Understanding Space in Resolution: Optimal Lower Bounds and Exponential Trade-offs

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Joint work with Eli Ben-Sasson
A Fundamental Problem in Computer Science

Problem

Given a propositional logic formula $F$, is it true no matter how we assign values to its variables?

TAUTOLOGY: Fundamental problem in Theoretical Computer Science since Cook’s NP-completeness paper (1971)

Last decade or so: also intense applied interest

Enormous progress on algorithms (although still exponential time in worst case)
Proof Complexity

Proof search algorithm: proof system with derivation rules

Proof complexity: study of proofs in such systems

- **Lower bounds**: no algorithm can do better (even optimal one always guessing the right move)
- **Upper bounds**: gives hope for good algorithms if we can search for proofs in system efficiently
Resolution

- Prove tautologies ⇔ refute unsatisfiable formulas in conjunctive normal form (CNF)
- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Basis of current state-of-the-art SAT-solvers (winners in SAT 2008 competition: resolution + clause learning)
- Key resources: time and space
- What are the connections between these resources? Are time and space correlated? Are there time/space trade-offs?
Outline

1 Resolution
   • Basics
   • Some Previous Work
   • Our Results

2 Outline of Proofs
   • Substitution Space Theorem
   • Pebble Games and Pebbling Contradictions
   • Putting the Pieces Together

3 Open Problems
Some Notation and Terminology

- **Literal** \( a \): variable \( x \) or its negation \( \overline{x} \)

- **Clause** \( C = a_1 \lor \ldots \lor a_k \): disjunction of literals
  At most \( k \) literals: **\( k \)-clause**

- **CNF formula** \( F = C_1 \land \ldots \land C_m \): conjunction of clauses
  **\( k \)-CNF formula**: CNF formula consisting of \( k \)-clauses
  (assume \( k \) fixed)

- Refer to clauses of CNF formula as **axioms**
  (as opposed to derived clauses)
Resolution Rule

Resolution rule:

\[
\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}
\]

Observation

If \( F \) is a satisfiable CNF formula and \( D \) is derived from clauses \( C_1, C_2 \in F \) by the resolution rule, then \( F \land D \) is satisfiable.

Prove \( F \) unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from \( F \) by resolution.
Resolution Rule:

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Resolution Rule

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Prove \( F \) unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from \( F \) by resolution.
Example CNF Formula

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Defined in terms of directed acyclic graph (DAG):
- source vertices true
- truth propagates upwards
- but sink vertex is false
Example CNF Formula

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Defined in terms of directed acyclic graph (DAG):
- source vertices true
- truth propagates upwards
- but sink vertex is false
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
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Can write down axioms, erase used clauses or infer new clauses (but only from clauses currently on the board!)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
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Write down axiom 1: \( u \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

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Write down axiom 1: $u$

Write down axiom 2: $v$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Write down axiom 1: $u$
Write down axiom 2: $v$
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Write down axiom 1: $u$
Write down axiom 2: $v$
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer clause $\overline{v} \lor x$ from clauses $u$ and $\overline{u} \lor \overline{v} \lor x$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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Write down axiom 1: $u$
Write down axiom 2: $v$
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer clause $\overline{v} \lor x$ from clauses $u$ and $\overline{u} \lor \overline{v} \lor x$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Write down axiom 2: $v$
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer clause $\overline{v} \lor x$ from clauses $u$ and $\overline{u} \lor \overline{v} \lor x$
Erase clause $\overline{u} \lor \overline{v} \lor x$
### Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( x \lor \overline{y} \lor z \)
7. \( z \)

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Write down axiom 2: \( v \)

Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)

Infer clause \( \overline{v} \lor x \) from clauses \( u \) and \( \overline{u} \lor \overline{v} \lor x \)

Erase clause \( \overline{u} \lor \overline{v} \lor x \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $x \lor \overline{y} \lor z$
7. $\overline{z}$

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Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer clause $\overline{v} \lor x$ from clauses $u$ and $\overline{u} \lor \overline{v} \lor x$
Erase clause $\overline{u} \lor \overline{v} \lor x$
Erase clause $u$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)
Infer clause \( \overline{v} \lor x \) from clauses \( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Erase clause \( \overline{u} \lor \overline{v} \lor x \)
Erase clause \( u \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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clauses $u$ and $\overline{u} \lor \overline{v} \lor x$
Erase clause $\overline{u} \lor \overline{v} \lor x$
Erase clause $u$
Infer clause $x$ from clauses $v$ and $\overline{v} \lor x$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $u \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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clauses $u$ and $u \lor \overline{v} \lor x$
Erase clause $u \lor \overline{v} \lor x$
Erase clause $u$
Infer clause $x$ from
clauses $v$ and $\overline{v} \lor x$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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\(v\)

