A Beautiful General Survey on Hardness Condensation

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Humboldt-Universität zu Berlin

Dagstuhl workshop 18051
Proof Complexity
Friday February 2, 2018

A Special Case of Hardness Condensation

Jakob Nordström

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Supercritical Space-Width Trade-offs for Resolution

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Joint work with Christoph Berkholz

Proof Complexity

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

Input: Unsatisfiable formula in conjunctive normal form (CNF)
Output: Polynomial-time verifiable certificate of unsatisfiability

Proof of unsatifiability = **refutation** of formula

Want to measure efficiency of proof system in terms of different complexity measures (size, space, et cetera)

Can be viewed as proving upper and lower bounds for weak nondeterministic models of computation

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

- Done when empty clause $\bot$ derived

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Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
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4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2, 4)$
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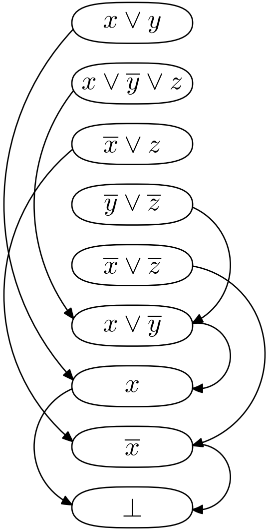

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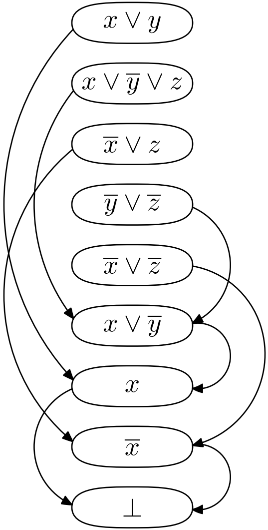
Tree-like resolution if DAG is tree



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Length of refuting F $=$ min length over all proofs for F

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But for resolution don't care too much about linear factors here
Set size = length

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Size should strictly speaking measure # symbols
But for resolution don't care too much about linear factors here
Set size = length

Width of proof = # literals in largest clause (3 in our example)
Width of refuting F = min width over all proofs for F

Width at most linear, so here obviously care about linear factors

Resolution Space

Space = amount of memory needed when performing refutation

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Clause space at step t: # clauses at steps $\leq t$ used at steps $\geq t$
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Example: Clause space at step 7

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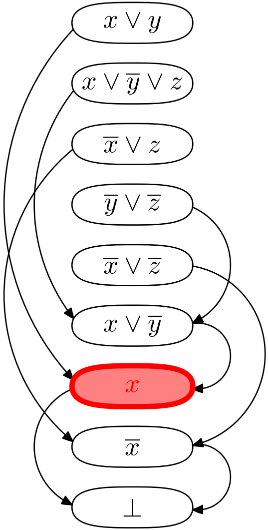

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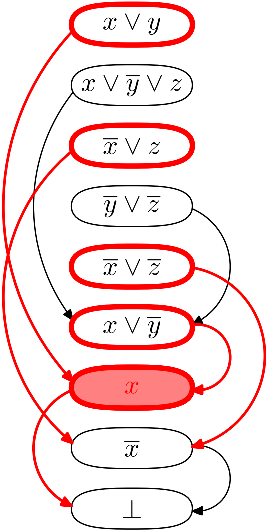

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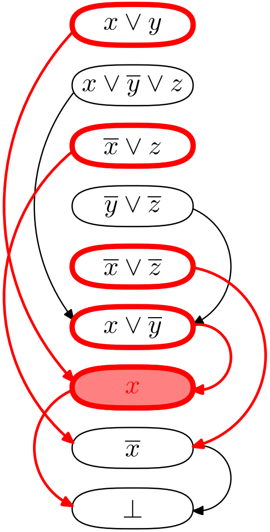

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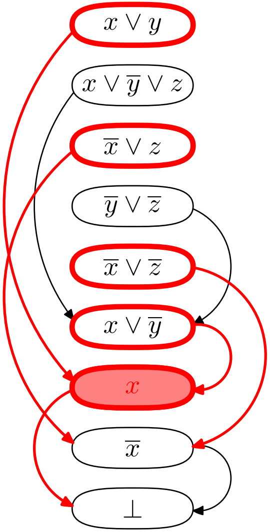
Space of proof = max over all steps
Space of refuting F = min over all proofs



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This talk: focus on width and clause space
But results translate to total space by:

$$\text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width}$$

Lower Bounds via Resolution Width

For n-variable k-CNFs (k constant) it holds that:

width $\leq$ Ω(clause space) [Atserias & Dalmau '03]

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- Can have width $\Theta(\sqrt{n})$ and still size $\mathrm{poly}(n)$ [Bonet & Galesi '99]
- Can have width $\mathcal{O}(1)$ and still clause space $\Omega(n/\log n)$ [Ben-Sasson & Nordström '08]

