Understanding Conflict-Driven SAT Solving Through the Lens of Proof Complexity

Jakob Nordström

Theoretical Computer Science Group
KTH Royal Institute of Technology

Theory reading group
November 20, 2017

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The Satisfiability Problem (SAT)

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Can we use computers to solve the SAT problem efficiently?

SAT Solving in Theory and Practice

The unreasonable effectiveness of SAT solvers

- The Boolean satisfiability problem (SAT) is NP-complete and so should be exponentially hard
- Yet conflict-driven clause learning (CDCL) SAT solvers can deal with formulas containing millions of variables
- How can they work so well? What are their limits?

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- Community structure
- Parameterized complexity
- This talk: **proof complexity**
  Rigorous analysis of underlying method of reasoning

Purpose of This Presentation

- Survey some of the research in the area (most of it **not** mine) including some ongoing work (of mine)
- Discuss some theoretical "benchmark formulas" used to understand potential and limitations of SAT solvers
- Highlight some (of the many) remaining open problems

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Caveats:

- By necessity, selective and somewhat subjective coverage
- Will sweep some technical details under the rug — happy to discuss offline
- Full references for all papers at end of slides

Some More Caveats and Clarifications

Limitations of proof complexity

- Asking for rigorous analysis is asking a lot. . .
- In addition, proof complexity considers optimal algorithms (so restrict focus to unsatisfiable formulas)
- Still possible to prove some highly nontrivial theorems
- Separate question how to interpret these theoretical theorems

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Why focus on theory benchmarks?

- See what SAT solvers can do (sometimes very neat things)
- See what SAT solvers cannot do (provably hard instances)
- See what SAT solvers "should be able" to do (formulas easy for proof system but hard for corresponding SAT solvers)

Outline

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$ (or $\neg x$)
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically
  $\Theta(h(N))$ grows equally quickly as $h(N)$ asymptotically

The Resolution Proof System Underlying CDCL

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Done when empty clause $\bot$ derived

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- annotated list or
- directed acyclic graph

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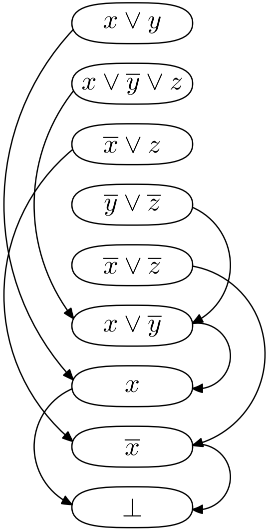

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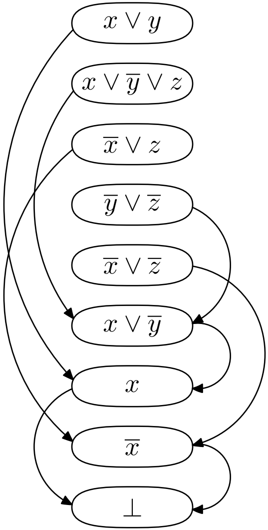
Tree-like resolution if DAG is tree



Making the Connection to DPLL

Basis of best modern SAT solvers still DPLL method [DP60, DLL62]

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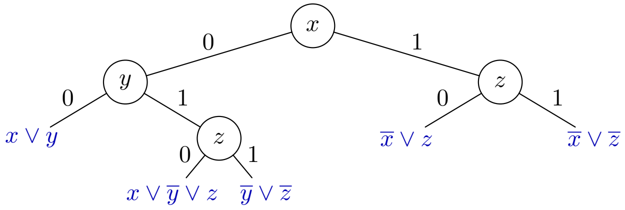
Visualize execution of DPLL algorithm as search tree

- Branch on variable assignments in internal nodes
- Stop in leaves when falsfied clause found



DPLL Execution as Resolution Proof

A DPLL execution is a resolution proof

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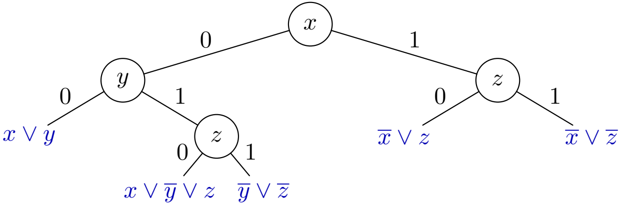

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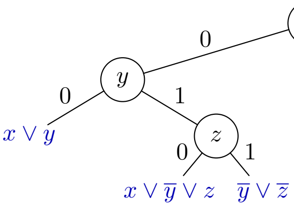

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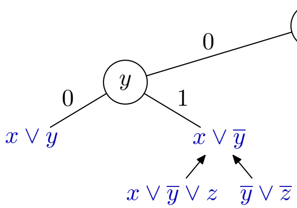

