A Generalized Method for Proving Polynomial Calculus Degree Lower Bounds

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The Satisfiability Problem (SAT)

\[(x \lor y) \land (x \lor \neg y \lor z) \land (\neg x \lor z) \land (\neg y \lor \neg z) \land (\neg x \lor \neg z)\]
The Satisfiability Problem (SAT)

Variables should be set to true or false

\[(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})\]
The Satisfiability Problem (SAT)

\[(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})\]

- Variables should be set to true or false
- Constraint \((x \lor \overline{y} \lor z)\): means \(x\) or \(z\) should be true or \(y\) false
The Satisfiability Problem (SAT)

\[(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})\]

- Variables should be set to true or false
- Constraint \((x \lor \overline{y} \lor z):\) means \(x\) or \(z\) should be true or \(y\) false
- \(\land\) means all constraints should hold simultaneously
The Satisfiability Problem (SAT)

\[(x \lor y) \land (x \lor \overline{y} \lor z) \land (x \lor z) \land (\overline{y} \lor \overline{z}) \land (x \lor \overline{z})\]

- Variables should be set to true or false
- Constraint \((x \lor \overline{y} \lor z)\): means \(x\) or \(z\) should be true or \(y\) false
- \(\land\) means all constraints should hold simultaneously

Is there a truth value assignment satisfying all these conditions? Or is it always the case that some constraint must fail to hold?
Satisfiable formulas have short, efficiently verifiable certificates (satisfying assignments)
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What about unsatisfiable formulas?
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What about unsatisfiable formulas?

**Proof system**
Formal specification of method for reasoning about formulas
Given formula $\mathcal{F}$, can produce certificate $\pi$ of unsatisfiability
Proof $\pi$ should be polynomial-time verifiable (in size of $\pi$, not $\mathcal{F}$)
Satisfiable formulas have short, efficiently verifiable certificates (satisfying assignments)

What about unsatisfiable formulas?

**Proof system**
Formal specification of method for reasoning about formulas
Given formula \( F \), can produce certificate \( \pi \) of unsatisfiability
Proof \( \pi \) should be polynomial-time verifiable (in size of \( \pi \), not \( F \))

**Proof complexity**
Study of upper and lower bounds for concrete proof systems
Motivations for Proof Complexity

**Program for showing $P \neq NP$**

Original motivation in [Cook & Reckhow '79]

Superpolynomial lower bounds for all proof systems $\Rightarrow NP \neq \text{co-NP}$

Still very distant goal...
Motivations for Proof Complexity

Program for showing $P \neq NP$
Original motivation in [Cook & Reckhow '79]
Superpolynomial lower bounds for all proof systems $\Rightarrow NP \neq co-NP$
Still very distant goal...

Quantify power of mathematical reasoning
Study efficient proofs of different mathematical principles
Determine how strong proof systems are needed
Measures “mathematical depth” of corresponding principle
Motivations for Proof Complexity

**Program for showing P ≠ NP**
Original motivation in [Cook & Reckhow ’79]
Superpolynomial lower bounds for all proof systems ⇒ NP ≠ co-NP
Still very distant goal…

**Quantify power of mathematical reasoning**
Study efficient proofs of different mathematical principles
Determine how strong proof systems are needed
Measures “mathematical depth” of corresponding principle

**Connections to SAT solving and combinatorial optimization**
Can formalize and study proof systems behind state-of-the-art SAT solvers
Sheds light on potential and limitations of such solvers
Also extends to combinatorial optimization (e.g., LP and SDP hierarchies)
Outline of This Presentation

1. Overview of some proof complexity basics
2. Discuss two proof systems
   - Resolution ($\iff$ state-of-the-art conflict-driven clause learning solvers)
   - Polynomial calculus ($\iff$ algebraic Gröbner basis computations)
3. Present framework for proving polynomial calculus lower bounds
   - Based on degree lower bounds via expansion
   - Expressed in terms of combinatorial game played on formula
   - Unifies previous lower bounds and yields some new ones
Some Notation and Terminology

- **Literal** $a$: variable $x$ or its negation $\overline{x}$

- **Clause** $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)

- **CNF formula** $F = C_1 \land \cdots \land C_m$: conjunction of clauses

- **$k$-CNF formula**: CNF formula with clauses of size $\leq k$
  (where $k$ is some constant)

- $N =$ size of formula ($\#$ literals, which is $\approx \#$ clauses for $k$-CNF)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**  

\[
 \frac{C \lor x \quad D \lor \bar{x}}{C \lor D}
\]

Refutation ends when empty clause \( \bot \) derived
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**  

\[
\frac{C \lor x \quad \quad D \lor \overline{x}}{C \lor D}
\]