Upper Bounds via Resolution Width

$$\text{size} \quad \leq \quad n^{\mathcal{O}(\text{width})}$$

Upper Bounds via Resolution Width

$$\begin{aligned} \text{size} &\leq n^{\mathcal{O}(\text{width})} \\ \text{time to find refutation} &\leq n^{\mathcal{O}(\text{width})} \end{aligned}$$

for $w \leftarrow 3 \dots n$ **do**
    Resolve all clauses & keep resolvents with at most w literals
    If $\bot$ has been derived, then output `UNSAT`
end for
Output `SAT`

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Recall: can always do clause space $\mathcal{O}(n)$

A Supercritical Space-Width Tradeoff

Theorem

For any $\varepsilon > 0$ *and* $6 \leq w \leq n^{\frac{1}{2}-\varepsilon}$ *exist* n*-variable* CNF*s* F_n *s.t.*

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Key components:

- Expander graphs
- XORification (substitution with exclusive or)

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- We feel "supercritical" is more descriptive

Expanders

Very well-connected so-called expander graphs play leading role in many proof complexity lower bounds

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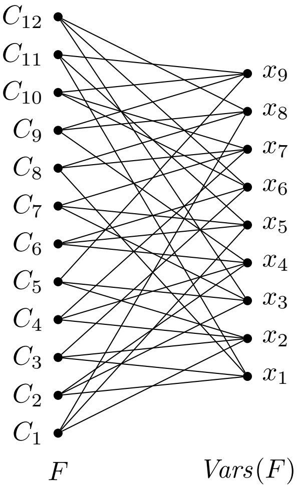

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- Clauses on the left
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- Edge if variable $\in$ clause (ignore signs)

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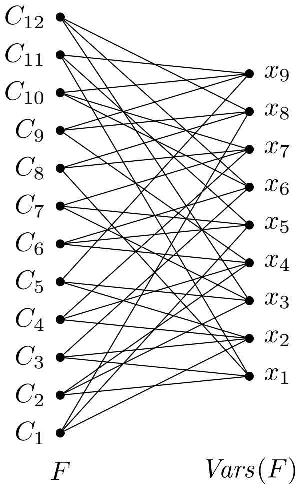

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- width, size, and space in resolution [Ben-Sasson & Wigderson '99, Ben-Sasson & Galesi '03]
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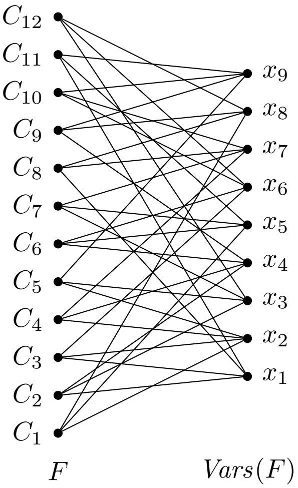

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Can also define more general graphs that capture "underlying combinatorial structure" and extend results [Mikša & Nordström '15]

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- # vars in memory $\geq s$ for $F \Longrightarrow$ clause space $\geq \Omega(s)$ for $F[\oplus_2]$ [Ben-Sasson & Nordström '08]

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Intuition behind proof

- Given resolution refutation π of $F[\oplus_2]$
- Extract the refutation π' of F that π is simulating
- Prove that extraction preserves complexity measures of interest

Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

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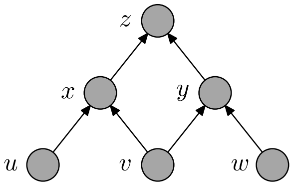

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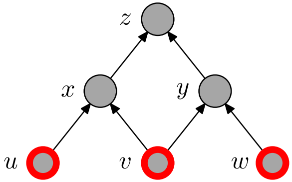

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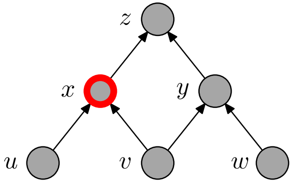

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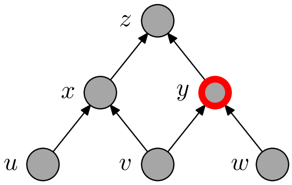

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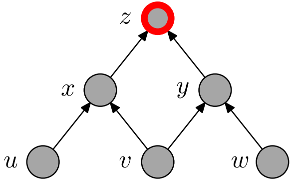

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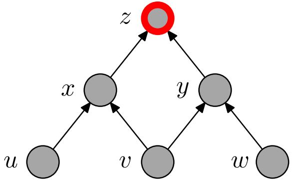

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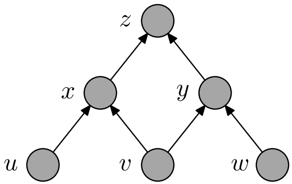