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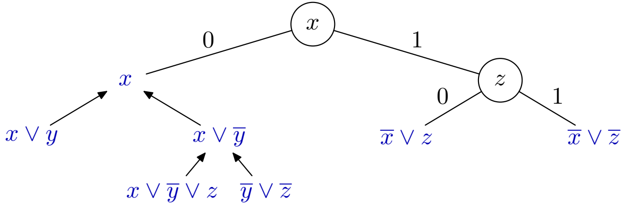

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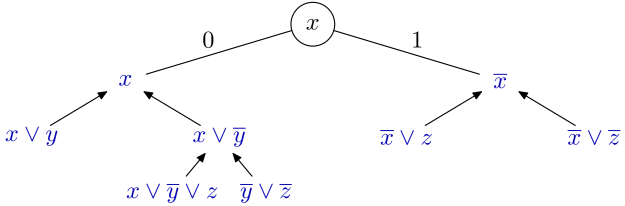

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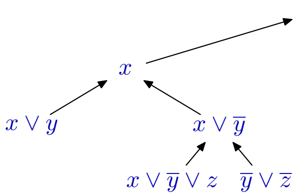

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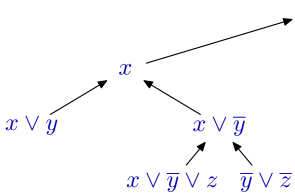

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(Slightly more needed to turn this into formal theorem, but this is essentially it)

CDCL Execution as Resolution Proof

Many more ingredients in modern CDCL SAT solvers [BS97, MS99, MMZ+01], for instance:

- Choice of branching variables crucial
- In leaf, compute & add reason for failure (clause learning)
- Restart every once in a while (saving learned clauses)

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But CDCL still yields resolution proofs (though clause learning $\Rightarrow$ general DAGs instead of trees)

Will talk more about this later in the presentation

Resolution Size/Length

Size/length of proof = # clauses (9 in our example)
Length of refuting F = min over all proofs for F

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Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Some Examples of Hard Formulas w.r.t. Length (1/2)

Pigeonhole principle (PHP) [Hak85]
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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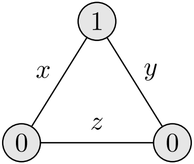
Even onto functional PHP formula is hard for resolution
"Resolution cannot count"

Some Examples of Hard Formulas w.r.t. Length (2/2)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



$$\begin{array}{rr} (x \lor y) & \land\ (\overline{x} \lor z) \\ \land\ (\overline{x} \lor \overline{y}) & \land\ (y \lor \overline{z}) \\ \land\ (x \lor \overline{z}) & \land\ (\overline{y} \lor z) \end{array}$$

Some Examples of Hard Formulas w.r.t. Length (2/2)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

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- Label every vertex 0/1 so that sum of labels odd
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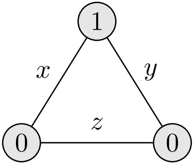

$$
\begin{aligned}
&(x \lor y) && \land (\overline{x} \lor z) \\
\land\ &(\overline{x} \lor \overline{y}) && \land (y \lor \overline{z}) \\
\land\ &(x \lor \overline{z}) && \land (\overline{y} \lor z)
\end{aligned}
$$

Requires length $\exp(\Omega(N))$ on well-connected so-called expanders
"Resolution cannot count $\bmod 2$"

Resolution Space

Space $=$ max $\#$ clauses in memory when performing refutation

Motivated by solver memory usage (but also of intrinsical theory interest)

Can be measured in different ways — makes most sense here to focus on clause space

Space at step t $=$ $\#$ clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2,4)$
7.	x	Res$(1,6)$
8.	$\overline{x}$	Res$(3,5)$
9.	$\perp$	Res$(7,8)$

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Example: Space at step 7 ...

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2,4)$
7.	x	Res$(1,6)$
8.	$\overline{x}$	Res$(3,5)$
9.	$\bot$	Res$(7,8)$

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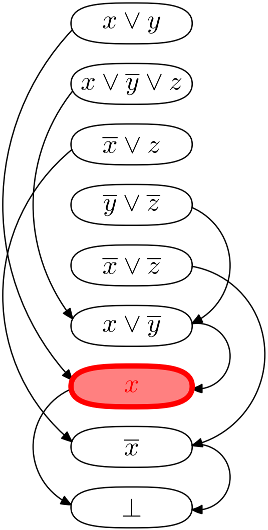

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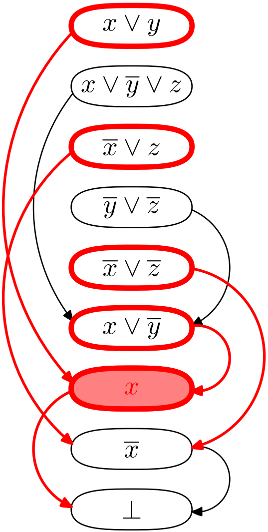