\(\overline{v} \lor x\)

\(x\)

Erase clause \(\overline{u} \lor \overline{v} \lor x\)

Erase clause \(u\)

Infer clause \(x\) from clauses \(v\) and \(\overline{v} \lor x\)

Erase clause \(\overline{v} \lor x\)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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- Erase clause $\overline{u} \lor \overline{v} \lor x$
- Erase clause $u$
- Infer clause $x$ from clauses $v$ and $\overline{v} \lor x$
- Erase clause $\overline{v} \lor x$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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Infer clause $x$ from clauses $v$ and $\overline{v} \lor x$

Erase clause $u$

Erase clause $\overline{v} \lor x$

Erase clause $v$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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Erase clause $u$
Infer clause $x$ from clauses $v$ and $\overline{v} \lor x$
Erase clause $\overline{v} \lor x$
Erase clause $v$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Infer clause $x$ from clauses $v$ and $\overline{v} \lor x$

Erase clause $\overline{v} \lor x$

Erase clause $v$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor v \lor x$
5. $v \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Erase clause $\overline{v} \lor x$
Erase clause $v$
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer clause $\overline{y} \lor z$ from clauses $x$ and $\overline{x} \lor \overline{y} \lor z$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \vee \overline{v} \vee x \)
5. \( \overline{v} \vee \overline{w} \vee y \)
6. \( \overline{x} \vee \overline{y} \vee z \)
7. \( \overline{z} \)

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Erase clause \( \overline{v} \vee x \)
Erase clause \( v \)
Write down axiom 6: \( \overline{x} \vee \overline{y} \vee z \)
Infer clause \( \overline{y} \vee z \) from clauses \( x \) and \( \overline{x} \vee \overline{y} \vee z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Erase clause \( v \)
Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)
Erase clause \( \overline{x} \lor \overline{y} \lor z \)
Example Resolution Refutation

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1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Erase clause \( v \)

Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( \overline{x} \lor \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor v \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( x \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[ \overline{y} \lor z \]

Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( x \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor v \lor x \)
5. \( v \lor \overline{w} \lor y \)
6. \( x \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\( \overline{y} \lor z \)

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( x \)

Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Erase clause \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( x \)

Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)

Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \bar{u} \lor \bar{v} \lor x \)
5. \( \bar{v} \lor \bar{w} \lor y \)
6. \( \bar{x} \lor \bar{y} \lor z \)
7. \( \bar{z} \)

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\( \bar{y} \lor z \)
\( \bar{v} \lor \bar{w} \lor y \)
\( \bar{v} \lor \bar{w} \lor z \)

Erase clause \( \bar{x} \lor \bar{y} \lor z \)
Erase clause \( x \)
Write down axiom 5: \( \bar{v} \lor \bar{w} \lor y \)
Infer clause \( \bar{v} \lor \bar{w} \lor z \) from clauses \( \bar{y} \lor z \) and \( \bar{v} \lor \bar{w} \lor y \)
### Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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#### Erase clause \( x \)

Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)

Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)

#### Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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$y \lor z$

$\overline{v} \lor \overline{w} \lor y$

Erase clause $x$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer clause $\overline{v} \lor \overline{w} \lor z$ from clauses $y \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase clause $\overline{v} \lor \overline{w} \lor y$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[ \overline{y} \lor z \]
\[ \overline{v} \lor \overline{w} \lor z \]

Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( \neg v \)
3. \( w \)
4. \( \neg u \lor \neg v \lor x \)
5. \( \neg v \lor \neg w \lor y \)
6. \( \neg x \lor \neg y \lor z \)
7. \( \neg z \)

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\[ \neg v \lor \neg w \lor z \]
\[ \neg v \lor \neg w \lor y \]

Infer clause \( \neg v \lor \neg w \lor z \) from clauses \( \neg y \lor z \) and \( \neg v \lor \neg w \lor y \)

Erase clause \( \neg v \lor \neg w \lor y \)

Erase clause \( \neg y \lor z \)

Write down axiom 2: \( v \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\( v \lor w \lor z \)  
\( v \lor y \lor z \)  
Write down axiom 2: \( v \)  
Write down axiom 3: \( w \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping

