

# Short Proofs May Be Spacious: An Optimal Separation of Space and Length in Resolution

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# Executive Summary of Talk

- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Basis of current state-of-the-art SAT-solvers (winners in SAT 2008 competition: resolution + clause learning)
- Key resources: **time** and **space**
- What are the connections between these resources?  
Are time and space correlated?  
Are there time/space trade-offs?

# Some Notation and Terminology

- **Literal**  $a$ : variable  $x$  or its negation  $\bar{x}$
- **Clause**  $C = a_1 \vee \dots \vee a_k$ : disjunction of literals  
At most  $k$  literals:  **$k$ -clause**
- **CNF formula**  $F = C_1 \wedge \dots \wedge C_m$ : conjunction of clauses  
 **$k$ -CNF formula**: CNF formula consisting of  $k$ -clauses  
(assume  $k$  fixed)
- Refer to clauses of CNF formula as **axioms**  
(as opposed to derived clauses)

# Resolution Rule

Resolution rule:

$$\frac{B \vee x \quad C \vee \bar{x}}{B \vee C}$$

Prove  $F$  **unsatisfiable** by deriving the unsatisfiable empty clause 0 (the clause with no literals) from  $F$  by resolution

# Resolution Rule

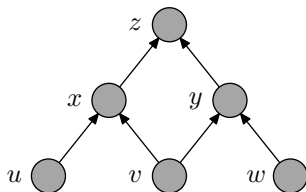
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$$\frac{B \vee x \quad C \vee \bar{x}}{B \vee C}$$

Prove  $F$  **unsatisfiable** by deriving the unsatisfiable empty clause 0 (the clause with no literals) from  $F$  by resolution

# Example CNF Formula

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

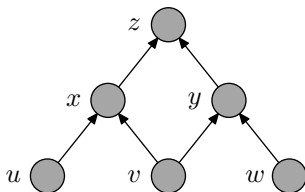


Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

# Example CNF Formula

1.  $u$
2.  $v$
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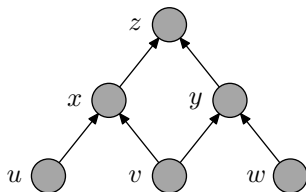


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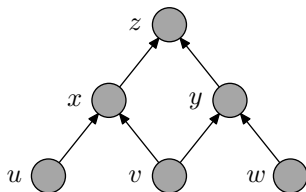
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- source vertices true
- **truth propagates upwards**
- but sink vertex is false



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7.  $\bar{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- **but sink vertex is false**

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	0
# literals in largest clause	0
# lines on blackboard used	0



Can **write down axioms**,  
**erase used clauses** or  
**infer new clauses** (but only from  
clauses currently on the board!)

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	1
# literals in largest clause	1
# lines on blackboard used	1

$u$

Write down axiom 1:  $u$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	2
# literals in largest clause	1
# lines on blackboard used	2

$u$
$v$

Write down axiom 1:  $u$

Write down axiom 2:  $v$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

$u$
$v$
$\bar{u} \vee \bar{v} \vee x$

Write down axiom 1:  $u$

Write down axiom 2:  $v$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

 $u$  $v$  $\bar{u} \vee \bar{v} \vee x$ Write down axiom 1:  $u$ Write down axiom 2:  $v$ Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$ 

Infer clause  $\bar{v} \vee x$  from  
clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

$$u$$

$$v$$

$$\bar{u} \vee \bar{v} \vee x$$

$$\bar{v} \vee x$$

Write down axiom 1:  $u$

Write down axiom 2:  $v$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$

**Infer clause  $\bar{v} \vee x$**  from  
clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

$$u$$

$$v$$

$$\bar{u} \vee \bar{v} \vee x$$

$$\bar{v} \vee x$$

Write down axiom 2:  $v$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$

Infer clause  $\bar{v} \vee x$  from

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $\bar{u} \vee \bar{v} \vee x$



# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

$$u$$

$$v$$

$$\bar{v} \vee x$$

Write down axiom 2:  $v$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$

Infer clause  $\bar{v} \vee x$  from

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $\bar{u} \vee \bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

$$u$$

$$v$$

$$\bar{v} \vee x$$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$

Infer clause  $\bar{v} \vee x$  from

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $u$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

$$v$$

$$\bar{v} \vee x$$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$

Infer clause  $\bar{v} \vee x$  from

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $u$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

$v$   
 $\bar{v} \vee x$

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$   
 Erase clause  $\bar{u} \vee \bar{v} \vee x$   
 Erase clause  $u$   
**Infer clause  $x$**  from  
 clauses  $v$  and  $\bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

$v$   
 $\bar{v} \vee x$   
 $x$

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$   
 Erase clause  $\bar{u} \vee \bar{v} \vee x$   
 Erase clause  $u$   
Infer clause  $x$  from  
 clauses  $v$  and  $\bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

$v$
$\bar{v} \vee x$
$x$

Erase clause  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $u$

Infer clause  $x$  from

clauses  $v$  and  $\bar{v} \vee x$

Erase clause  $\bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

 $v$ 
 $x$ 

Erase clause  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $u$

Infer clause  $x$  from

clauses  $v$  and  $\bar{v} \vee x$

Erase clause  $\bar{v} \vee x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

 $v$ 
 $x$ 

Erase clause  $u$

Infer clause  $x$  from

clauses  $v$  and  $\bar{v} \vee x$

Erase clause  $\bar{v} \vee x$

Erase clause  $v$



# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

$x$

Erase clause  $u$   
 Infer clause  $x$  from  
     clauses  $v$  and  $\bar{v} \vee x$   
 Erase clause  $\bar{v} \vee x$   
 Erase clause  $v$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