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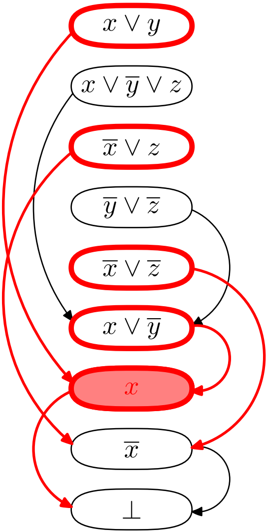

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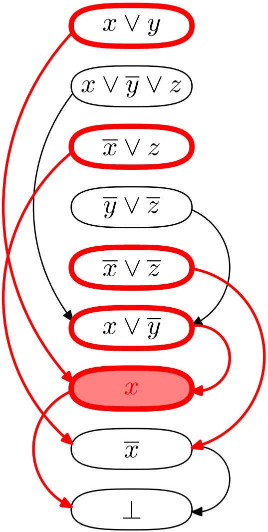
Space of proof = max over all steps
Space of refuting F = min over all proofs



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Matching $\Omega(N)$ lower bounds known [ABRW02, BG03, ET01]

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Matching $\Omega(N)$ lower bounds known [ABRW02, BG03, ET01]

Linear space lower bounds might not seem so impressive. . .

But:

- Apply for space on top of storing formula
- Hold even for optimal algorithms that magically know exactly which clauses to throw away or keep
- So significantly more space might be needed in practice
- And linear space upper bound holds only for proofs of exponential size

Length and Space

Exist space-efficient proofs $\Rightarrow$ exist short proofs [AD08]
(for k-CNF formulas, to be precise)

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Pebbling formulas [Nor09, NH13, BN08]

- Can be refuted in length $\mathcal{O}(N)$
- May require space $\Omega(N/\log N)$

Length-Space Trade-offs

Length $\approx$ running time; space $\approx$ memory consumption

SAT solvers aggressively try to minimize both — is this possible?

Length-Space Trade-offs

Length ≈ running time; space ≈ memory consumption
SAT solvers aggressively try to minimize both — is this possible?

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

Holds for

- Pebbling formulas on the right graphs
- Tseitin formulas on long, narrow rectangular grids

So simultaneous optimization not possible [at least in theory]

Abstract Description of CDCL (1/2)

Trail: a stack of decisions $x_i \stackrel{\mathsf{d}}{=} b$ and unit propagations $x_i \stackrel{C}{=} b$

$$\Big(\underbrace{x_7 \stackrel{\mathsf{d}}{=} 0}_{\text{dec. level 1}}, \underbrace{x_2 \stackrel{\mathsf{d}}{=} 1, x_{12} \stackrel{C_1}{=} 0}_{\text{decision level 2}}, \underbrace{x_6 \stackrel{\mathsf{d}}{=} 1, x_4 \stackrel{C_2}{=} 1, x_1 \stackrel{C_3}{=} 0}_{\text{decision level 3}}, \underbrace{x_{11} \stackrel{\mathsf{d}}{=} 0, x_{59} \stackrel{C_4}{=} 1}_{\text{decision level 4}}\Big)$$

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else if all variables assigned, output `SAT`;
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else if restart, set trail to () and move to **Case**;
else

1. decide if to apply database reduction to $\mathcal{D}$;
2. move to **Decision**

Abstract Description of CDCL (2/2)

Unit Pick clause $C \in \mathcal{D}$ that is unit w.r.t. trail
(All literals except one is falsified)
Add propagated assignment $x \stackrel{C}{=} b$ to trail
Move to **Case**

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Conflict **If** trail contains no decisions, output `UNSAT`;
else

- apply learning scheme to derive clause C;
- backjump, i.e., remove assignments from trail until C not false but still unit propagates;
- move to **Unit**

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Description from [EJL+16] drawing heavily on [AFT11, BHJ08, PD11]

CDCL Execution Example

Too small formula for interesting example. . .

$$(x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z})$$

CDCL Execution Example

Too small formula for interesting example… So expand slightly:

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

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$w \stackrel{\mathsf{d}}{=} 0$

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$y \stackrel{u \lor x \lor y}{=} 1$

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$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

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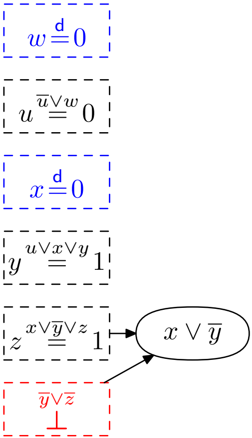