Refutation ends when empty clause $\bot$ derived

1. $x \lor y$
2. $x \lor \overline{y} \lor z$
3. $\overline{x} \lor z$
4. $\overline{y} \lor \overline{z}$
5. $\overline{x} \lor \overline{z}$
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

\[
\frac{C \lor x}{C \lor D}
\]

\[
\frac{D \lor \bar{x}}{C \lor D}
\]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as

- annotated list or
- directed acyclic graph

1. \( x \lor y \) Axiom
2. \( x \lor \bar{y} \lor z \) Axiom
3. \( \bar{x} \lor z \) Axiom
4. \( \bar{y} \lor \bar{z} \) Axiom
5. \( \bar{x} \lor \bar{z} \) Axiom
6. \( x \lor \bar{y} \) Res(2, 4)
7. \( x \) Res(1, 6)
8. \( \bar{x} \) Res(3, 5)
9. \( \bot \) Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

$\begin{align*}
1. & \quad x \lor y & \text{Axiom} \\
2. & \quad x \lor \overline{y} \lor z & \text{Axiom} \\
3. & \quad \overline{x} \lor z & \text{Axiom} \\
4. & \quad \overline{y} \lor \overline{z} & \text{Axiom} \\
5. & \quad \overline{x} \lor \overline{z} & \text{Axiom} \\
6. & \quad x \lor \overline{y} & \text{Res}(2, 4) \\
7. & \quad x & \text{Res}(1, 6) \\
8. & \quad \overline{x} & \text{Res}(3, 5) \\
9. & \quad \bot & \text{Res}(7, 8)
\end{align*}$

Derive new clauses by **resolution rule**

$\frac{C \lor x \quad D \lor \overline{x}}{C \lor \overline{D}}$

Repetition ends when empty clause $\bot$ derived

Can represent refutation as

- **annotated list** or
- **directed acyclic graph**
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

\[
\frac{C \lor x}{C \lor D}
\]

\[
\frac{D \lor \bar{x}}{C \lor D}
\]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as

- annotated list or
- directed acyclic graph

1. \( x \lor y \) Axiom
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The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \bar{x}}{C \lor D}
\]

Refutation ends when empty clause \( \bot \) derived

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Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

\[ \frac{C \lor x \quad D \lor \overline{x}}{C \lor D} \]

Refutation ends when empty clause \( \bot \) derived

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9. \( \bot \) Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)

Derive new clauses by **resolution rule**

\[ \frac{C \lor x}{\therefore C \lor D} \quad \text{Res} \]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as

- **annotated list** or
- **directed acyclic graph**

1. \( x \lor y \)  \hspace{1cm} Axiom
2. \( x \lor \overline{y} \lor z \)  \hspace{1cm} Axiom
3. \( \overline{x} \lor z \)  \hspace{1cm} Axiom
4. \( \overline{y} \lor \overline{z} \)  \hspace{1cm} Axiom
5. \( \overline{x} \lor \overline{z} \)  \hspace{1cm} Axiom
6. \( x \lor \overline{y} \)  \hspace{1cm} Res(2, 4)
7. \( x \)  \hspace{1cm} Res(1, 6)
8. \( \overline{x} \)  \hspace{1cm} Res(3, 5)
9. \( \bot \)  \hspace{1cm} Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

\[ \frac{C \lor x}{C \lor D} \]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as

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1. \( x \lor y \)   Axiom
2. \( x \lor \overline{y} \lor z \)   Axiom
3. \( \overline{x} \lor z \)   Axiom
4. \( \overline{y} \lor \overline{z} \)   Axiom
5. \( \overline{x} \lor \overline{z} \)   Axiom
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The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)

Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D}
\]

Refutation ends when empty clause \( \bot \) derived

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Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**  

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\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D}
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Refutation ends when empty clause \(\bot\) derived

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2. \(x \lor \overline{y} \lor z\) Axiom
3. \(\overline{x} \lor z\) Axiom
4. \(\overline{y} \lor \overline{z}\) Axiom
5. \(\overline{x} \lor \overline{z}\) Axiom
6. \(x \lor \overline{y}\) Res(2, 4)
7. \(x\) Res(1, 6)
8. \(\overline{x}\) Res(3, 5)
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Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**  

\[ \frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D} \]

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Can represent refutation as  
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3. \( \overline{x} \lor z \)  
4. \( \overline{y} \lor \overline{z} \)  
5. \( \overline{x} \lor \overline{z} \)  
6. \( x \lor \overline{y} \)  
7. \( x \)  
8. \( \overline{x} \)  
9. \( \bot \)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor \overline{D}}
\]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as