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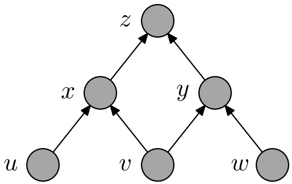

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Easy to refute pebbling formulas in size $\mathcal{O}(n)$ and width $\mathcal{O}(1)$
Pebbling space lower bounds $\Rightarrow$ clause space lower bounds
[Ben-Sasson & Nordström '08, '11]

XOR Substitution with Recycling (1/2)

Suppose

- F CNF formula over variables U
- $\mathcal{G} = (U \mathbin{\dot\cup} V, E)$ bipartite graph

Substituted formula $F[\mathcal{G}]$ over variables V:

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$

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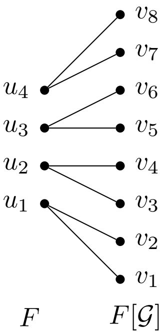

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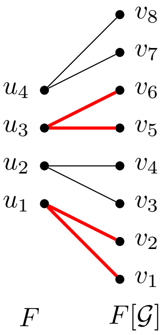

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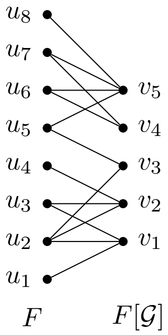

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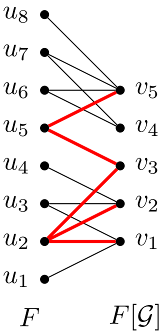

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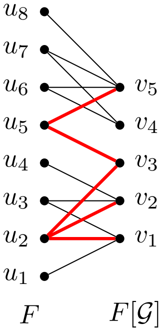

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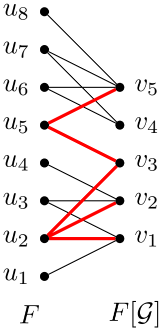

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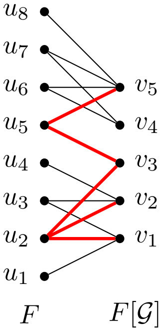

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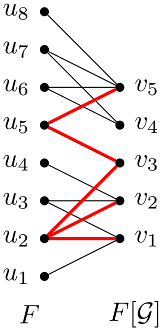

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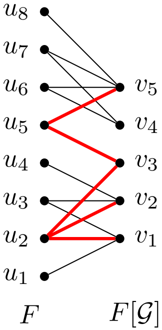

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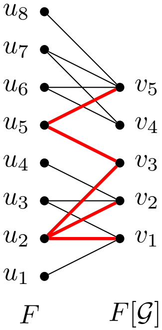

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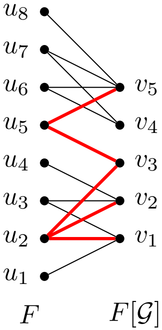

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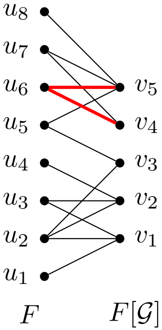

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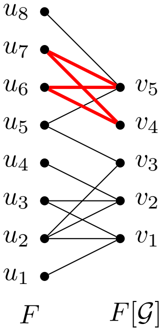

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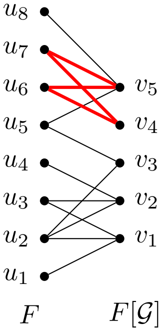

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Solution: Use expander graphs!

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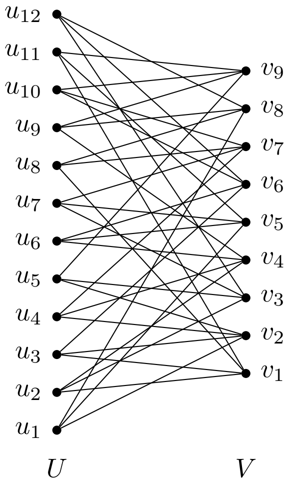
Bipartite Boundary Expander



$\mathcal{G} = (U \dot\cup V, E)$ is (d, r, c)-boundary expander if

- left-degree $\leq d$
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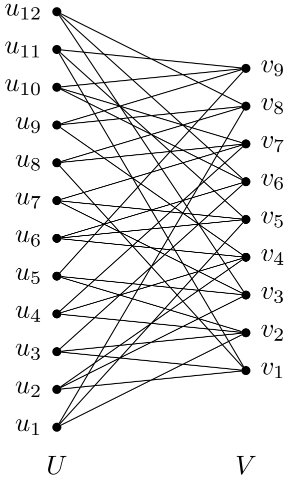