 $x$ 
 $\bar{x} \vee \bar{y} \vee z$ 

Infer clause  $x$  from  
clauses  $v$  and  $\bar{v} \vee x$

Erase clause  $\bar{v} \vee x$

Erase clause  $v$

**Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$**

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

$x$   
 $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{v} \vee x$

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

**Infer clause  $\bar{y} \vee z$**  from  
 clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

$x$ $\bar{x} \vee \bar{y} \vee z$ $\bar{y} \vee z$
--

Erase clause  $\bar{v} \vee x$

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

**Infer clause  $\bar{y} \vee z$**  from  
 clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

 $x$ 
 $\bar{x} \vee \bar{y} \vee z$ 
 $\bar{y} \vee z$ 

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

Infer clause  $\bar{y} \vee z$  from

clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

$$x$$

$$\bar{y} \vee z$$

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

Infer clause  $\bar{y} \vee z$  from

clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

$x$   
 $\bar{y} \vee z$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

Infer clause  $\bar{y} \vee z$  from

clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $x$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

Infer clause  $\bar{y} \vee z$  from

clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $x$



# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

Infer clause  $\bar{y} \vee z$  from  
clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

**Infer clause  $\bar{v} \vee \bar{w} \vee z$**  from  
clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

$$\bar{v} \vee \bar{w} \vee z$$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

**Infer clause  $\bar{v} \vee \bar{w} \vee z$**  from

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

$$\bar{v} \vee \bar{w} \vee z$$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee z$$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee z$$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{y} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{v} \vee \bar{w} \vee z$$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$   
 Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
 clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{y} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	10
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
 clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{y} \vee z$   
 Write down axiom 2:  $v$



# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	11
# literals in largest clause	3
# lines on blackboard used	4

$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{y} \vee z$

Write down axiom 2:  $v$

Write down axiom 3:  $w$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	12
# literals in largest clause	3
# lines on blackboard used	4

$\bar{v} \vee \bar{w} \vee z$
$v$
$w$
$\bar{z}$

Erase clause  $\bar{v} \vee \bar{w} \vee z$

Erase clause  $\bar{y} \vee z$

Write down axiom 2:  $v$

Write down axiom 3:  $w$

Write down axiom 7:  $\bar{z}$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	12
# literals in largest clause	3
# lines on blackboard used	4

$\bar{v} \vee \bar{w} \vee z$   
 $v$   
 $w$   
 $\bar{z}$

Write down axiom 2:  $v$   
 Write down axiom 3:  $w$   
 Write down axiom 7:  $\bar{z}$   
**Infer clause  $\bar{w} \vee z$  from**  
 clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 2:  $v$

Write down axiom 3:  $w$

Write down axiom 7:  $\bar{z}$

Infer clause  $\bar{w} \vee z$  from  
clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 3:  $w$

Write down axiom 7:  $\bar{z}$

Infer clause  $\bar{w} \vee z$  from

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

Erase clause  $v$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$$\bar{v} \vee \bar{w} \vee z$$

$$w$$

$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 3:  $w$

Write down axiom 7:  $\bar{z}$

Infer clause  $\bar{w} \vee z$  from

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

Erase clause  $v$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$$\bar{v} \vee \bar{w} \vee z$$

$$w$$

$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 7:  $\bar{z}$   
 Infer clause  $\bar{w} \vee z$  from  
 clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
 Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

 $w$ 
 $\bar{z}$ 
 $\bar{w} \vee z$ 

Write down axiom 7:  $\bar{z}$   
 Infer clause  $\bar{w} \vee z$  from  
 clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
 Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$



# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

 $w$  $\bar{z}$  $\bar{w} \vee z$ 

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

Erase clause  $v$

Erase clause  $\bar{v} \vee \bar{w} \vee z$

**Infer clause  $z$**  from

clauses  $w$  and  $\bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

 $w$  $\bar{z}$  $\bar{w} \vee z$  $z$ 

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

Erase clause  $v$

Erase clause  $\bar{v} \vee \bar{w} \vee z$

**Infer clause  $z$**  from

clauses  $w$  and  $\bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

 $w$  $\bar{z}$  $\bar{w} \vee z$  $z$ Erase clause  $v$ Erase clause  $\bar{v} \vee \bar{w} \vee z$ Infer clause  $z$  fromclauses  $w$  and  $\bar{w} \vee z$ Erase clause  $w$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

$\bar{z}$   
 $\bar{w} \vee z$   
 $z$

Erase clause  $v$

Erase clause  $\bar{v} \vee \bar{w} \vee z$

Infer clause  $z$  from

clauses  $w$  and  $\bar{w} \vee z$

Erase clause  $w$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

$\bar{z}$   
 $\bar{w} \vee z$   
 $z$

Erase clause  $\bar{v} \vee \bar{w} \vee z$

Infer clause  $z$  from

clauses  $w$  and  $\bar{w} \vee z$

Erase clause  $w$

Erase clause  $\bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

 $\bar{z}$ 
 $z$ 

Erase clause  $\bar{v} \vee \bar{w} \vee z$

Infer clause  $z$  from

clauses  $w$  and  $\bar{w} \vee z$

Erase clause  $w$

Erase clause  $\bar{w} \vee z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

 $\bar{z}$ 
 $z$ 

clauses  $w$  and  $\bar{w} \vee z$

Erase clause  $w$

Erase clause  $\bar{w} \vee z$

**Infer clause 0** from  
clauses  $\bar{z}$  and  $z$

# Example Resolution Refutation

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

## Blackboard bookkeeping

total # clauses on board	15
# literals in largest clause	3
# lines on blackboard used	5