- **annotated list** or
- **directed acyclic graph**

1. \( x \lor y \) \quad Axiom
2. \( x \lor \overline{y} \lor z \) \quad Axiom
3. \( \overline{x} \lor z \) \quad Axiom
4. \( \overline{y} \lor \overline{z} \) \quad Axiom
5. \( \overline{x} \lor \overline{z} \) \quad Axiom
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8. \( \overline{x} \) \quad Res(3, 5)
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The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{\frac{D \lor \bar{x}}{C \lor D}}
\]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as
- annotated list or
- directed acyclic graph

1. \( x \lor y \)  \hspace{1em}  Axiom
2. \( x \lor \bar{y} \lor z \)  \hspace{1em}  Axiom
3. \( \bar{x} \lor z \)  \hspace{1em}  Axiom
4. \( \bar{y} \lor \bar{z} \)  \hspace{1em}  Axiom
5. \( \bar{x} \lor \bar{z} \)  \hspace{1em}  Axiom
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7. \( x \)  \hspace{1em}  Res(1, 6)
8. \( \bar{x} \)  \hspace{1em}  Res(3, 5)
9. \( \bot \)  \hspace{1em}  Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (**axioms**)  

Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{D \lor \overline{x}} \quad \frac{C \lor x}{D \lor \overline{x}}
\]

Refutation ends when empty clause \( \perp \) derived

Can represent refutation as

- annotated list or
- directed acyclic graph

1. \( x \lor y \)  \hspace{1cm} Axiom
2. \( x \lor \overline{y} \lor z \)  \hspace{1cm} Axiom
3. \( \overline{x} \lor z \)  \hspace{1cm} Axiom
4. \( \overline{y} \lor \overline{z} \)  \hspace{1cm} Axiom
5. \( \overline{x} \lor \overline{z} \)  \hspace{1cm} Axiom
6. \( x \lor \overline{y} \)  \hspace{1cm} Res(2, 4)
7. \( x \)  \hspace{1cm} Res(1, 6)
8. \( \overline{x} \)  \hspace{1cm} Res(3, 5)
9. \( \perp \)  \hspace{1cm} Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor \overline{D}}
\]

Refutation ends when empty clause \( \bot \) derived

Can represent refutation as

- annotated list or
- directed acyclic graph
The Resolution Proof System

Goal: refute \textbf{unsatisfiable} CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D}
\]

Refutation ends when empty clause $$\bot$$ derived

Can represent refutation as
- annotated list or
- directed acyclic graph

Tree-like resolution if DAG is tree
Resolution Size/Length

**Size/length** = \# clauses in refutation

Most fundamental measure in proof complexity

Never worse than $\exp(O(N))$

Matching $\exp(\Omega(N))$ lower bounds known
**Pigeonhole principle (PHP)** [Haken ’85]
“$n + 1$ pigeons don’t fit into $n$ holes”

Variables $p_{i,j} =$ “pigeon $i$ goes into hole $j$”

$$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n} \quad \text{every pigeon $i$ gets a hole}$$

$$\overline{p}_{i,j} \lor \overline{p}_{i',j} \quad \text{no hole $j$ gets two pigeons $i \neq i'$}$$

Can also add “functionality” and “onto” axioms

$$\overline{p}_{i,j} \lor \overline{p}_{i,j'} \quad \text{no pigeon $i$ gets two holes $j \neq j'$}$$

$$p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j} \quad \text{every hole $j$ gets a pigeon}$$
Examples of Hard Formulas w.r.t Resolution Size (1/2)

**Pigeonhole principle (PHP) [Haken ’85]**

“$n + 1$ pigeons don’t fit into $n$ holes”

Variables $p_{i,j} =$ “pigeon $i$ goes into hole $j$”

\[
p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n} \quad \text{every pigeon $i$ gets a hole}
\]

\[
\overline{p}_{i,j} \lor \overline{p}_{i',j} \quad \text{no hole $j$ gets two pigeons $i \neq i'$}
\]

Can also add “functionality” and “onto” axioms

\[
\overline{p}_{i,j} \lor \overline{p}_{i,j'}
\]

\[
p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j} \quad \text{no pigeon $i$ gets two holes $j \neq j'$}
\]

\[
\text{every hole $j$ gets a pigeon}
\]

Even onto functional PHP formula is hard for resolution

“Resolution cannot count”
Tseitin formulas [Urquhart ’87]
“Sum of degrees of vertices in graph is even”

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label

\[
(x \lor y) \land (\overline{x} \lor z) \\
\land (\overline{x} \lor \overline{y}) \land (y \lor \overline{z}) \\
\land (x \lor \overline{z}) \land (\overline{y} \lor z)
\]
**Tseitin formulas** [Urquhart ’87]

“Sum of degrees of vertices in graph is even”

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of \# true incident edges = label

\[
\begin{align*}
(x \lor y) & \land (\overline{x} \lor z) \\
\land (\overline{x} \lor \overline{y}) & \land (y \lor \overline{z}) \\
\land (x \lor \overline{z}) & \land (\overline{y} \lor z)
\end{align*}
\]

