,6} \lor x_{8,7}) \land (\overline{x}_{1,1} \lor \overline{x}_{2,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{3,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{4,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{5,1}) \land (\overline{x}_{1,1} \lor 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Three Problems. . .

Colouring

Does graph $G = (V, E)$ have a colouring with k colours so that neighbours have distinct colours?

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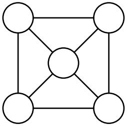

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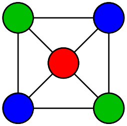

3-colouring exists

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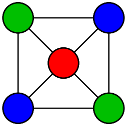

3-colouring exists but no 2-colouring

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Clique

Is there a clique in graph $G = (V, E)$ with k vertices that are all pairwise connected?

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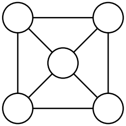

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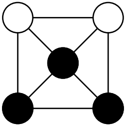

3-clique exists

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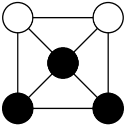

3-clique exists but no 4-clique

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Sat

Given propositional logic formula F, is there a satisfying assignment?

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Sat

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Colouring:	frequency allocation for mobile base stations
Clique:	bioinformatics, computational chemistry
Sat:	easily models these and many other problems

. . . That Are Impossible to Solve in Theory. . .

All three problems NP-complete [Coo71, Lev73, Kar72]

Conventional wisdom $\Rightarrow$ infeasible to solve in practice

Even practically impossible to find approximate solution in any meaningful sense [Kho01, Zuc07, Hås99, Hås01]

. . . But Easy in Practice?!

SAT

Conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ+01]
Deal with real-world instances containing millions of variables
Often run in (close to) linear time!

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Colouring

Award-winning sequence of papers [DLMM08, DLMO09, DLMM11]

Relatively simple linear algebra methods

Authors report being unable to find hard instances!

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- Have exponential hardness results for worst-case running time under plausible mathematical assumptions

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- Have exponential hardness results for worst-case running time under plausible mathematical assumptions
- But these worst-case lower bounds don't seem very relevant for "real-case" problems and algorithms
- For some of these algorithms we can't even rule out that they would solve NP-complete problems in linear time (also seems preposterous)
- Since we're not really able to analyse these algorithms, it's very hard to understand
  - when and why they sometimes fail miserably
  - how to improve them

Long-Term Research Goals

- Strengthen the mathematical analysis of algorithmic methods
- Construct stronger algorithms for combinatorial problems
- Develop a better understanding of real-world efficient computation

Mathematical Analysis of Algorithmic Methods

Study methods of reasoning that are powerful enough to capture state-of-the-art algorithms used in practice

Use mathematical tools to establish theorems abut the power and limitations of such algorithms and methods

Recent examples:

- Lower bound $\gtrapprox n^k$ for algorithms [Pro12, McC17] for k-CLIQUE in [ABdR$^+$18]
- Exponential lower bounds for algebraic algorithms [DLMM08, DLMO09, DLMM11] for COLOURING in [MN15, LN17]

Stronger Algorithms for Combinatorial Problems

Use insights into stronger mathemathical methods of reasoning to build algorithms for SAT and other NP-complete problems

Goal: More efficient algorithms having the potential to go significantly beyond state of the art

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- Pseudo-Boolean solver [EN18] performing very well in the pseudo-Boolean competitions 2015 and 2016 [Pse15, Pse16]
- Try to push further to, e.g.,
  - Pseudo-Boolean optimization
  - Integer linear programming (ILP)
  - Mixed integer linear programming (MIP)
  - Constraint programming (CP)
  - Satisfiability modulo theories (SMT)

Better Understanding of Real-World Efficient Computation

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1. Model algorithms as closely as possible and prove rigorous theorems
2. Using these theoretical insights, carefully construct *extremal* benchmarks w.r.t. different complexity-theoretic properties
3. Cannot prove anything *formally*, but theory intuition tells us that instances are likely to be challenging for different heuristics
4. So run experiments on these benchmarks to shed light on
   - what impact each heuristic has on performance
   - how this correlates with theoretical properties
5. Since benchmarks are crafted they are also *scalable*, meaning we can study *how performance scales as the instance size increases*

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Thank you for your attention!

References I

[ABdR+18] Albert Atserias, Ilario Bonacina, Susanna F. de Rezende, Massimo Lauria, Jakob Nordström, and Alexander Razborov. Clique is hard on average for regular resolution. In *Proceedings of the 50th Annual ACM Symposium on Theory of Computing (STOC '18)*, June 2018. To appear.

[BS97] Roberto J. Bayardo Jr. and Robert Schrag. Using CSP look-back techniques to solve real-world SAT instances. In *Proceedings of the 14th National Conference on Artificial Intelligence (AAAI '97)*, pages 203–208, July 1997.