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Erase clause $\overline{v} \lor \overline{w} \lor y$
Erase clause $\overline{y} \lor z$
Write down axiom 2: $v$
Write down axiom 3: $w$
Write down axiom 7: $\overline{z}$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor \overline{x} \)
5. \( \overline{v} \lor \overline{w} \lor \overline{y} \)
6. \( \overline{x} \lor \overline{y} \lor \overline{z} \)
7. \( \overline{z} \)

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Write down axiom 2: \( v \)
Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Infer clause \( \overline{w} \lor \overline{z} \) from clauses \( v \) and \( \overline{v} \lor \overline{w} \lor \overline{z} \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor v \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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- Write down axiom 2: $v$
- Write down axiom 3: $w$
- Write down axiom 7: $\overline{z}$
- Infer clause $\overline{w} \lor z$ from clauses $v$ and $\overline{v} \lor \overline{w} \lor z$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[ \overline{V} \lor \overline{W} \lor z \]
\[ \overline{V} \]
\[ \overline{W} \]
\[ \overline{Z} \]
\[ \overline{W} \lor z \]

Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Infer clause \( \overline{w} \lor z \) from clauses \( v \) and \( \overline{V} \lor \overline{W} \lor z \)
Erase clause \( v \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor v \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( x \lor y \lor z \)
7. \( \overline{z} \)

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\[ \overline{v} \lor \overline{w} \lor z \]
\[ w \]
\[ \overline{z} \]
\[ \overline{w} \lor z \]

Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Infer clause \( \overline{w} \lor z \) from clauses \( v \) and \( \overline{v} \lor \overline{w} \lor z \)

Erase clause \( v \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor v \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Write down axiom 7: \( \overline{z} \)
Infer clause \( \overline{w} \lor z \) from clauses \( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Erase clause \( v \)
Erase clause \( \overline{v} \lor \overline{w} \lor z \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Write down axiom 7: $\overline{z}$

Infer clause $\overline{w} \lor z$ from clauses $v$ and $\overline{v} \lor \overline{w} \lor z$

Erase clause $v$

Erase clause $\overline{v} \lor \overline{w} \lor z$
## Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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### Blackboard:
- $w$
- $\overline{z}$
- $\overline{w} \lor z$

### Inferences:
- clauses $v$ and $\overline{v} \lor \overline{w} \lor z$
- Erase clause $v$
- Erase clause $\overline{v} \lor \overline{w} \lor z$
- Infer clause $z$ from clauses $w$ and $\overline{w} \lor z$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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clauses $v$ and $\overline{v} \lor \overline{w} \lor z$

Erase clause $v$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Infer clause $z$ from

clauses $w$ and $\overline{w} \lor z$
**Example Resolution Refutation**

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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- **Erase clause** $v$
- **Erase clause** $\overline{v} \lor \overline{w} \lor z$
- **Infer clause** $z$ from clauses $w$ and $\overline{w} \lor z$
- **Erase clause** $w$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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7. \( \overline{z} \)

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\[
\begin{array}{c}
\overline{z} \\
\overline{w} \lor z \\
z
\end{array}
\]

Erase clause \( v \)
Erase clause \( \overline{v} \lor \overline{w} \lor z \)
Infer clause \( z \) from clauses \( w \) and \( \overline{w} \lor z \)
Erase clause \( w \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \bar{u} \lor v \lor x \)
5. \( \bar{v} \lor \bar{w} \lor y \)
6. \( \bar{x} \lor \bar{y} \lor z \)
7. \( \bar{z} \)

**Blackboard bookkeeping**

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>total # clauses on board</td>
<td>14</td>
</tr>
<tr>
<td>max # lines on board</td>
<td>5</td>
</tr>
<tr>
<td>max # literals on board</td>
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\[
\begin{array}{c}
\bar{z} \\
\bar{w} \lor z \\
z
\end{array}
\]

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- clauses \( w \) and \( \bar{w} \lor z \)
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\( z \)
\( 0 \)

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Definition of Length and Space

- **Length** $L(\pi)$ of refutation $\pi : F \vdash 0$
  
  total # clauses in all of $\pi$
  
  (in our example 15)

- **(Clause) Space** $Sp(\pi)$ of refutation $\pi : F \vdash 0$
  
  max # clauses on blackboard simultaneously
  
  (in our example 5)

- **Variable space** $VarSp(\pi)$ of refutation $\pi : F \vdash 0$
  
  max # literals on blackboard simultaneously
  
  (in our example 8)
Length and Space of Refuting $F$

- **Length of refuting $F$ is**
  \[
  L(F \vdash 0) = \min_{\pi:F \vdash 0} \{ L(\pi) \}
  \]

- **Clause space of refuting $F$ is**
  \[
  Sp(F \vdash 0) = \min_{\pi:F \vdash 0} \{ Sp(\pi) \}
  \]

- **Variable space of refuting $F$ is**
  \[
  VarSp(F \vdash 0) = \min_{\pi:F \vdash 0} \{ VarSp(\pi) \}
  \]
Why Should We Care About These Measures?

