How Limited Interaction Hinders Real Communication
(and What It Means for Proof and Circuit Complexity)

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Joint work with Susanna F. de Rezende and Marc Vinyals

The SAT Problem in Theory and Practice

Complexity theory

- Satisfiability of formulas in propositional logic foundational problem
- SAT proven NP-complete in [Coo71, Lev73]
- Hence most likely totally intractable
- Just remains to prove this — one of the million-dollar "Millennium Problems"

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Applied SAT solving

- Dramatic performance increase last 15–20 years
- State-of-the-art SAT solvers can deal with millions of variables
- But we also know tiny formulas that are totally beyond reach
- Why do SAT solvers work so well? And why do they sometimes miserably fail?

SAT solving and Proof Complexity

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- **This work:** First such strong trade-offs capturing also cutting planes

Informal Statement of Results

Theorem (Main)

First time-space trade-offs holding uniformly for resolution, polynomial calculus, and cutting planes for formulas such that:

- $\exists$ *proofs in small size*
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Theorem (By-product)

Exponential separation in monotone-AC^i *hierarchy (improving on [RM99])*

Conjunctive Normal Form

$$(x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z})$$

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- Task: Refute given CNF formula (i.e., prove it is unsatisfiable)

The Theoretical Model

- Proof system operates with formulas of some syntactic form
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
- Refutation ends when (explicit) contradiction is derived

Cutting Planes (CP)

Clauses interpreted as linear inequalities

E.g., $x \vee y \vee \overline{z} \;\rightsquigarrow\; x + y + (1 - z) \geq 1 \;\rightsquigarrow\; x + y - z \geq 0$

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Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

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Exact derivation rules not too important for our work — just need to know that we operate with linear inequalities

Complexity Measures for Cutting Planes

Length = total # lines/inequalities in refutation

Size = sum also sizes of coefficients

Line space = max # lines in memory during refutation

Total space = sum of sizes of coefficients of lines in memory

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Worst-case bounds size $\leq 2^{\mathcal{O}(n)}$ and total space $\leq \mathcal{O}(n^2)$ for CNF formula over n variables, so mindset should be

- large size $\approx \exp(n^\delta)$
- large space $\approx n^\delta$

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What about "true" trade-offs?

Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)

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Theorem (Informal sample)

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Remarks:

- Upper bounds for # bits; lower bounds for # formulas/lines
- Analogous bounds also for resolution & polynomial calculus
- Even for semantic versions of proof systems where anything implied by blackboard can be inferred in just one step

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4. Construct graphs G with strong round-cost trade-offs for Dymond–Tompa pebbling

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- Strictly stronger than standard deterministic communication

Falsified Clause Search Problem

Fix:

- unsatisfiable CNF formula F
- (devious) partition of $Vars(F)$ between Alice and Bob

Falsified clause search problem $Search(F)$

Input: Assignment α to $Vars(F)$ split between Alice and Bob

Output: Clause $C \in F$ such that α falsifies C

Actually, computing not function but relation — will mostly ignore this for simplicity

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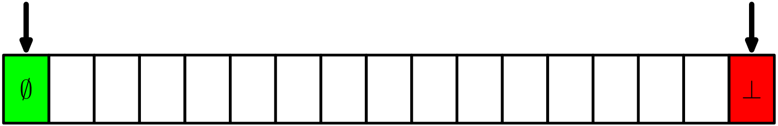

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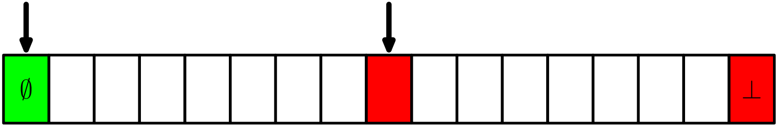

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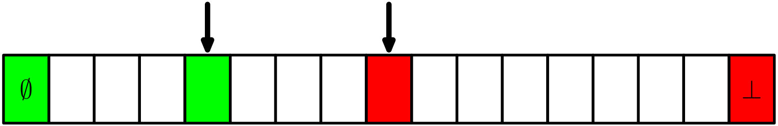

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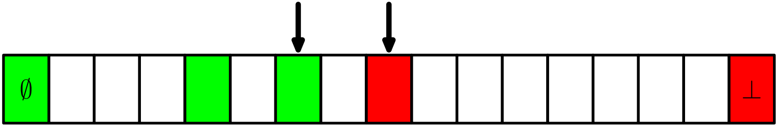