$\bar{z}$
$z$
0

clauses  $w$  and  $\bar{w} \vee z$

Erase clause  $w$

Erase clause  $\bar{w} \vee z$

**Infer clause 0** from  
clauses  $\bar{z}$  and  $z$



# Length, Width and Space

- **Length**  $L(\pi)$  of refutation  $\pi : F \vdash 0$   
total # clauses in all of  $\pi$   
(in our example 15)
- **Width**  $W(\pi)$  of refutation  $\pi : F \vdash 0$   
# literals in largest clause in  $\pi$   
(in our example 3)
- **Space**  $Sp(\pi)$  of refutation  $\pi : F \vdash 0$   
max # clauses on blackboard simultaneously  
(in our example 5)

# Length, Width and Space of Refuting $F$

- Length of refuting  $F$  is

$$L(F \vdash 0) = \min_{\pi: F \vdash 0} \{L(\pi)\}$$

- Width of refuting  $F$  is

$$W(F \vdash 0) = \min_{\pi: F \vdash 0} \{W(\pi)\}$$

- Space of refuting  $F$  is

$$Sp(F \vdash 0) = \min_{\pi: F \vdash 0} \{Sp(\pi)\}$$

# Why Should We Care About These Measures?

- **Length:** Lower bound on **time** for proof search algorithm
- **Space:** Lower bound on **memory** for proof search algorithm
- **Width:** Intimately connected to length and space 😊

# Results for Length and Width

## Length

- Easy exponential upper bound
- [Haken 1985] and [Urquhart 1987]: exponential lower bounds on refutation length

## Width

- Always  $W(F \vdash 0) \leq \# \text{ variables in } F$
- [Ben-Sasson & Wigderson 1999]: **strong correlation** between **length and width** of refuting formula

# Results for Width and Space

Always  $Sp(F \vdash 0) \leq \text{size of } F$  [Esteban & Torán 1999]

All space and width bounds for “the usual suspects” coincide!?

Theorem (Atserias & Dalmau 2003)

*For any unsatisfiable  $k$ -CNF formula  $F$  it holds that  
space  $Sp(F \vdash 0) \geq \text{width } W(F \vdash 0) - \mathcal{O}(1)$ .*

Theorem (Nordström 2006)

*There are  $k$ -CNF formula families  $\{F_n\}_{n=1}^{\infty}$  of size  $\mathcal{O}(n)$  with*

- *refutation width  $W(F_n \vdash 0) = \mathcal{O}(1)$  and*
- *refutation space  $Sp(F_n \vdash 0) = \Theta(\log n)$ .*

# Results for Width and Space

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- refutation space  $Sp(F_n \vdash 0) = \Theta(\log n)$ .

# Connection Between Length and Space?

Current state of knowledge	
Length vs. width	strongly correlated
Width vs. space	separated
Length vs. space	???

- Small space  $\Rightarrow$  short length (easy)
- But does short length imply small space?  
(For tree-like resolution: yes! [Esteban & Torán 1999])
- Or are there formulas with short, easy refutations that must require large space?

No consensus on what the “right answer” should be



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# Getting Closer...

## Theorem (Nordström & Håstad 2008)

There are  $k$ -CNF formula families  $\{F_n\}_{n=1}^{\infty}$  of size  $\mathcal{O}(n)$  with

- refutation length  $L(F_n \vdash 0) = \mathcal{O}(n)$ ,
- refutation width  $W(F_n \vdash 0) = \mathcal{O}(1)$  and
- refutation space  $Sp(F_n \vdash 0) = \Theta(\sqrt{n})$ .

Exponential improvement of previous space-width separation

But **does not answer space-length question**—space  $\sqrt{n}$  is just where it gets interesting!

Above result still **consistent with strong space-length correlation**  
à la Ben-Sasson & Wigderson

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But **does not answer space-length question**—space  $\sqrt{n}$  is just where it gets interesting!

Above result still **consistent with strong space-length correlation**  
à la Ben-Sasson & Wigderson

# Our result: An Optimal Space-Length Separation

Length and space are “completely uncorrelated”

## Theorem

There are  $k$ -CNF formula families  $\{F_n\}_{n=1}^{\infty}$  of size  $\mathcal{O}(n)$  with

- refutation length  $L(F_n \vdash 0) = \mathcal{O}(n)$ ,
- refutation width  $W(F_n \vdash 0) = \mathcal{O}(1)$  and
- refutation space  $Sp(F_n \vdash 0) = \Theta(n/\log n)$ .

**Optimal separation of space and length**—given length  $n$ ,  
always possible to achieve space  $\mathcal{O}(n/\log n)$

Also optimal space-width separation up to logarithmic factor

# In a Decade Far, Far Away...

Want to find formulas that

- can be quickly refuted
- but require large space

Such time-space trade-off questions well-studied for pebble games modelling calculations described by DAGs

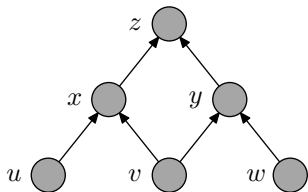
- **Time** needed for calculation: # pebbling moves
- **Space** needed for calculation: max # pebbles required

Known result:  $\exists$  DAGs requiring many pebbles in terms of size

Look at **CNF formulas encoding pebble games** on DAGs!