Requires size \(\exp(\Omega(N))\) on well-connected so-called expanders

“Resolution cannot count mod 2”
Resolution Width

**Width** = size of largest clause in refutation (always $\leq N$)
Resolution Width

**Width** = size of largest clause in refutation (always \( \leq N \))

**Width upper bound \( \Rightarrow \) size upper bound**

**Proof:** at most \((2 \cdot \#\text{variables})^{\text{width}}\) distinct clauses
(This simple counting argument is essentially tight [Atserias et al.’14])
Resolution Width

**Width** = size of largest clause in refutation *(always $\leq N$)*

Width upper bound $\Rightarrow$ size upper bound

**Proof:** at most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses
(This simple counting argument is essentially tight [Atserias et al.'14])

Width lower bound $\Rightarrow$ size lower bound

Much less obvious...
Width Lower Bounds Imply Size Lower Bounds

**Theorem ([Ben-Sasson & Wigderson ’99])**

\[
\text{size} \geq \exp \left( \Omega \left( \frac{(\text{width})^2}{(\text{formula size } N)} \right) \right)
\]
Width Lower Bounds Imply Size Lower Bounds

Theorem ([Ben-Sasson & Wigderson ’99])

\[
\text{size} \geq \exp \left( \Omega \left( \frac{(width)^2}{(formula \ size \ N)} \right) \right)
\]

Yields superpolynomial size bounds for width \(\omega(\sqrt{N \log N})\)
Almost all known lower bounds on size derivable via width
Width Lower Bounds Imply Size Lower Bounds

**Theorem ([Ben-Sasson & Wigderson ’99])**

\[
    \text{size} \geq \exp \left( \Omega \left( \frac{(\text{width})^2}{(\text{formula size } N)} \right) \right)
\]

Yields superpolynomial size bounds for width \( \omega(\sqrt{N \log N}) \)

Almost all known lower bounds on size derivable via width

For **tree-like resolution** have size \( \geq 2^{\text{width}} \) [Ben-Sasson & Wigderson ’99]

General resolution: width up to \( O(\sqrt{N \log N}) \) implies no size lower bounds — possible to tighten analysis? **No!**
**Ordering principles** [Stålmarck '96, Bonet & Galesi '99]

“Every (partially) ordered set \( \{e_1, \ldots, e_n\} \) has minimal element”

Variables \( x_{i,j} = “e_i < e_j” \)

\[
\overline{x}_{i,j} \lor \overline{x}_{j,i} \quad \text{anti-symmetry; not both } e_i < e_j \text{ and } e_j < e_i
\]

\[
\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k} \quad \text{transitivity; } e_i < e_j \text{ and } e_j < e_k \text{ implies } e_i < e_k
\]

\[
\bigvee_{1 \leq i \leq n, i \neq j} x_{i,j} \quad e_j \text{ is not a minimal element}
\]
Optimality of the Size-Width Lower Bound

**Ordering principles** [Stålmarck '96, Bonet & Galesi '99]

“Every (partially) ordered set \(\{e_1, \ldots, e_n\}\) has minimal element”

Variables \(x_{i,j} = “e_i < e_j”\)

\[
\begin{align*}
x_{i,j} \lor x_{j,i} & \quad \text{anti-symmetry; not both } e_i < e_j \text{ and } e_j < e_i \\
x_{i,j} \lor x_{j,k} \lor x_{i,k} & \quad \text{transitivity; } e_i < e_j \text{ and } e_j < e_k \text{ implies } e_i < e_k \\
\lor_{1 \leq i \leq n, i \neq j} x_{i,j} & \quad e_j \text{ is not a minimal element}
\end{align*}
\]

Refutable in resolution in size \(O(N)\)

Requires resolution width \(\Omega\left(\frac{3}{\sqrt{N}}\right)\) (converted to 3-CNF)
Polynomial Calculus

Introduced in [Clegg et al. '96]; slightly modified in [Alekhnovich et al. '00]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\text{GF}(2)$ in practice

**Example:** $x \lor y \lor \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv \text{true}$ and $1 \equiv \text{false}$)
Polynomial Calculus

Introduced in [Clegg et al. ’96]; slightly modified in [Alekhnovich et al. ’00]

Clauses interpreted as **polynomial equations over finite field**
Any field in theory; GF(2) in practice

**Example:** \( x \lor y \lor \overline{z} \) gets translated to \( xy\overline{z} = 0 \)
(Think of \( 0 \equiv \text{true} \) and \( 1 \equiv \text{false} \))