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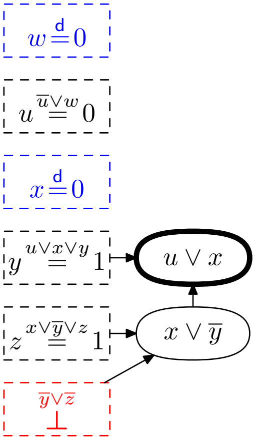

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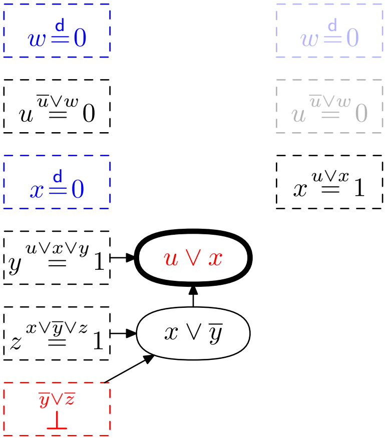

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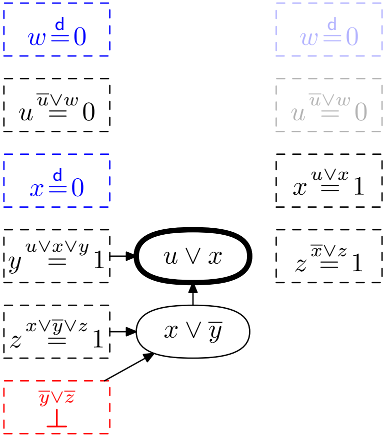

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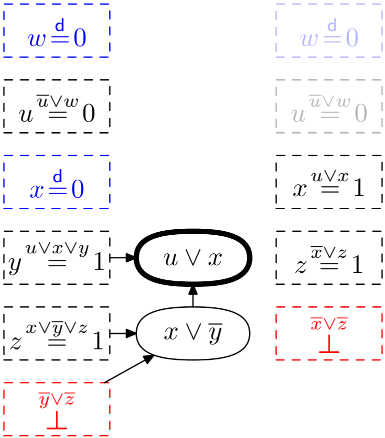

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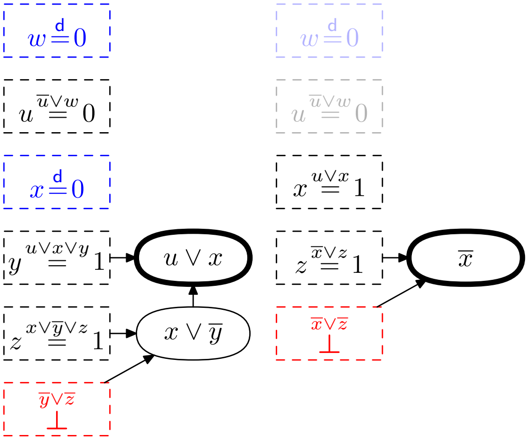

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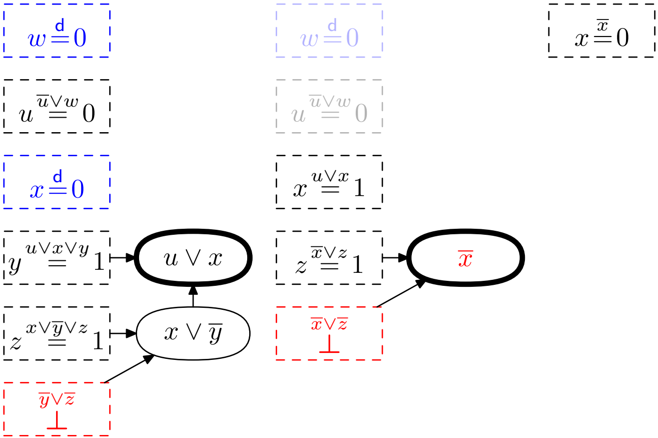

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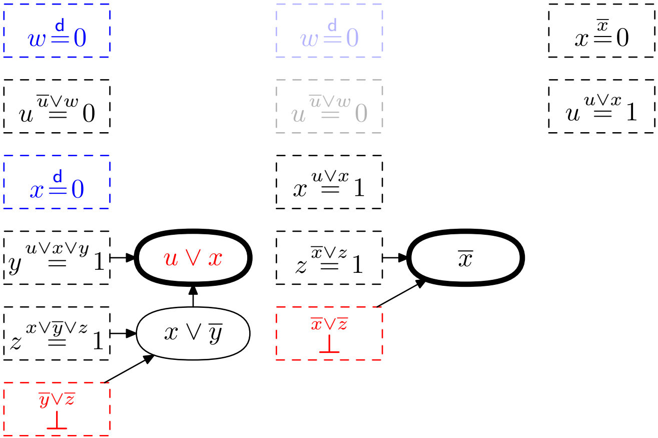