# The Black-White Pebble Game

**Goal:** get **single black pebble** on **sink vertex** of  $G$

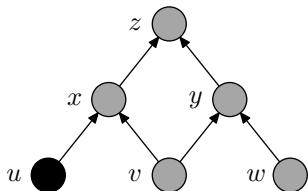


Number of pebbles	
Current	0
Max so far	0

- 1 Can **place black pebble** on (empty) vertex  $v$  if all immediate predecessors have pebbles on them
- 2 Can always **remove black pebble** from vertex
- 3 Can always **place white pebble** on (empty) vertex
- 4 Can **remove white pebble** from  $v$  if all immediate predecessors have pebbles on them

# The Black-White Pebble Game

Goal: get **single black pebble** on **sink vertex** of  $G$



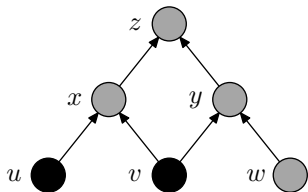
Number of pebbles	
Current	1
Max so far	1

- 1 Can **place black pebble** on (empty) vertex  $v$  if all immediate predecessors have pebbles on them
- 2 Can always **remove black pebble** from vertex
- 3 Can always **place white pebble** on (empty) vertex
- 4 Can **remove white pebble** from  $v$  if all immediate predecessors have pebbles on them



# The Black-White Pebble Game

Goal: get **single black pebble** on **sink vertex** of  $G$

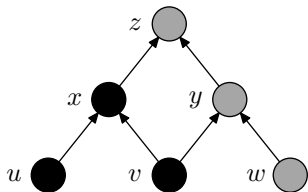


Number of pebbles	
Current	2
Max so far	2

- 1 Can **place black pebble** on (empty) vertex  $v$  if all immediate predecessors have pebbles on them
- 2 Can always **remove black pebble** from vertex
- 3 Can always **place white pebble** on (empty) vertex
- 4 Can **remove white pebble** from  $v$  if all immediate predecessors have pebbles on them

# The Black-White Pebble Game

Goal: get **single black pebble** on **sink vertex** of  $G$

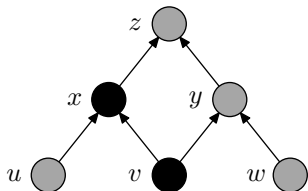


Number of pebbles	
Current	3
Max so far	3

- 1 Can **place black pebble** on (empty) vertex  $v$  if all immediate predecessors have pebbles on them
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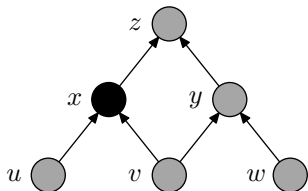


Number of pebbles	
Current	2
Max so far	3

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- 2 Can always **remove black pebble** from vertex
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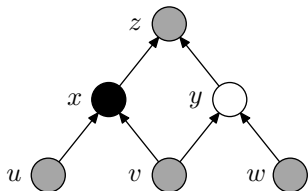


Number of pebbles	
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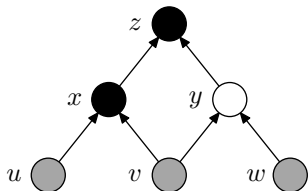


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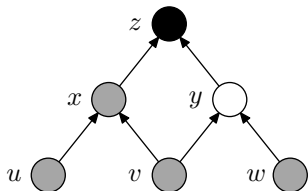


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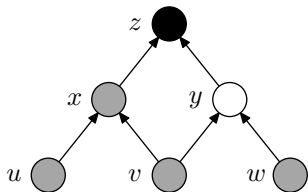


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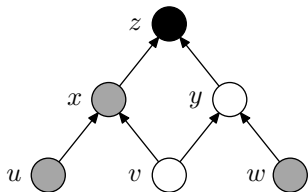
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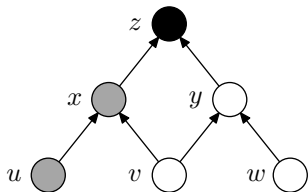


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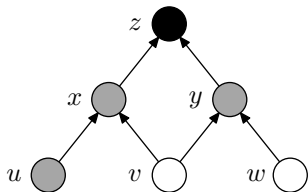


Number of pebbles	
Current	4
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# The Black-White Pebble Game

Goal: get **single black pebble** on **sink vertex** of  $G$

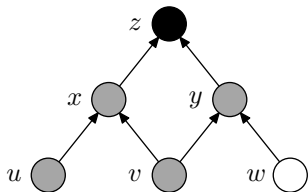


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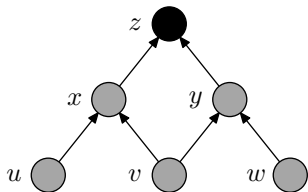


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# Black-White Pebbling Price

- Cost of pebbling:  
max # pebbles simultaneously in  $G$   
(in our example 4)
- **Black-white pebbling price  $BW\text{-Peb}(G)$**  of DAG  $G$ :  
minimal cost of any pebbling

Many bounds on pebbling price known. We will use:

- Always at most  $\mathcal{O}(n/\log n)$  pebbles needed [Hopcroft, Paul & Valiant 1977]
- There are (explicit) DAGs requiring  $\Omega(n/\log n)$  pebbles [Gilbert & Tarjan 1978]

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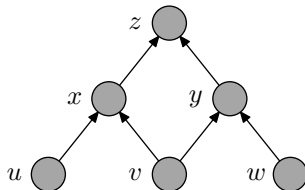
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# Pebbling Contradiction

CNF formula encoding pebble game on DAG  $G$

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



- sources are true
- truth propagates upwards
- but sink is false

Studied by [Bonet et al. 1998, Raz & McKenzie 1999, Ben-Sasson & Wigderson 1999] and others

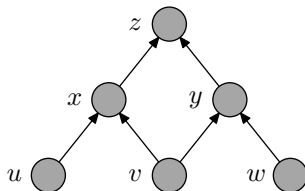
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# Proof Outline

Resolution	Pebbling
Translate sets of clauses...	into black and white pebbles
...then the clause set must contain at least $N$ clauses	Prove that if the translation results in $N$ pebbles...
Show that consecutive sets of clauses on blackboard in a resolution refutation...	translates into a black-white pebbling of corresponding DAG
...yielding same lower bound on space in resolution	Plug in lower bound on black-white pebbling price...