**Derivation rules**

<table>
<thead>
<tr>
<th>Boolean axioms</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - x = 0 )</td>
<td>( x + \overline{x} = 1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Linear combination</th>
<th>Multiplication</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 0 )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( q = 0 )</td>
<td>( xp = 0 )</td>
</tr>
<tr>
<td>( \alpha p + \beta q = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

**Goal:** Derive \( 1 = 0 \) \( \iff \) no common root \( \iff \) formula unsatisfiable
**Clauses turn into monomials**

Write out all polynomials as sums of monomials

W.l.o.g. all polynomials multilinear (because of Boolean axioms)
Clauses turn into monomials
Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation
Example: Resolution step:

\[
\begin{array}{c}
x \lor \overline{y} \lor z \\
\overline{y} \lor \overline{z} \\
x \lor \overline{y}
\end{array}
\]
Polynomial Calculus Can Mimic Resolution Steps

**Example:** Resolution step:

\[
\begin{align*}
x \lor \overline{y} \lor z & \quad \overline{y} \lor \overline{z} \\
\hline
x \lor \overline{y}
\end{align*}
\]

simulated by polynomial calculus derivation:

\[
\begin{align*}
\overline{y}z = 0 & \quad \overline{y}z + \overline{y}z - \overline{y} = 0 \\
x\overline{y}z = 0 & \quad x\overline{y}z + x\overline{y}z - x\overline{y} = 0 \\
x\overline{y}z = 0 & \quad -x\overline{y}z + x\overline{y} = 0 \\
\hline
x\overline{y} = 0
\end{align*}
\]
Polynomial calculus simulates resolution efficiently

- Can mimic refutation step by step as shown on previous slide
- Essentially no increase in length/size or width/degree
- Hence worst-case upper bounds for resolution carry over

Polynomial calculus is strictly stronger w.r.t. both size and degree

- Consider, e.g., Tseitin formulas on expanders
- Over GF(2) can just do Gaussian elimination
- Also other examples not depending on field characteristic
Size vs. Degree

- **Degree upper bound ⇒ size upper bound** [Clegg et al.'96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [Atserias et al.'14]

- **Degree lower bound ⇒ size lower bound** [Impagliazzo et al.'99]
  Precursor of [Ben-Sasson & Wigderson '99] — can do same proof to get same bound

- Size-degree lower bound essentially optimal [Galesi & Lauria ’10]
  Example: same ordering principle formulas

- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery much less developed)
Lower Bounds via Expansion

**Standard approach:** Lower bounds from expansion

Simplest example: Clause-variable incidence graph (CVIG)
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**Boundary expansion:**
Subsets of left vertices have many unique neighbours on right
**Standard approach:** Lower bounds from expansion

Simplest example: Clause-variable incidence graph (CVIG)

**Boundary expansion:**
Subsets of left vertices have many unique neighbours on right

Clauses

- $x_3 \lor \overline{x}_5 \lor x_9$
- $\overline{x}_1 \lor \overline{x}_8 \lor \overline{x}_{10}$
- $x_1 \lor x_4 \lor \overline{x}_9$
- $\overline{x}_3 \lor x_8 \lor \overline{x}_{10}$
- $\overline{x}_3 \lor x_6 \lor x_9$
- $\overline{x}_2 \lor \overline{x}_5 \lor x_7$
- $\overline{x}_4 \lor x_5 \lor \overline{x}_9$
- $x_2 \lor x_7 \lor x_{10}$
- $\overline{x}_1 \lor x_3 \lor \overline{x}_6$
- $x_2 \lor x_4 \lor \overline{x}_8$

Variables

- $x_1$
- $x_2$
- $x_3$
- $x_4$
- $x_5$
- $x_6$
- $x_7$
- $x_8$
- $x_9$
- $x_{10}$
**Standard approach:** Lower bounds from expansion

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**Boundary expansion:**
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**Standard approach:** Lower bounds from expansion