- **Length**: Lower bound on **time** for proof search algorithm
- **Space**: Lower bound on **memory** for proof search algorithm

Can also give ideas for proof search heuristics
Which space measure is “the right one”?

Potentially long discussion...  

Short answer: Clause space more studied but both are interesting

Technical aside: When comparing different measures, for simplicity consider only \( k \)-CNF formulas (during this talk)
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Upper and Lower Bounds on Length

Easy upper bound: \( L(F \vdash \bot) \leq 2^{(# \text{ variables in } F + 1)} \)

**Theorem (Haken 1985)**

Polynomial-size CNF formula family with (weakly) exponential lower bound on refutation length (*pigeonhole principle*)

Later improved to truly exponential lower bounds for different formula families ([Urquhart 1987, Chvátal & Szemerédi 1988] and others)

But resolution used widely in practice anyway
Amenable to proof search because of its simplicity
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Easy upper bound on clause space: $Sp(F \vdash 0) \leq \text{size of } F$, or more precisely $\leq \min(\# \text{ variables in } F, \# \text{ clauses in } F) + O(1)$ [Esteban & Torán 1999]

Theorem (Torán 1999, Alekhnovich et al. 2000)

There are polynomial-size CNF formula families matching this upper bound on clause space up to multiplicative constants

Easy bound on variable space: $VarSp(F \vdash 0) \leq \text{(size of } F)^2$

No matching lower bound known! Not even superlinear bound
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Are Short Proofs Simple?

Does the length of refuting a formula tell us anything about the space?

- Does short length imply small space?
- Or are there formulas with short, easy refutations that must require large space?

For restricted form of so called tree-like resolution

$Sp(F \vdash 0) \leq \log L(F \vdash 0) + \mathcal{O}(1)$ [Esteban & Torán 1999]

General case has remained open, with no consensus on what the “right answer” should be

Results in [Nordström 2006, Nordström & Håstad 2008] can be interpreted as giving a clue but do not rule anything out
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Formulas refutable in small space are refutable in short length—easy corollary of [Atserias & Dalmau 2003]

- But can space-efficient proofs always be carried out quickly?
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What Trade-off Parameters Are of Interest?

- For what values of refutation space?
  Constant? Sublinear? Linear? Superlinear?
- How robust?
  Given minimal refutation space $S$, how much larger space is needed to get short length?
- How dramatic?
  Polynomial? Superpolynomial? Exponential?
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Short Proofs May Be Spacious

Length and clause space are “completely uncorrelated”

**Theorem**

There are $k$-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- refutation length $L(F_n \vdash 0) = \mathcal{O}(n)$ and
- refutation clause space $\text{Sp}(F_n \vdash 0) = \Omega(n/\log n)$.

Optimal separation—given length $n$, always possible to achieve space $\mathcal{O}(n/\log n)$
Simple Proofs Can Be (Very) Long

**Theorem**

There are \( k \)-CNF formulas \( \{ F_n \} \) of size \( \mathcal{O}(n) \) refutable in linear length \( L(F_n \vdash 0) = \mathcal{O}(n) \) such that

1. \( \text{Sp}(F_n \vdash 0) = \mathcal{O}(1) \) but \( L(\pi) = \Omega((n/\text{Sp}(\pi))^2) \) for any refutation \( \pi \)

2. \( \text{Sp}(F_n \vdash 0) = \omega(1) \) but for space \( \leq 3\sqrt{n} \) superpolynomial length is needed

3. \( \text{Sp}(F_n \vdash 0) = \mathcal{O}(\log^2 n) \) but all the way up to space \( \mathcal{O}(n/\log n), \text{ length } n^{\Omega(\log \log n)} \) is needed

4. \( \text{Sp}(F_n \vdash 0) \) up to \( \mathcal{O}(n/\log n) \) but even getting within multiplicative factor requires exponential length

NB! Results hold for both space flavours.
Simple Proofs Can Be (Very) Long

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Any Practical Implications?