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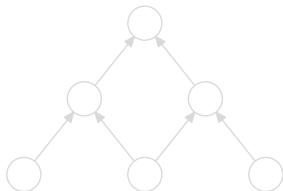
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# Interpreting Clauses in Terms of Pebbles

Black-white pebbling models non-deterministic computation

- black pebbles  $\Leftrightarrow$  computed results
- white pebbles  $\Leftrightarrow$  guesses needing to be verified



“We know  $z$  assuming  $v, w$ ”

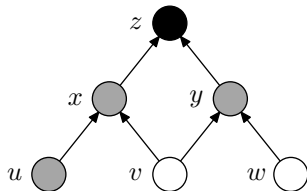
Corresponds to  $(v \wedge w) \rightarrow z$

I.e., clause  $\boxed{\bar{v} \vee \bar{w} \vee z}$  on  
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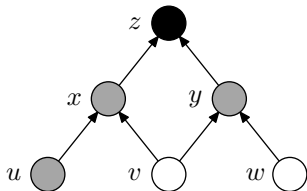
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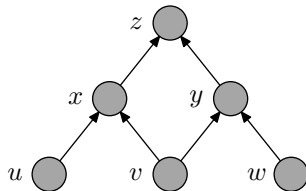
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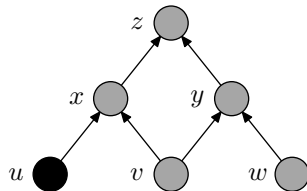
# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
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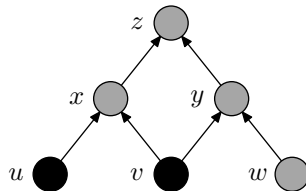


$u$

Write down axiom 1:  $u$

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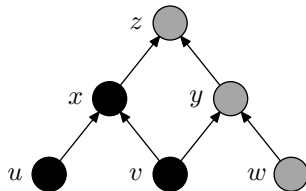
$u$   
 $v$

Write down axiom 1:  $u$   
Write down axiom 2:  $v$



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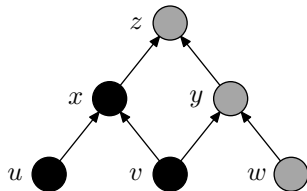
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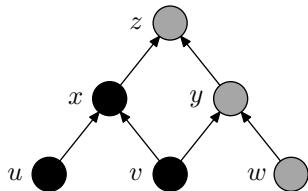


$u$   
 $v$   
 $\bar{u} \vee \bar{v} \vee x$

Write down axiom 1:  $u$   
Write down axiom 2:  $v$   
Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$   
**Infer clause  $\bar{v} \vee x$**  from  
clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$

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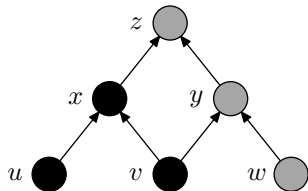


$u$   
 $v$   
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$$u$$

$$v$$

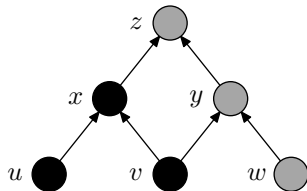
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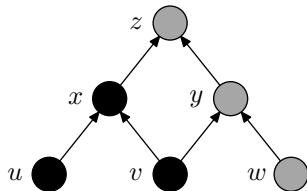
$$v$$

$$\bar{v} \vee x$$

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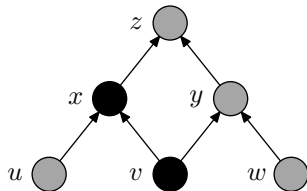


$u$   
 $v$   
 $\bar{v} \vee x$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$   
Infer clause  $\bar{v} \vee x$  from  
clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $u$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

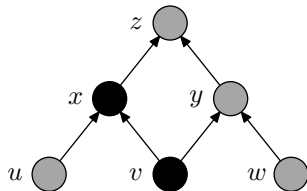


$v$   
 $\bar{v} \vee x$

Write down axiom 4:  $\bar{u} \vee \bar{v} \vee x$   
Infer clause  $\bar{v} \vee x$  from  
clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $u$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



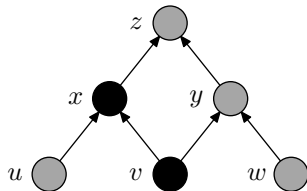
$v$   
 $\bar{v} \vee x$

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$   
 Erase clause  $\bar{u} \vee \bar{v} \vee x$   
 Erase clause  $u$   
**Infer clause  $x$**  from  
 clauses  $v$  and  $\bar{v} \vee x$



# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

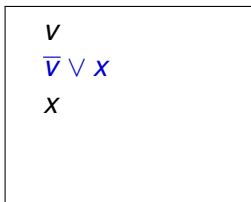
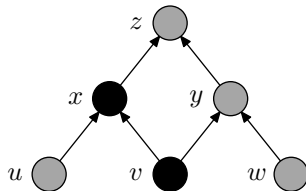


$v$   
 $\bar{v} \vee x$   
 $x$

clauses  $u$  and  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $u$   
**Infer clause  $x$**  from  
clauses  $v$  and  $\bar{v} \vee x$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