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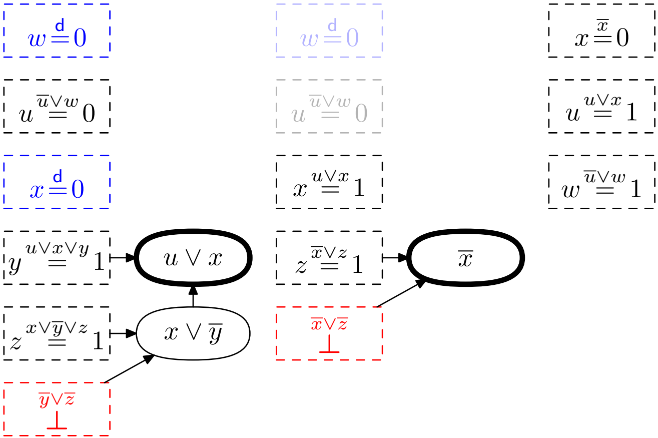

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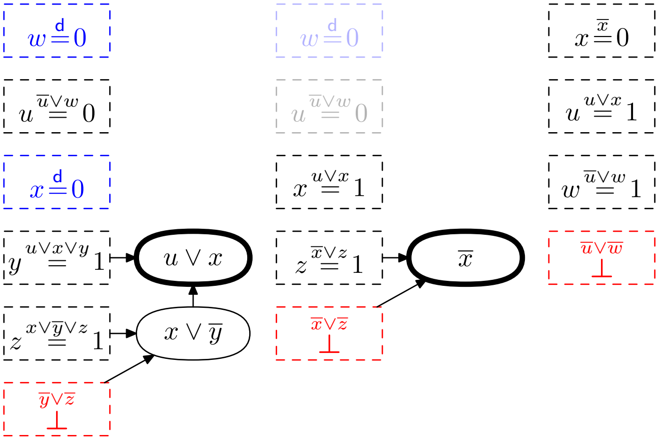

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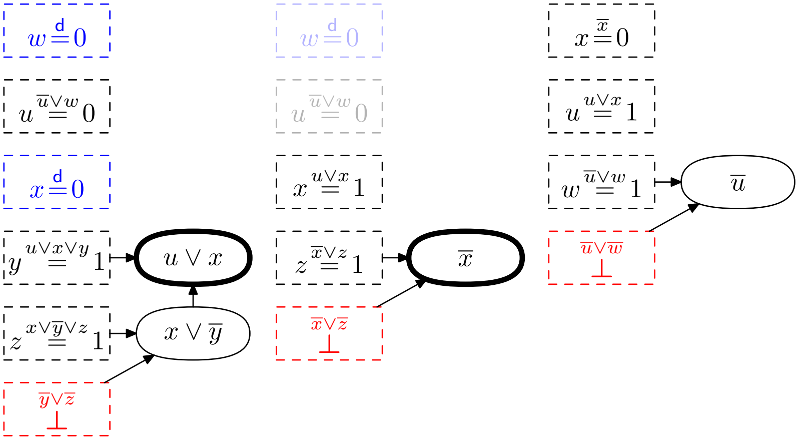

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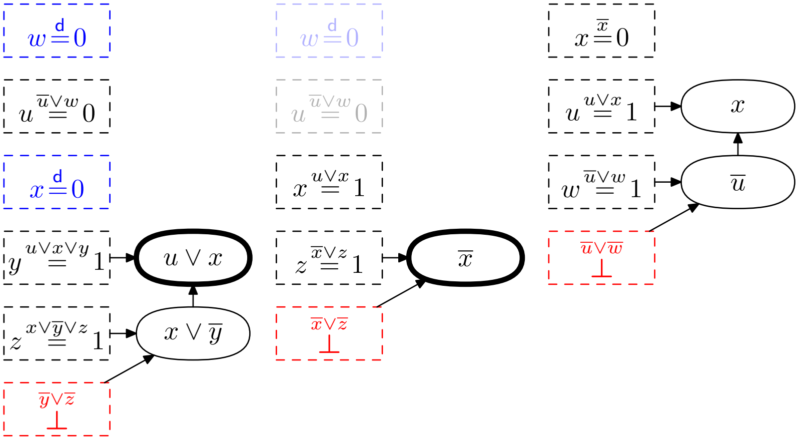