Yes and no

Space measures memory consumption for clause learning algorithms but $\text{space} \leq \text{formula size}$—practical applications usually will have much more memory available than that.

But maybe lower bounds on space can give clue about hardness anyway.

[Sabharwal et al. 2003] exhibits formulas with very short refutations that state-of-the-art SAT-solver cannot find.

Same kind of formulas that we have been studying.
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Rest of This Talk

- Prove new theorem about variable substitution and proof space
- Study old combinatorial game from the 1970s
- Combine the two
Key Idea: Variable Substitution

Substitute $x_1 \oplus x_2$ for every variable $x$ in formula. Example:

$$x \lor \overline{y}$$

$\Downarrow$

$$(x_1 \oplus x_2) \lor \neg(y_1 \oplus y_2)$$

$\Downarrow$

$$(x_1 \lor x_2 \lor y_1 \lor \overline{y}_2) \land (x_1 \lor x_2 \lor \overline{y}_1 \lor y_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor y_1 \lor \overline{y}_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor y_2)$$
Key Technical Result: Substitution Space Theorem

**Theorem**

For any CNF formula $F$, let $F[⊕]$ denote formula with $\text{XOR } x_1 ⊕ x_2$ substituted for $x$, written in CNF in canonical way. Then any refutation $\pi$ of $F[⊕]$ can be transformed into refutation $\pi'$ of $F$ such that

- **Length of $\pi$ ≥ length of $\pi'$** (sort of but not quite—actually $\#$ axiom downloads in $\pi$ ≥ $\#$ axiom downloads in $\pi'$)
- **Clause space of $\pi$ ≥ maximal $\#$ variables mentioned simultaneously in $\pi'$**

(Full statement is slightly more general)
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For any CNF formula $F$, let $F[\oplus]$ denote formula with $\text{XOR } x_1 \oplus x_2$ substituted for $x$, written in CNF in canonical way. Then any refutation $\pi$ of $F[\oplus]$ can be transformed into refutation $\pi'$ of $F$ such that

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Given refutation $\pi$ of $F[\oplus]$, make "shadow refutation" $\pi'$ of $F$

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**Sketch of Proof of Substitution Space Theorem**

Given refutation $\pi$ of $F[\oplus]$, make "shadow refutation" $\pi'$ of $F$

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How to Get a Handle on Length-Space Relations?

Want to find formulas that
- can be quickly refuted but require large space
- have space-efficient refutations requiring much time

Such time-space trade-off questions well-studied for pebble games modelling calculations described by DAGs ([Cook & Sethi 1976] and many others)

- **Time** needed for calculation: \# pebbling moves
- **Space** needed for calculation: max \# pebbles required
The Black-White Pebble Game

**Goal:** get single black pebble on sink vertex of $G$

![Diagram of a graph with vertices labeled $u, v, w, x, y, z$.]

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1. Can place black pebble on (empty) vertex $v$ if all immediate predecessors have pebbles on them.
2. Can always remove black pebble from vertex.
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Jakob Nordström (MIT CSAIL)
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Pebbling Price

- Cost of pebbling:
  max # pebbles simultaneously in $G$
  (in our example 4)

- Black-white pebbling price $BW-Peb(G)$ of DAG $G$:
  minimal cost of any pebbling

- Black pebbling price $Peb(G)$ of DAG $G$:
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- Black pebbling price at most square of black-white pebbling price but known to coincide within multiplicative factor for many DAGs
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Pebbling Contradiction

**CNF formula encoding pebble game on DAG $G$**

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

- sources are true
- truth propagates upwards
- but sink is false

Studied by [Bonet et al. 1998, Raz & McKenzie 1999, Ben-Sasson & Wigderson 1999] and others

Our hope is that pebbling properties of DAG somehow carry over to resolution refutations of pebbling contradictions
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Interpreting Refutations as Black-White Pebblings

Black-white pebbling models non-deterministic computation

- black pebbles ⇔ computed results
- white pebbles ⇔ guesses needing to be verified

"Know $z$ assuming $v$, $w$"

Corresponds to $(v \land w) \rightarrow z$, i.e., blackboard clause $\overline{v} \lor \overline{w} \lor z$

So translate clauses to pebbles by:
- unnegated variable ⇒ black pebble
- negated variable ⇒ white pebble
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7. $\overline{z}$

Write down axiom 1: $u$
Example of Refutation-Pebbling Correspondence

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2. $v$
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4. $\overline{u} \lor \overline{v} \lor x$
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Write down axiom 1: $u$
Write down axiom 2: $v$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Write down axiom 1: \( u \)
Write down axiom 2: \( v \)
Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)
Example of Refutation-Pebbling Correspondence