Erase clause  $\bar{u} \vee \bar{v} \vee x$

Erase clause  $u$

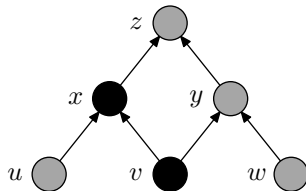
Infer clause  $x$  from

clauses  $v$  and  $\bar{v} \vee x$

Erase clause  $\bar{v} \vee x$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

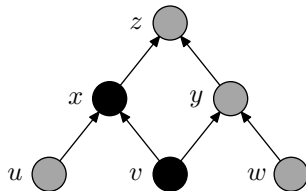


$v$   
 $x$

Erase clause  $\bar{u} \vee \bar{v} \vee x$   
Erase clause  $u$   
Infer clause  $x$  from  
clauses  $v$  and  $\bar{v} \vee x$   
Erase clause  $\bar{v} \vee x$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

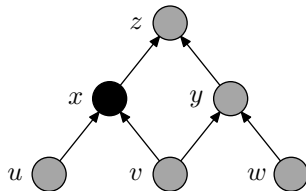


$v$   
 $x$

Erase clause  $u$   
Infer clause  $x$  from  
clauses  $v$  and  $\bar{v} \vee x$   
Erase clause  $\bar{v} \vee x$   
Erase clause  $v$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

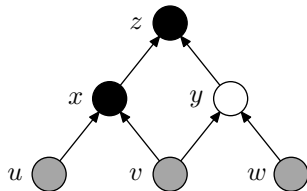


$x$

Erase clause  $u$   
Infer clause  $x$  from  
clauses  $v$  and  $\bar{v} \vee x$   
Erase clause  $\bar{v} \vee x$   
Erase clause  $v$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$x$   
 $\bar{x} \vee \bar{y} \vee z$

Infer clause  $x$  from  
clauses  $v$  and  $\bar{v} \vee x$

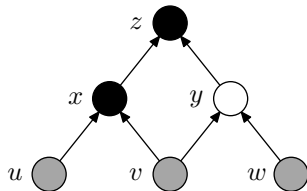
Erase clause  $\bar{v} \vee x$

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$x$   
 $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{v} \vee x$

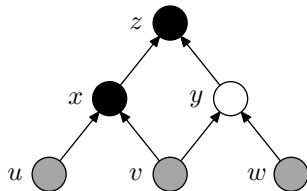
Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

**Infer clause  $\bar{y} \vee z$**  from  
clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$x$$

$$\bar{x} \vee \bar{y} \vee z$$

$$\bar{y} \vee z$$

Erase clause  $\bar{v} \vee x$

Erase clause  $v$

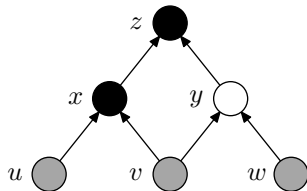
Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

Infer clause  $\bar{y} \vee z$  from  
clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$



# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$x$$

$$\bar{x} \vee \bar{y} \vee z$$

$$\bar{y} \vee z$$

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

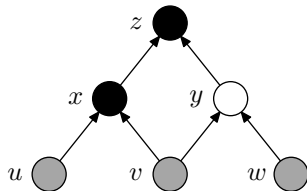
Infer clause  $\bar{y} \vee z$  from

clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$x$$

$$\bar{y} \vee z$$

Erase clause  $v$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$

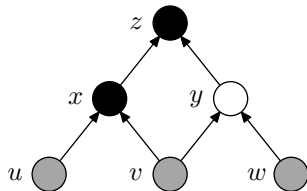
Infer clause  $\bar{y} \vee z$  from

clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

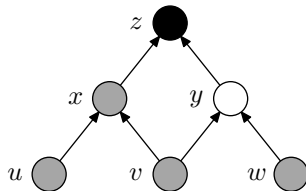


$x$   
 $\bar{y} \vee z$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$   
Infer clause  $\bar{y} \vee z$  from  
clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$   
Erase clause  $\bar{x} \vee \bar{y} \vee z$   
Erase clause  $x$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

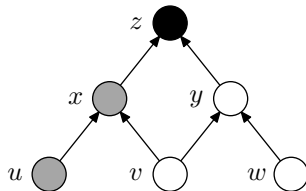


$$\bar{y} \vee z$$

Write down axiom 6:  $\bar{x} \vee \bar{y} \vee z$   
Infer clause  $\bar{y} \vee z$  from  
clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$   
Erase clause  $\bar{x} \vee \bar{y} \vee z$   
Erase clause  $x$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

Infer clause  $\bar{y} \vee z$  from  
clauses  $x$  and  $\bar{x} \vee \bar{y} \vee z$

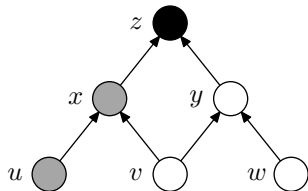
Erase clause  $\bar{x} \vee \bar{y} \vee z$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

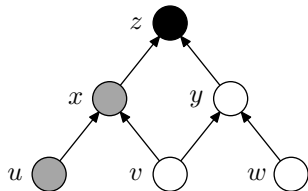
Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

$$\bar{v} \vee \bar{w} \vee z$$

Erase clause  $\bar{x} \vee \bar{y} \vee z$

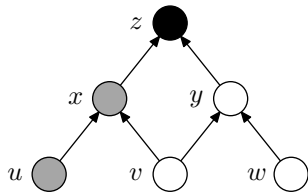
Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee y$$

$$\bar{v} \vee \bar{w} \vee z$$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from