[Coo71] Stephen A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the 3rd Annual ACM Symposium on Theory of Computing (STOC '71)*, pages 151–158, 1971.

[DLMM08] Jesús A. De Loera, Jon Lee, Peter N. Malkin, and Susan Margulies. Hilbert's Nullstellensatz and an algorithm for proving combinatorial infeasibility. In *Proceedings of the 21st International Symposium on Symbolic and Algebraic Computation (ISSAC '08)*, pages 197–206, July 2008.

[DLMM11] Jesús A. De Loera, Jon Lee, Peter N. Malkin, and Susan Margulies. Computing infeasibility certificates for combinatorial problems through Hilbert's Nullstellensatz. *Journal of Symbolic Computation*, 46(11):1260–1283, November 2011.

References II

[DLMO09] Jesús A. De Loera, Jon Lee, Susan Margulies, and Shmuel Onn. Expressing combinatorial problems by systems of polynomial equations and Hilbert's Nullstellensatz. *Combinatorics, Probability and Computing*, 18(04):551–582, July 2009.

[EGG+18] Jan Elffers, Jesús Giráldez-Cru, Stephan Gocht, Jakob Nordström, and Laurent Simon. Seeking practical CDCL insights from theoretical SAT benchmarks. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence and the 23rd European Conference on Artificial Intelligence (IJCAI-ECAI '18)*, July 2018. To appear.

[EGNV18] Jan Elffers, Jesús Giráldez-Cru, Jakob Nordström, and Marc Vinyals. Using combinatorial benchmarks to probe the reasoning power of pseudo-Boolean solvers. In *Proceedings of the 21st International Conference on Theory and Applications of Satisfiability Testing (SAT '18)*, July 2018. To appear.

[EN18] Jan Elffers and Jakob Nordström. Divide and conquer: Towards faster pseudo-Boolean solving. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence and the 23rd European Conference on Artificial Intelligence (IJCAI-ECAI '18)*, July 2018. To appear.

References III

[Hås99] Johan Håstad. Clique is hard to approximate within $n^{1-\epsilon}$. *Acta Mathematica*, 182:105–142, 1999. Preliminary version in *FOCS '96*.

[Hås01] Johan Håstad. Some optimal inapproximability results. *Journal of the ACM*, 48(4):798–859, July 2001. Preliminary version in *STOC '97*.

[Kar72] Richard M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, The IBM Research Symposia Series, pages 85–103. Springer, 1972.

[Kho01] Subhash Khot. Improved inapproximability results for MaxClique, chromatic number and approximate graph coloring. In *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science (FOCS '01)*, pages 600–609, October 2001.

[Lev73] Leonid A. Levin. Universal sequential search problems. *Problemy peredachi informatsii*, 9(3):115–116, 1973. In Russian. Available at `http://mi.mathnet.ru/ppi914`.

References IV

[LN17] Massimo Lauria and Jakob Nordström. Graph colouring is hard for algorithms based on Hilbert's Nullstellensatz and Gröbner bases. In *Proceedings of the 32nd Annual Computational Complexity Conference (CCC '17)*, volume 79 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 2:1–2:20, July 2017.

[McC17] Ciaran McCreesh. *Solving Hard Subgraph Problems in Parallel*. PhD thesis, University of Glasgow, 2017.

[MMZ+01] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 38th Design Automation Conference (DAC '01)*, pages 530–535, June 2001.

[MN15] Mladen Mikša and Jakob Nordström. A generalized method for proving polynomial calculus degree lower bounds. In *Proceedings of the 30th Annual Computational Complexity Conference (CCC '15)*, volume 33 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 467–487, June 2015.

[MS99] João P. Marques-Silva and Karem A. Sakallah. GRASP: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers*, 48(5):506–521, May 1999. Preliminary version in *ICCAD '96*.

References V

[Pro12] Patrick Prosser. Exact algorithms for maximum clique: A computational study. *Algorithms*, 5(4):545–587, 2012.

[Pse15] Pseudo-Boolean evaluation 2015. `http://pbeva.computational-logic.org/`, September 2015.

[Pse16] Pseudo-Boolean competition 2016. `http://www.cril.univ-artois.fr/PB16/`, July 2016.

[VEG+18] Marc Vinyals, Jan Elffers, Jesús Giráldez-Cru, Stephan Gocht, and Jakob Nordström. In between resolution and cutting planes: A study of proof systems for pseudo-Boolean SAT solving. In *Proceedings of the 21st International Conference on Theory and Applications of Satisfiability Testing (SAT '18)*, July 2018. To appear.

[Zuc07] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6):103–128, August 2007. Preliminary version in *STOC '06*.