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7. \( \overline{z} \)

| \( u \) | \( v \) | \( \overline{u} \lor \overline{v} \lor x \) |

Write down axiom 1: \( u \)
Write down axiom 2: \( v \)
Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)

Infer clause \( \overline{v} \lor x \) from clauses \( u \) and \( \overline{u} \lor \overline{v} \lor x \)
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1. $u$
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Example of Refutation-Pebbling Correspondence

1.  \( u \)
2.  \( v \)
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4.  \( \overline{u} \lor \overline{v} \lor x \)
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6.  \( x \lor \overline{y} \lor z \)
7.  \( \overline{z} \)

Write down axiom 2: \( v \)
Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)
Infer clause \( \overline{v} \lor x \) from clauses \( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Erase clause \( \overline{u} \lor \overline{v} \lor x \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \neg u \lor \neg v \lor x \)
5. \( \neg v \lor \neg w \lor y \)
6. \( \neg x \lor \neg y \lor z \)
7. \( \neg z \)

Write down axiom 2: \( v \)
Write down axiom 4: \( \neg u \lor \neg v \lor x \)
Infer clause \( \neg v \lor x \) from clauses \( u \) and \( \neg u \lor \neg v \lor x \)
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Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)

Infer clause \( \overline{v} \lor x \) from clauses \( u \) and \( \overline{u} \lor \overline{v} \lor x \)

Erase clause \( \overline{u} \lor \overline{v} \lor x \)

Erase clause \( u \)
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\[
\begin{array}{c}
\text{\( v \)} \\
\text{\( \overline{V} \lor x \)}
\end{array}
\]

Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)

Infer clause \( \overline{v} \lor x \) from clauses \( u \) and \( \overline{u} \lor \overline{v} \lor x \)

Erase clause \( \overline{u} \lor \overline{v} \lor x \)

Erase clause \( u \)
Example of Refutation-Pebbling Correspondence

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clauses $u$ and $\overline{u} \lor \overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Erase clause $u$

Infer clause $x$ from

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Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\begin{array}{c}
\text{Erase clause } \overline{u} \lor \overline{v} \lor x \\
\text{Erase clause } u \\
\text{Infer clause } x \text{ from clauses } v \text{ and } \overline{v} \lor x \\
\text{Erase clause } \overline{v} \lor x
\end{array}
\]
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[ \begin{align*}
\text{Erase clause } & \overline{u} \lor \overline{v} \lor x \\
\text{Erase clause } & u \\
\text{Infer clause } & x \text{ from clauses } v \text{ and } \overline{v} \lor x \\
\text{Erase clause } & \overline{v} \lor x 
\end{align*} \]
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Erase clause \( u \)
Infer clause \( x \) from clauses \( v \) and \( \overline{v} \lor x \)
Erase clause \( \overline{v} \lor x \)
Erase clause \( v \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Erase clause $u$
Infer clause $x$ from clauses $v$ and $\overline{v} \lor x$
Erase clause $\overline{v} \lor x$
Erase clause $v$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( x \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Infer clause \( x \) from clauses \( v \) and \( \overline{v} \lor x \)
Erase clause \( \overline{v} \lor x \)
Erase clause \( v \)
Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

- Erase clause \( \overline{v} \lor x \)
- Erase clause \( v \)
- Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
- Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

| \( x \) | \( x \lor \overline{y} \lor z \) |
| \( \overline{x} \lor \overline{y} \lor z \) | \( \overline{y} \lor z \) |

Erase clause \( \overline{v} \lor x \)
Erase clause \( v \)
Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)
**Example of Refutation-Pebbling Correspondence**

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

**Diagram**

- Node labeling:
  - \( u \)
  - \( v \)
  - \( w \)
  - \( x \)
  - \( y \)
  - \( z \)

**Steps**
- Erase clause \( v \)
- Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
- Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)
- Erase clause \( \overline{x} \lor \overline{y} \lor z \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Erase clause $v$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer clause $\overline{y} \lor z$ from clauses $x$ and $\overline{x} \lor \overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( \overline{x} \lor \overline{y} \lor z \)

Erase clause \( x \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer clause $\overline{y} \lor z$ from clauses $x$ and $\overline{x} \lor \overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Erase clause $x$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\begin{align*}
\overline{y} \lor z \\
\overline{v} \lor \overline{w} \lor y
\end{align*}
\]