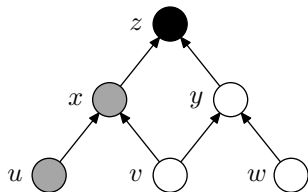
clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$



# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee z$$

Erase clause  $x$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$

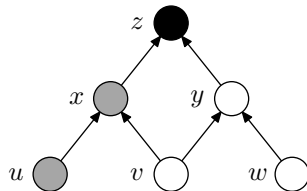
Infer clause  $\bar{v} \vee \bar{w} \vee z$  from

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



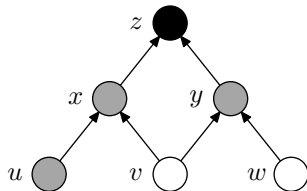
$$\bar{y} \vee z$$

$$\bar{v} \vee \bar{w} \vee z$$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$   
 Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
 clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{y} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

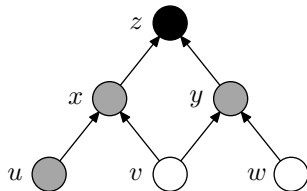


$$\bar{v} \vee \bar{w} \vee z$$

Write down axiom 5:  $\bar{v} \vee \bar{w} \vee y$   
Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$   
Erase clause  $\bar{v} \vee \bar{w} \vee y$   
Erase clause  $\bar{y} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



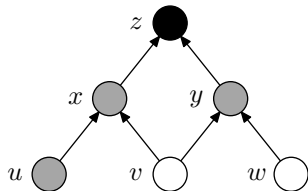
$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

Infer clause  $\bar{v} \vee \bar{w} \vee z$  from  
 clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{v} \vee \bar{w} \vee y$   
 Erase clause  $\bar{y} \vee z$   
 Write down axiom 2:  $v$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

clauses  $\bar{y} \vee z$  and  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

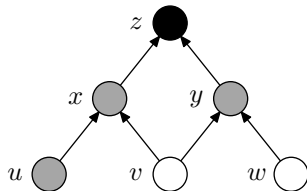
Erase clause  $\bar{y} \vee z$

Write down axiom 2:  $v$

Write down axiom 3:  $w$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

$$\bar{z}$$

Erase clause  $\bar{v} \vee \bar{w} \vee y$

Erase clause  $\bar{y} \vee z$

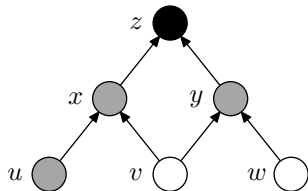
Write down axiom 2:  $v$

Write down axiom 3:  $w$

Write down axiom 7:  $\bar{z}$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

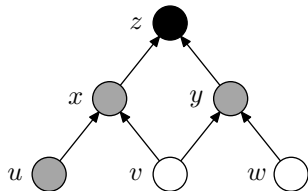


$\bar{v} \vee \bar{w} \vee z$   
 $v$   
 $w$   
 $\bar{z}$

Write down axiom 2:  $v$   
Write down axiom 3:  $w$   
Write down axiom 7:  $\bar{z}$   
**Infer clause  $\bar{w} \vee z$  from**  
clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 2:  $v$

Write down axiom 3:  $w$

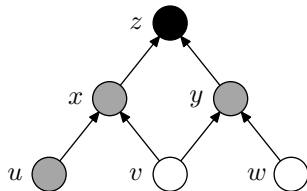
Write down axiom 7:  $\bar{z}$

Infer clause  $\bar{w} \vee z$  from  
clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$



# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{v} \vee \bar{w} \vee z$$

$$v$$

$$w$$

$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 3:  $w$

Write down axiom 7:  $\bar{z}$

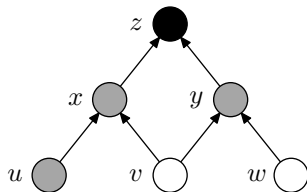
Infer clause  $\bar{w} \vee z$  from

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$

Erase clause  $v$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

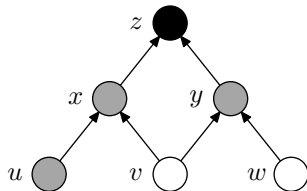


$\bar{v} \vee \bar{w} \vee z$   
 $w$   
 $\bar{z}$   
 $\bar{w} \vee z$

Write down axiom 3:  $w$   
Write down axiom 7:  $\bar{z}$   
Infer clause  $\bar{w} \vee z$  from  
clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
Erase clause  $v$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{v} \vee \bar{w} \vee z$$

$$w$$

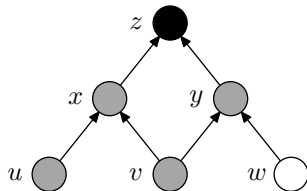
$$\bar{z}$$

$$\bar{w} \vee z$$

Write down axiom 7:  $\bar{z}$   
 Infer clause  $\bar{w} \vee z$  from  
 clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
 Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

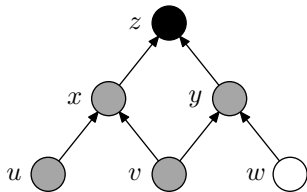


$w$   
 $\bar{z}$   
 $\bar{w} \vee z$

Write down axiom 7:  $\bar{z}$   
 Infer clause  $\bar{w} \vee z$  from  
 clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
 Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$

# Example of Refutation-Pebbling Correspondence

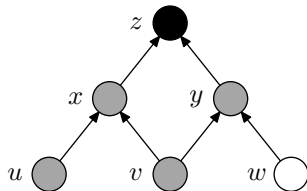
1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