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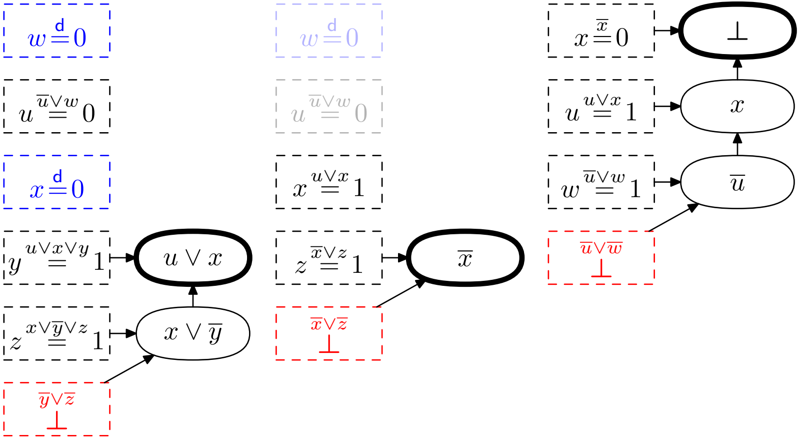

CDCL Execution Example as Resolution Refutation

Obtain resolution refutation. . .

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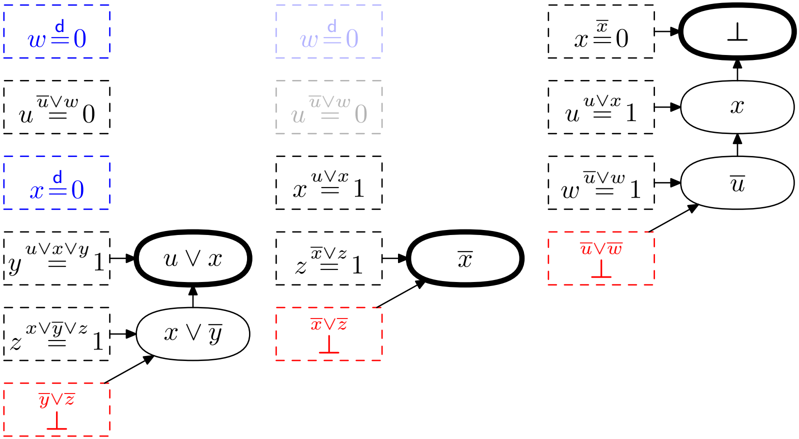
Obtain resolution refutation from CDCL execution...



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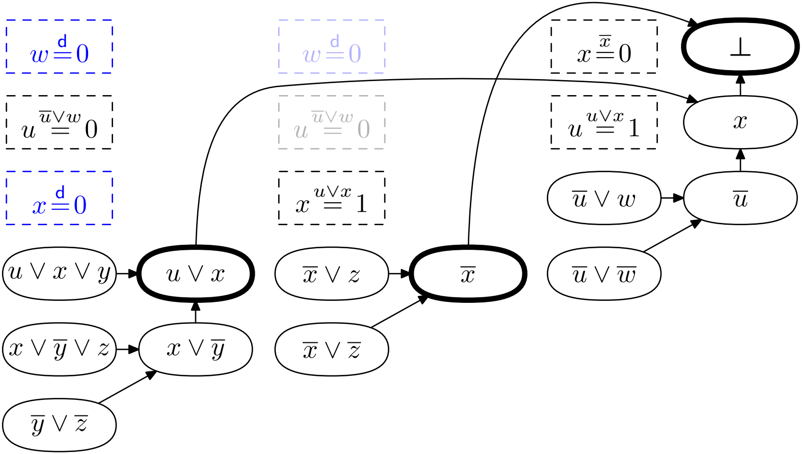
Obtain resolution refutation from CDCL execution by stringing together conflict analyses:



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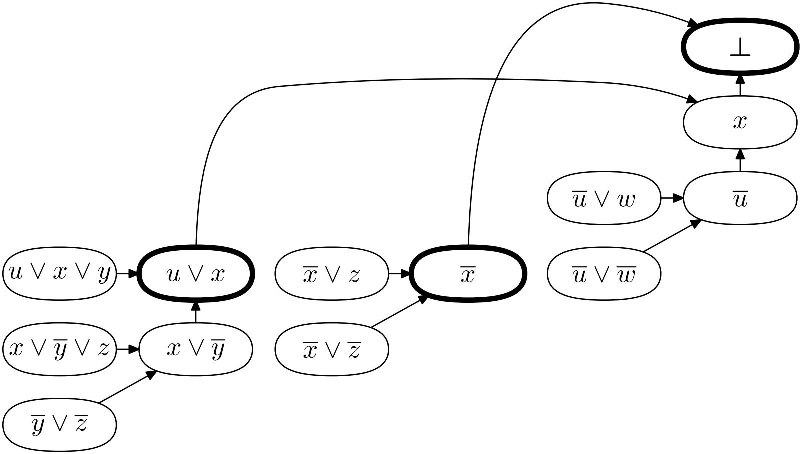

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- Long line of work aimed at proving that CDCL explores resolution search space efficiently, e.g., [BKS04, Van05, BHJ08, HBPV08]
- Challenging problem — progress only by making assumptions such as
  - artificial preprocessing
  - decisions past conflicts
  - non-standard learning scheme
  - no unit propagation(!)