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**Boundary expansion:**
Subsets of left vertices have many unique neighbours on right

```
x3 ∨ x5 ∨ x9
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x1 ∨ x8 ∨ x10</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>x1 ∨ x4 ∨ x9</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
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<td>x3 ∨ x6 ∨ x9</td>
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<tr>
<td>-----------------</td>
</tr>
<tr>
<td>x2 ∨ x5 ∨ x7</td>
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<tr>
<td>-----------------</td>
</tr>
<tr>
<td>x4 ∨ x5 ∨ x9</td>
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<tr>
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</tr>
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</tr>
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Subsets of left vertices have many unique neighbours on right

**Problem:**
CVIG might lose expansion of combinatorial problem
Lower Bounds via Expansion

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Subsets of left vertices have many unique neighbours on right

**Problem:**
CVIG might lose expansion of combinatorial problem

Need graph capturing underlying principle!
Our Results

Main Theorem (Informal)

Graph structure on formula such that expansion implies hardness in polynomial calculus

Extends an approach from [Alekhnovich, Razborov ’01]

Unifies many previous lower bounds for polynomial calculus

Corollary: New lower bound resolving open question in [Razborov ’02]
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Warm-up: Use resolution to present main ideas and challenges
Revisiting Tseitin Formulas

Given set of equations over $\mathbb{F}_2$

\[
\begin{align*}
    x + w &= 0 \\
    x + y &= 0 \\
    y + w + z &= 1 \\
    z &= 0
\end{align*}
\]
Revisiting Tseitin Formulas

Given set of equations over $\mathbb{F}_2$

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\begin{align*}
    x + w &= 0 \\
    x + y &= 0 \\
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    z &= 0
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\]

Encode as clauses

\[
\begin{align*}
    x \lor w \\
    x \lor w \\
    x \lor y \\
    x \lor y \\
    y \lor w \lor z \\
    y \lor w \lor z \\
    y \lor w \lor z \\
    y \lor w \lor z \\
    z \\
    z \\
\end{align*}
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Does CVIG expand?
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Encode as clauses

Does CVIG expand? No!
Revisiting Tseitin Formulas

Given set of equations over $\mathbb{F}_2$

\[
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\]

Encode as clauses

Does CVIG expand?  No!

Graph should encode equations, not clauses!
Use one vertex per equation on the left

Put edge if variable appears in equation
Use **one vertex per equation** on the left

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Put edge if variable appears in equation
**Constraint-Variable Incidence Graph**

Use **one vertex per equation** on the left

Put edge if variable appears in equation

Use one vertex per equation on the left

Put edge if variable appears in equation

### Clauses
- $x \lor \overline{w}$
- $\overline{x} \lor w$
- $x \lor \overline{y}$
- $\overline{x} \lor y$
- $y \lor w \lor z$
- $y \lor w \lor \overline{z}$
- $y \lor \overline{w} \lor z$
- $y \lor \overline{w} \lor \overline{z}$
- $\overline{z}$

### Variables
- $x$
- $y$
- $w$
- $z$

### Constraints
- $x + w = 0$
- $x + y = 0$
- $y + w + z = 1$
- $z = 0$

### Variables
- $x$
- $y$
- $w$
- $z$
Use one vertex per equation on the left

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Use one vertex per equation on the left

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Now the constraint-variable incidence graph expands!
Proof Sketch of Tseitin Lower Bound

Constraints

\[ \begin{align*}
  x + w &= 0 \\
  x + y &= 0 \\
  y + w + z &= 1 \\
  z &= 0
\end{align*} \]

Variables

\[ \begin{align*}
  x \\
  y \\
  w \\
  z
\end{align*} \]
For each clause, look at minimal set of constraints implying it
Proof Sketch of Tseitin Lower Bound

\[
x + w = 0
\]
\[
x + y = 0
\]
\[
y + w + z = 1
\]
\[
z = 0
\]

For each clause, look at minimal set of constraints implying it

1. For each clause, look at minimal set of constraints implying it
   - Axioms: 1 constraint needed
   - Contradiction \( \bot \): All constraints needed
Proof Sketch of Tseitin Lower Bound

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   Halfway through: Clause $C$ depending on medium-sized set $S$
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Proof Sketch of Tseitin Lower Bound

For each clause, look at minimal set of constraints implying it

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3. Proof: Suppose not $\Rightarrow$ not all of $S$ needed for $C$