 $w$  $\bar{z}$  $\bar{w} \vee z$ 

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
 Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$   
**Infer clause  $z$**  from  
 clauses  $w$  and  $\bar{w} \vee z$

# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
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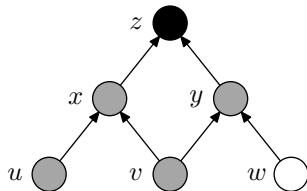


$w$   
 $\bar{z}$   
 $\bar{w} \vee z$   
 $z$

clauses  $v$  and  $\bar{v} \vee \bar{w} \vee z$   
 Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$   
**Infer clause  $z$**  from  
 clauses  $w$  and  $\bar{w} \vee z$

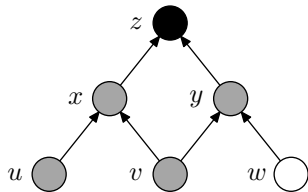
# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
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 $w$  $\bar{z}$  $\bar{w} \vee z$  $z$ Erase clause  $v$ Erase clause  $\bar{v} \vee \bar{w} \vee z$ Infer clause  $z$  fromclauses  $w$  and  $\bar{w} \vee z$ Erase clause  $w$

# Example of Refutation-Pebbling Correspondence

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6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{z}$$

$$\bar{w} \vee z$$

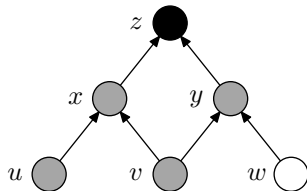
$$z$$

Erase clause  $v$   
 Erase clause  $\bar{v} \vee \bar{w} \vee z$   
 Infer clause  $z$  from  
 clauses  $w$  and  $\bar{w} \vee z$   
 Erase clause  $w$



# Example of Refutation-Pebbling Correspondence

1.  $u$
2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



$$\bar{z}$$

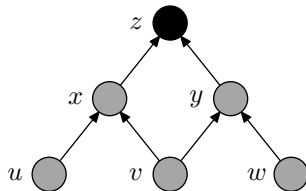
$$\bar{w} \vee z$$

$$z$$

Erase clause  $\bar{v} \vee \bar{w} \vee z$   
 Infer clause  $z$  from  
 clauses  $w$  and  $\bar{w} \vee z$   
 Erase clause  $w$   
 Erase clause  $\bar{w} \vee z$

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1.  $u$
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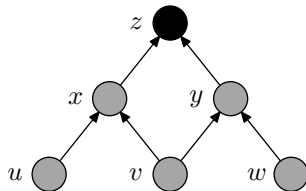


$\bar{z}$   
 $z$

Erase clause  $\bar{v} \vee \bar{w} \vee z$   
Infer clause  $z$  from  
clauses  $w$  and  $\bar{w} \vee z$   
Erase clause  $w$   
Erase clause  $\bar{w} \vee z$

# Example of Refutation-Pebbling Correspondence

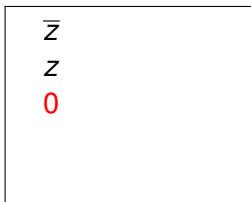
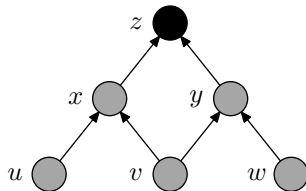
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4.  $\bar{u} \vee \bar{v} \vee x$
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6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$

 $\bar{z}$  $z$ 

clauses  $w$  and  $\bar{w} \vee z$   
Erase clause  $w$   
Erase clause  $\bar{w} \vee z$   
**Infer clause 0** from  
clauses  $\bar{z}$  and  $z$

# Example of Refutation-Pebbling Correspondence

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2.  $v$
3.  $w$
4.  $\bar{u} \vee \bar{v} \vee x$
5.  $\bar{v} \vee \bar{w} \vee y$
6.  $\bar{x} \vee \bar{y} \vee z$
7.  $\bar{z}$



clauses  $w$  and  $\bar{w} \vee z$   
Erase clause  $w$   
Erase clause  $\bar{w} \vee z$   
**Infer clause 0** from  
clauses  $\bar{z}$  and  $z$

# Sweeping the details under the rug. . .

Guiding intuition for [Nordström 2006] and [Nordström & Håstad 2008] as well as our work

Looks very nice, but doesn't work—in reality things get (much) messier

Refutations have no reason to derive nicely structured clauses  
⇒ different ideas needed to translate refutations to pebbings

Our key new idea: substitute  $x_1 \oplus x_2$  for  $x$  in pebbling formulas

- cleaner proofs
- improvement to optimal bounds

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# Length-Space Trade-offs

## Question

*Are there formulas refutable in **short length** and **small space**, but for which any **small-space refutation must be long**?*

Answer is **yes, and in a very strong sense**

Superpolynomial and even exponential trade-offs for wide range of different parameters

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# Length-Width Trade-offs?

[Ben-Sasson & Wigderson 1999] showed that given **short refutation**, can find **(reasonably) narrow refutation**

But not the same refutation!

**Exponential blow-up in length**—is this necessary?

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*Suppose that a  $k$ -CNF formula  $F$  has a **short** refutation. Does it then have a refutation that is **both short and narrow**?*

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# Conclusions

- **This work:** Optimal space-length separation with formulas refutable in **length**  $\mathcal{O}(n)$  and **space**  $\Omega(n/\log n)$
- **More recently:** Strong trade-offs between space and length
- **Open questions:** Trade-offs between **width and length?**  
Trade-offs for **superlinear space?** Extensions to **stronger proof systems?**

Thank you for your attention!