Infer clause \( \overline{y} \lor z \) from clauses \( x \) and \( \overline{x} \lor \overline{y} \lor z \)
Erase clause \( \overline{x} \lor \overline{y} \lor z \)
Erase clause \( x \)
Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Erase clause $\overline{x} \lor \overline{y} \lor z$
Erase clause $x$
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer clause $\overline{v} \lor \overline{w} \lor z$ from clauses $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Erase clause $\overline{x} \lor \overline{y} \lor z$
Erase clause $x$
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer clause $\overline{v} \lor \overline{w} \lor y$ from clauses $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( x \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[ \begin{align*}
\overline{y} \lor z \\
\overline{v} \lor \overline{w} \lor y \\
\overline{v} \lor \overline{w} \lor z
\end{align*} \]

Erase clause \( x \)
Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( x \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[ \overline{y} \lor z \]
\[ \overline{v} \lor \overline{w} \lor z \]

Erase clause \( x \)
Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer clause $\overline{v} \lor \overline{w} \lor z$ from clauses $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Erase clause $\overline{y} \lor z$
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer clause $\overline{v} \lor \overline{w} \lor z$ from clauses $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Erase clause $\overline{y} \lor z$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[ \overline{v} \lor \overline{w} \lor z \]
\[ v \]

Infer clause \( \overline{v} \lor \overline{w} \lor z \) from clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{y} \lor z \)
Write down axiom 2: \( v \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[ \overline{v} \lor \overline{w} \lor z \]

\[ v \]

\[ w \]

clauses \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)

Erase clause \( \overline{v} \lor \overline{w} \lor y \)

Erase clause \( \overline{y} \lor z \)

Write down axiom 2: \( v \)

Write down axiom 3: \( w \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\overline{v} \lor \overline{w} \lor z
\]
\[
v
\]
\[
w
\]
\[
\overline{z}
\]

Erase clause \( \overline{v} \lor \overline{w} \lor y \)
Erase clause \( \overline{y} \lor z \)
Write down axiom 2: \( v \)
Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Write down axiom 2: $v$
Write down axiom 3: $w$
Write down axiom 7: $\overline{z}$

Infer clause $\overline{w} \lor z$ from clauses $v$ and $\overline{v} \lor \overline{w} \lor z$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Write down axiom 2: \( v \)
Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)

Infer clause \( \overline{w} \lor z \) from clauses \( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\begin{align*}
\overline{v} \lor \overline{w} \lor z \\
v \\
w \\
\overline{z} \\
\overline{w} \lor z
\end{align*}
\]

Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Infer clause \( \overline{w} \lor z \) from clauses \( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Erase clause \( v \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\begin{align*}
\overline{v} \lor \overline{w} \lor z \\
w \\
\overline{z} \\
\overline{w} \lor z
\end{align*}
\]

Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Infer clause \( \overline{w} \lor z \) from clauses \( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Erase clause \( v \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Write down axiom 7: $\overline{z}$
Infer clause $\overline{w} \lor z$ from clauses $v$ and $\overline{v} \lor \overline{w} \lor z$
Erase clause $v$
Erase clause $\overline{v} \lor \overline{w} \lor z$
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $x \lor \overline{y} \lor z$
7. $\overline{z}$

Write down axiom 7: $\overline{z}$

Infer clause $\overline{w} \lor z$ from clauses $v$ and $\overline{v} \lor \overline{w} \lor z$

Erase clause $v$

Erase clause $\overline{v} \lor \overline{w} \lor z$
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

clauses $v$ and $\overline{v} \lor \overline{w} \lor z$
Erase clause $v$
Erase clause $\overline{v} \lor \overline{w} \lor z$
Infer clause $z$ from clauses $w$ and $\overline{w} \lor z$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\begin{array}{c}
w \\
\overline{z} \\
\overline{w} \lor z \\
z
\end{array}
\]

clauses \( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Erase clause \( v \)
Erase clause \( \overline{v} \lor \overline{w} \lor z \)
Infer clause \( z \) from clauses \( w \) and \( \overline{w} \lor z \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Erase clause \( v \)
Erase clause \( \overline{v} \lor \overline{w} \lor z \)
Infer clause \( z \) from clauses \( w \) and \( \overline{w} \lor z \)
Erase clause \( w \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\[
\begin{align*}
\overline{z} \\
\overline{w} \lor z \\
z
\end{align*}
\]

Erase clause \( v \)
Erase clause \( \overline{v} \lor \overline{w} \lor z \)
Infer clause \( z \) from clauses \( w \) and \( \overline{w} \lor z \)