CDCL Simulation of Resolution

General idea is obvious:

- Given resolution proof $(C_1, C_2, \ldots, C_\tau)$
- Force solver to efficiently learn C_t for $t = 1, 2, 3, \ldots$

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Not as easy as it seems. . .

- First result in clean model in [PD11]: CDCL as proof system polynomially simulates resolution w.r.t. time/size
- Constructive version in [AFT11]: $\exists$ resolution proof with clauses of bounded size $\Rightarrow$ CDCL will run fast
- Good, so then we're done understanding CDCL?
  Not quite. . .

Room for Further Improvement of [AFT11, PD11]?

- Very frequent **restarts** needed — no progress at all in between
  Restarts are important, but not quite *that* important?!
- **Decision strategy** in [PD11] needs (unknown) resolution proof or should be fully random in [AFT11]
  Probably inherent — fully algorithmic result unlikely [AR08]
- In **clause database** no learned clause must ever be forgotten
  But in practice something like 90–95% of clauses erased...
- Solvers typically have to run in (close to) linear time $\mathcal{O}(n)$
  But **simulation running time** something like $\mathcal{O}(n^5)$

What We Would Want

Want a more fine-grained and realistic CDCL model...

- Capture restarts, clause learning, memory management, etc.
- Modular design to allow study of different features
- Theoretical analogue of projects in [KSM11, SM11, ENSS16]

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. . . Leading to improved theoretical insights

- Can CDCL proof search be time *and space* efficient?
- And can it be *really* efficient? (No large polynomial blow-ups)
- How does memory management affect proof search quality?
- Do restarts increase reasoning power?
- How do other heuristics help or hinder proof search?

What We Have So Far (1/2)

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- **Proof:** Decisions + conflict analyses + erasures + restarts
- **Time/Size:** # decisions + propagations + conflict analysis steps
  Space: (Size of clause database) − (size of formula)

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- Only math theorems, but have some indications of similar behaviour in practical experiments [ENSS16]

Cutting Planes

Introduced in [CCT87] based on integer LP in [Gom63, Chv73]

Clauses interpreted as linear inequalities over the reals with integer coefficients (identifying $1 \equiv true$ and $0 \equiv false$)

Example: $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

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Derivation rules

Variable axioms $$\frac{}{0 \leq x \leq 1}$$

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$$

Addition $$\frac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation
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Hard Formulas w.r.t. Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with an m-clique is not $(m-1)$-colourable"

$p_{i,j}$ = indicator variables for edges in an n-vertex graph
$q_{k,i}$ = identifiers for members of m-clique in graph
$r_{i,\ell}$ = encoding of legal $(m-1)$-colouring of vertices

$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	some vertex is kth member of clique
$\overline{q}_{k,i} \lor \overline{q}_{k',i}$	clique members are uniquely defined
$p_{i,j} \lor \overline{q}_{k,i} \lor \overline{q}_{k',j}$	clique members are connected by edges
$r_{i,1} \lor r_{i,2} \lor \cdots \lor r_{i,m-1}$	every vertex i has a colour
$\overline{p}_{i,j} \lor \overline{r}_{i,\ell} \lor \overline{r}_{j,\ell}$	neighbours have distinct colours

Exponential lower bound via interpolation and circuit complexity
Technique very specifically tied to structure of formula

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Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)

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Theorem ([FPPR17, HP17])

Random CNF formulas of logarithmic width are exponentially hard for cutting planes

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Theorem ([dRNV16])

There exist flavours of pebbling formulas such that

- $\exists$ *small-size refutations with constant-size coefficients*
- $\exists$ *small-space refutations with constant-size coefficients*
- *Decreasing the space even for refutations with exponentially large coefficients causes exponential blow-up of length*

What About Conflict-Driven Cutting Planes Solvers?

So-called pseudo-Boolean SAT solvers use (a subset of) cutting planes — but seems hard to make them competitive with CDCL

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Several challenges:

- How detect unit propagation? Not enough to watch just 2 literals (or any finite number)
- Linear constraints more complicated than clauses — and integer arithmetic can become expensive
- Not obvious how to do conflict analysis
  - Can sometimes skip "resolution steps" in conflict analysis with propagating constraints on reason side — good or bad?
  - Can happen that "resolvent" is not conflicting — can be fixed in several ways, but what way is best?