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Resolution edge game on \((P, x)\)

1. Adversary provides assignment \(\rho\) to all variables
2. Can flip \(x\) to some \(b\) so that \(P\) is satisfied
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Theorem (Ben-Sasson & Wigderson ’99)

If from formula \(\mathcal{F} = \bigwedge_{P \in \mathcal{F}} P\) can form bipartite graph \(\mathcal{G}(\mathcal{F})\) such that

- \(\mathcal{G}(\mathcal{F})\) is expanding and
- for all edges \((P, x)\) we can satisfy \(P\) by flipping \(x\)

then refuting \(\mathcal{F}\) requires large width
**Tseitin**: linear equations $\Rightarrow$ easy over $\mathbb{F}_2$ (Gaussian elimination)

Need stronger guarantee from constraint-variable incidence graph!
Polynomial Calculus Edge Game

**Tseitin:** linear equations ⇒ easy over $\mathbb{F}_2$ (Gaussian elimination)

Need stronger guarantee from constraint-variable incidence graph!

**Resolution graph:**
- Graph is boundary expander
- Can play resolution edge game on every edge $(P, x)$

1. Commit to assignment $x = b$ ahead of time
2. Adversary provides assignment $\rho$ to all variables
3. Flipping $x = b$ satisfies $P$
Polynomial Calculus Edge Game

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For polynomial calculus we have to play a harder game
Polynomial Calculus Edge Game

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- Graph is boundary expander
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---

### Polynomial calculus edge game on $(P, x)$

1. Commit to assignment $x = b$ ahead of time
2. Adversary provides assignment $\rho$ to all variables
3. Flipping $x = b$ satisfies $P$

Easy to see we can’t win this game for Tseitin formulas
Main Theorem (Preliminary Version)

If from formula $\mathcal{F} = \bigwedge_{P \in \mathcal{F}} P$ can form bipartite graph $\mathcal{G}(\mathcal{F})$ such that:
- $\mathcal{G}(\mathcal{F})$ is expanding, and
- for all edges $(P, x)$ can fix $P$ to true by flipping $x$,
then refuting $\mathcal{F}$ requires large degree

Not enough to prove functional pigeonhole principle hard!
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Pigeonhole Principle (PHP)

“\( n + 1 \) pigeons don’t fit into \( n \) holes”
Pigeonhole Principle (PHP)

“$n + 1$ pigeons don’t fit into $n$ holes”
Pigeonhole Principle (PHP)

“\(n + 1\) pigeons don’t fit into \(n\) holes”

Very wide clauses — hit with restriction to decrease width

Restricts choices of holes for each pigeon — graph PHP formula
Pigeonhole Principle (PHP)

“\(n + 1\) pigeons don’t fit into \(n\) holes”

Very wide clauses — hit with restriction to decrease width

Restricts choices of holes for each pigeon — graph PHP formula

\[
\begin{align*}
\text{Pigeons} & : x_{1,1} \lor x_{1,3} & & \text{Holes} & : \overline{x}_{1,1} \lor \overline{x}_{2,1} & & \overline{x}_{2,1} \lor \overline{x}_{3,1} \\
& & \text{1} & & \overline{x}_{1,1} \lor \overline{x}_{3,1} \\
& x_{2,1} \lor x_{2,3} & & \text{2} & & \overline{x}_{2,1} \lor \overline{x}_{3,1} \\
& x_{3,1} \lor x_{3,2} & & \text{2} & & \overline{x}_{2,1} \lor \overline{x}_{3,1} \\
& x_{4,2} \lor x_{4,3} & & \text{3} & & \overline{x}_{3,2} \lor \overline{x}_{4,2} \\
\end{align*}
\]
Pigeonhole Principle (PHP)

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Restricts choices of holes for each pigeon — graph PHP formula

\[
\begin{align*}
&x_{1,1} \lor x_{1,3} \\
&x_{2,1} \lor x_{2,3} \\
&x_{3,1} \lor x_{3,2} \\
&x_{4,2} \lor x_{4,3}
\end{align*}
\]

\[
\begin{align*}
&\overline{x}_{1,1} \lor \overline{x}_{2,1} \\
&\overline{x}_{1,1} \lor \overline{x}_{3,1} \\
&\overline{x}_{2,1} \lor \overline{x}_{3,1} \\
&\overline{x}_{3,2} \lor \overline{x}_{4,2} \\
&\overline{x}_{1,3} \lor \overline{x}_{2,3} \\
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Restricts choices of holes for each pigeon — graph PHP formula

\[
\begin{align*}
\text{Pigeons:} & \quad x_{1,1} \lor x_{1,3} & \quad 1 \\
x_{2,1} \lor x_{2,3} & \quad 2 \\
x_{3,1} \lor x_{3,2} & \quad 3 \\
x_{4,2} \lor x_{4,3} & \quad 4 \\
\text{Holes:} & \quad \overline{x}_{1,1} \lor \overline{x}_{2,1} & \quad 1 \\
& \quad \overline{x}_{1,1} \lor \overline{x}_{3,1} & \\
& \quad \overline{x}_{2,1} \lor \overline{x}_{3,1} & \\
& \quad \overline{x}_{3,2} \lor \overline{x}_{4,2} & \\
& \quad \overline{x}_{1,3} \lor \overline{x}_{2,3} & \\
& \quad \overline{x}_{1,3} \lor \overline{x}_{4,3} & \\
& \quad \overline{x}_{2,3} \lor \overline{x}_{4,3} & \\
\end{align*}
\]

But again CVIG not expanding!
Proving PHP Lower Bound

Isolate hole axioms from graph and group hole variables together
Isolate hole axioms from graph and group hole variables together

Constraints

\[ x_{1,1} \lor x_{1,3} \]
\[ x_{2,1} \lor x_{2,3} \]
\[ x_{3,1} \lor x_{3,2} \]