Erase clause \( w \)
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Erase clause $\overline{v} \lor \overline{w} \lor z$
Infer clause $z$ from clauses $w$ and $\overline{w} \lor z$

Erase clause $w$
Erase clause $\overline{w} \lor z$
Example of Refutation-Pebbling Correspondence

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Erasure clause $\overline{v} \lor \overline{w} \lor z$
Infer clause $z$ from clauses $w$ and $\overline{w} \lor z$

Erasure clause $w$
Erasure clause $\overline{w} \lor z$
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

clauses \( w \) and \( \overline{w} \lor z \)
Erase clause \( w \)
Erase clause \( \overline{w} \lor z \)
Infer clause 0 from clauses \( \overline{z} \) and \( z \)
Example of Refutation-Pebbling Correspondence

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

\( \overline{z} \)
\( z \)
\( 0 \)

clauses \( w \) and \( \overline{w} \lor z \)
Erase clause \( w \)
Erase clause \( \overline{w} \lor z \)
Infer clause \( 0 \) from
clauses \( \overline{z} \) and \( z \)
Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson 2002)

Any refutation translates into black-white pebbling with

- \( \# \text{ moves} \leq \text{refutation length} \)
- \( \# \text{ pebbles} \leq \text{max } \# \text{ simultaneous variable occurrences} \)

Theorem (Ben-Sasson et al. 2000)

Any black-only pebbling translates into refutation with

- \( \text{refutation length} \leq \# \text{ moves} \)
- \( \text{variable space} \leq \# \text{ pebbles} \)
Formal Refutation-Pebbling Correspondence

**Theorem (Ben-Sasson 2002)**

Any refutation translates into black-white pebbling with

- \# moves ≤ refutation length
- \# pebbles ≤ max \# simultaneous variable occurrences

**Theorem (Ben-Sasson et al. 2000)**

Any black-only pebbling translates into refutation with

- refutation length ≤ \# moves
- variable space ≤ \# pebbles
Then Along Comes the Substitution Space Theorem

Applying the Substitution Space Theorem

- lifts lower bound from variable occurrences to clause space
- maintains upper bound in terms of variable space

Get our results by

- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results
- to get tight trade-offs, showing that resolution refutations can sometimes do better than black-only pebblings
Then Along Comes the Substitution Space Theorem

Applying the Substitution Space Theorem
- lifts lower bound from variable occurrences to clause space
- maintains upper bound in terms of variable space

Get our results by
- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results
- to get tight trade-offs, showing that resolution refutations can sometimes do better than black-only pebblings
Lower Bounds on Variable Space?

Open Question

Are there polynomial-size $k$-CNF formulas with variable refutation space $\text{VarSp}(F \vdash 0) = \Omega\left((\text{size of } F)^2\right)$?

Answer conjectured to be “yes” by (Alekhnovich et al. 2000)

Or can we at least prove a superlinear lower bound on variable space?
Length-Width Trade-offs?

**Width**: size of largest clause in refutation

[Ben-Sasson & Wigderson 1999] strong length-width correlation:
- A refutation in small width must be short (easy)
- Given short refutation, can find narrow refutation

But short and narrow refutations not the same! Exponential blow-up in length—is this necessary?

---

Open Question

*Suppose that a k-CNF formula F has a short refutation. Does it then have a refutation that is both short and narrow? Or are there formula families exhibiting length-width trade-offs?*
Length-Width Trade-offs?

**Width**: size of largest clause in refutation

[Ben-Sasson & Wigderson 1999] **strong length-width correlation**:

- A refutation in **small width** must be **short** (easy)
- Given **short refutation**, can find **narrow refutation**

But short and narrow refutations not the same! **Exponential blow-up in length**—is this necessary?

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Stronger Length-Space Trade-offs?

Open Question

Are there superpolynomial trade-offs for formulas refutable in constant space?

Open Question

Are there formulas with trade-offs in the range space > formula size? Or can every refutation be carried out in at most linear space?

Pebbling formulas cannot answer these questions—always refutable in linear time and linear space simultaneously.
Empirical Results?

Open Question

*Do our trade-off phenomena show up in real life for state-of-the-art SAT-solvers run on pebbling contradictions?*

(Possibly with some modifications to make easy refutation somewhat harder to discover)

Or are pebbling formulas of all flavours always easy in practice?
Summing up

- Optimal length-space separation in resolution
- Strong length-space trade-offs for wide range of parameters
- Many remaining open questions about space in resolution

Thank you for your attention!