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- Study tweaked versions of well-studied formulas with:
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  - without any preprocessing
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Yields huge number of potential combinations

- Not all combinations make sense, but many do
- Test also settings where "convential wisdom" knows answer

Some Preliminary Conclusions (1/2)

Importance of restarts

- Sometimes very frequent restarts very important
- Crucial in [AFT11, PD11] for CDCL to simulate resolution efficiently
- Also seems to matter in practice for some formulas which are hard for subsystems of resolution such as regular resolution (stone formulas [AJPU07])

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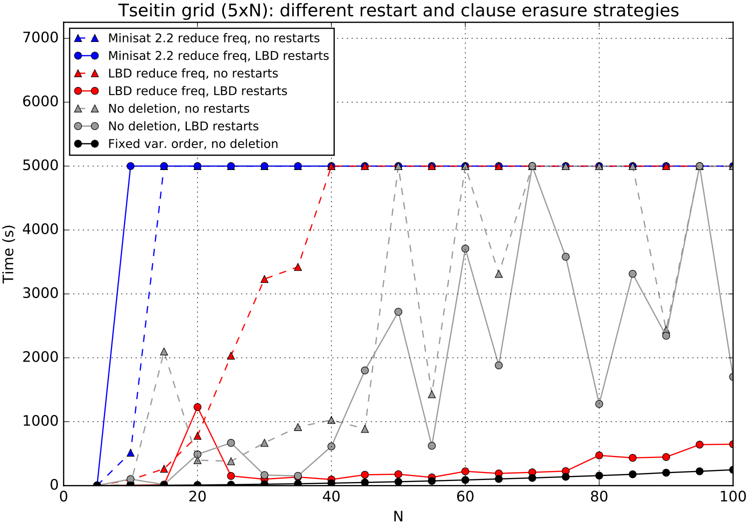
Clause erasure

- Theory says very aggressive clause removal could hurt badly
- Seem to see this on scaled-down versions of time-space trade-off formulas in [BBI12, BNT13] (Tseitin formulas)
- Even no erasure at all can be competitive for these formulas for frequent enough restarts

Plot 1: Tseitin Formulas on Grids



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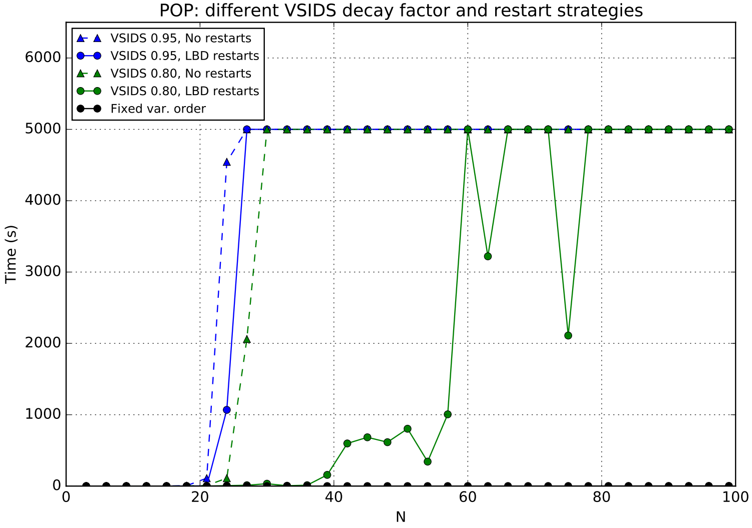
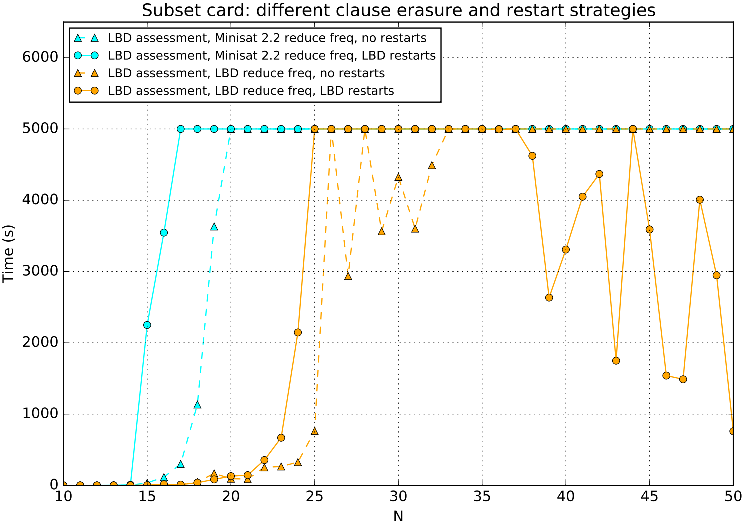
CDCL vs. resolution

- Sometimes CDCL fails miserably on easy formulas
- Sometimes strange easy-hard-easy patterns (subset cardinality formulas [Spe10, VS10, MN14])

Plot 2: Ordering Principle Formulas



Plot 3: Subset Cardinality Formulas



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Thank you for your attention!

References I

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