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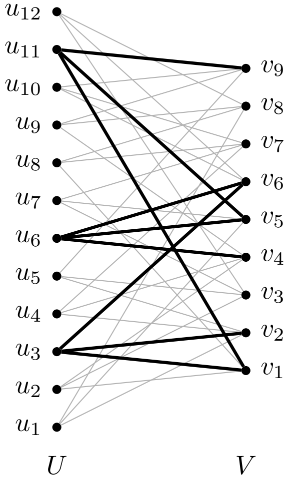

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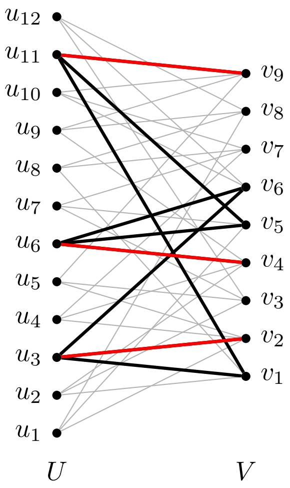

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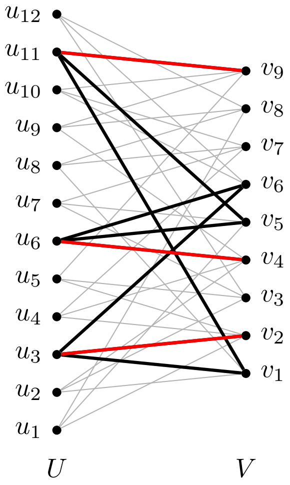

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Lemma ([Razborov '16])

For $\varepsilon > 0$ and n, d with $d \leq |V|^{\frac{1}{2}-\varepsilon}$, $|U| = n$, $|V| = n^{\mathcal{O}(1/d)}$ there are $(d, r, 2)$-boundary expanders $\mathcal{G}$ with $r = d \log n$

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Look at clauses $\mathcal{C}$ in memory in width-w refutation of $F[\mathcal{G}]$

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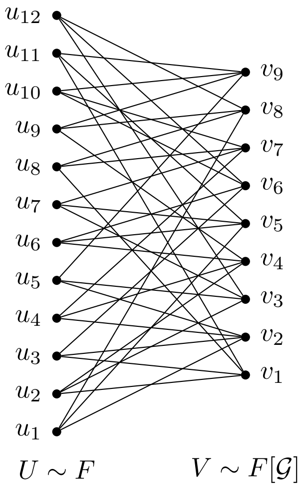

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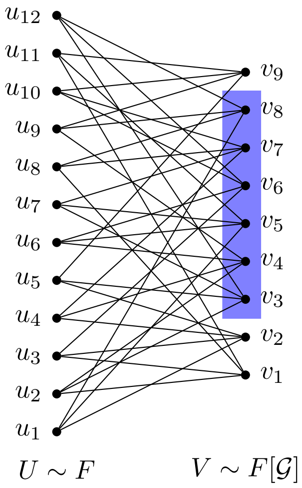

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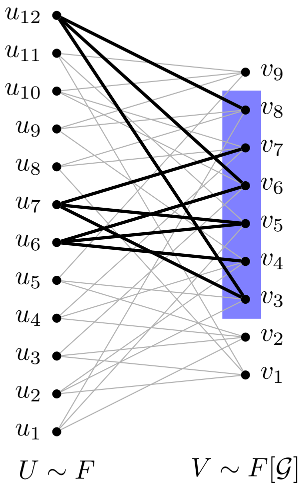

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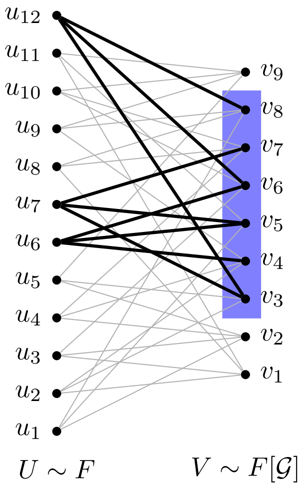

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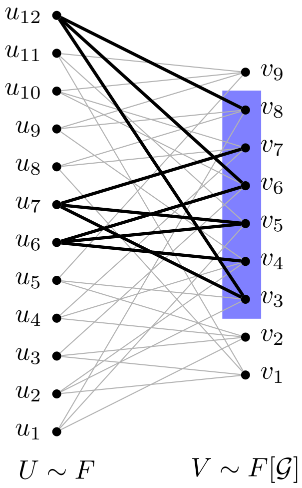

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Actual details very different

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Some further technical twists needed, but this is main idea of proof

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Introduced in [Razborov JACM '16] to show that treelike resolution refutations of width w can require doubly exponential size $2^{n^{\Omega(w)}}$

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Has also been used to establish

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Where else can this technique be useful?

Concluding Remarks

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Are there supercritical tradeoffs for 3-CNFs?

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Thank you for your attention!