\[ x_{4,2} \lor x_{4,3} \]

Variable groups

\{x_{1,1}, x_{2,1}, x_{3,1}\}
\{x_{3,2}, x_{4,2}\}
\{x_{1,3}, x_{2,3}, x_{4,3}\}
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\[ x_2,1 \lor x_2,3 \]
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Variable groups

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\{ x_3,2, x_4,2 \}
\{ x_1,3, x_2,3, x_4,3 \}

Pigeons

\[ x_1,1 \lor x_1,3 \]
\[ x_2,1 \lor x_2,3 \]
\[ x_3,1 \lor x_3,2 \]
\[ x_4,2 \lor x_4,3 \]

Holes

\[ \overline{x}_1,1 \lor \overline{x}_2,1 \]
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Variable groups

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Isolate hole axioms from graph and group hole variables together

- Change game to play on assignments to **groups of variables**
- Assignments must **satisfy any hole axioms touched**
Proving PHP Lower Bound

Isolate hole axioms from graph and group hole variables together

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Polynomial calculus edge game on \((P, V)\) with side constraint \(E\)

1. Commit to \(\rho_V : V \rightarrow \{0, 1\}\) satisfying any clauses touched in \(E\)
2. Adversary provides total assignment \(\rho\) not violating \(E\)
3. Flipping \(V\) to \(\rho_V\) should now satisfy \(P \land E\)
Proving PHP Lower Bound

Isolate hole axioms from graph and group hole variables together

Constraints

Variable groups

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Main Theorem

If from formula $\mathcal{F} = \mathcal{F}' \land E$ we can form bipartite graph $\mathcal{G}(\mathcal{F}')$ such that:

- $\mathcal{G}(\mathcal{F}')$ is expanding, and
- For all edges $(P, V)$ there is an assignment to $V$
  - fixing $P$ to true and
  - satisfying any touched clause in $E$

then refuting $\mathcal{F}$ requires large degree.
Main Theorem

If from formula $F = F' \land E$ we can form bipartite graph $G(F')$ such that:

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  - fixing $P$ to true and
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then refuting $F$ requires large degree

Provides common framework for previous lower bounds:

- CNF formulas with expanding CVIGs [Alekhnovich & Razborov '01]
- Pigeonhole principle [Alekhnovich & Razborov '01]
- Graph ordering principle [Galesi & Lauria '10]
Main Theorem

If from formula $\mathcal{F} = \mathcal{F}' \land E$ we can form bipartite graph $\mathcal{G}(\mathcal{F}')$ such that:

- $\mathcal{G}(\mathcal{F}')$ is expanding, and
- For all edges $(P, V)$ there is an assignment to $V$
  - fixing $P$ to true and
  - satisfying any touched clause in $E$

then refuting $\mathcal{F}$ requires large degree

Provides common framework for previous lower bounds:
- CNF formulas with expanding CVIGs [Alekhnovich & Razborov '01]
- Pigeonhole principle [Alekhnovich & Razborov '01]
- Graph ordering principle [Galesi & Lauria '10]

Allows us to establish that functional PHP is hard
Can have "fat pigeons" assigned to multiple holes
⇒ Add functionality axioms (makes mapping 1-to-1)

Can have holes with no pigeons
⇒ Add onto axioms (makes mapping onto)

Functional PHP = PHP + Functionality
Onto-PHP = PHP + Onto
Onto-FPHP = PHP + Functionality + Onto

Jakob Nordström (KTH)
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Singapore Feb '16 30/33
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Pigeons
\[ x_{1,1} \lor x_{1,3} \]
\[ x_{2,1} \lor x_{2,3} \]
\[ x_{3,1} \lor x_{3,2} \]
\[ x_{4,2} \lor x_{4,3} \]

Holes
\[ \overline{x}_{1,1} \lor \overline{x}_{2,1} \]
\[ \overline{x}_{1,1} \lor \overline{x}_{3,1} \]
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Observe that [AR '01] proves hardness of Onto-PHP

Prove that FPHP is hard in polynomial calculus
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- Observe that [AR '01] proves hardness of Onto-PHP
- Prove that FPHP is hard in polynomial calculus
Prove polynomial calculus lower bounds for other formulas
Open Problems

- Prove polynomial calculus lower bounds for other formulas
  - graph colouring formulas
  - independent set formulas

- Prove size lower bounds via technique that doesn’t use degree
  - clique formulas
  - weak pigeonhole principle formulas

- Find truly general framework capturing all PC degree lower bounds
  - We generalize only part of [Alekhnovich & Razborov ’01]
  - Cannot deal with lower bounds à la [Buss et al. ’99]

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Take-away Message

**Generalized method for polynomial calculus degree lower bounds**

- Unified framework for most previous lower bounds
- Exponential size lower bound for Functional PHP

**Future directions**

- Extend techniques further to other tricky formulas
- Develop non-degree-based size lower bound techniques
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Thank you for your attention!