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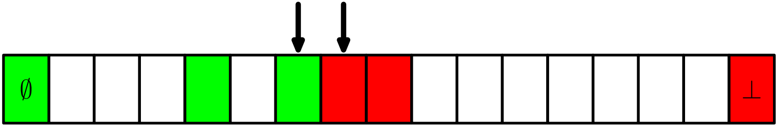

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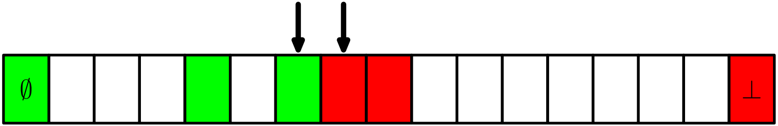

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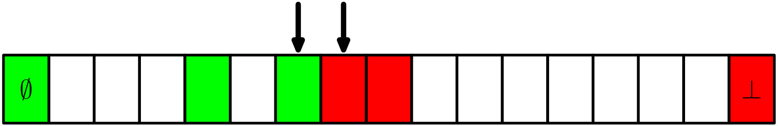

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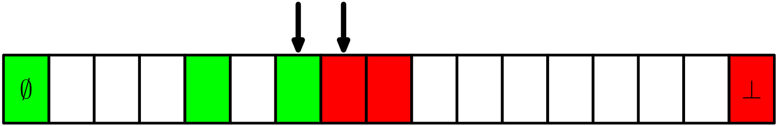

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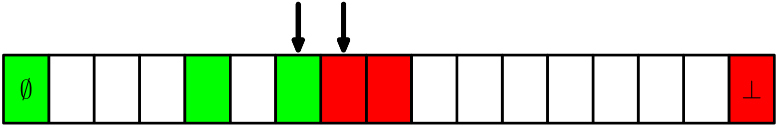

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Only one round per blackboard evaluation

(Alice and Bob simply evaluate their parts of each inequality and ask referee to compare)

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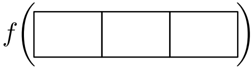

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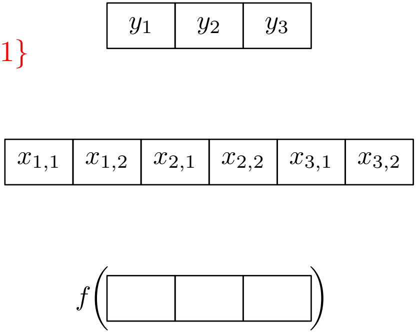

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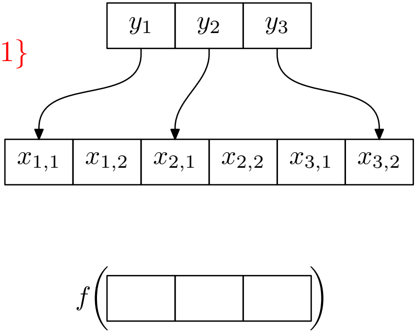

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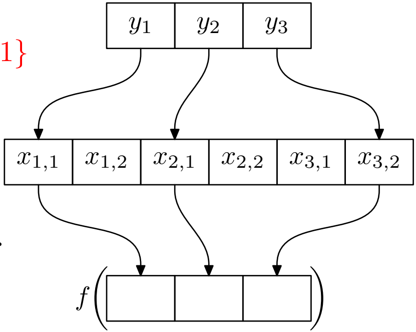

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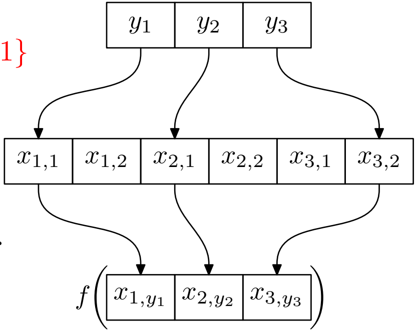

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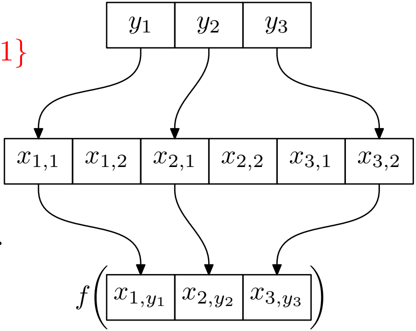

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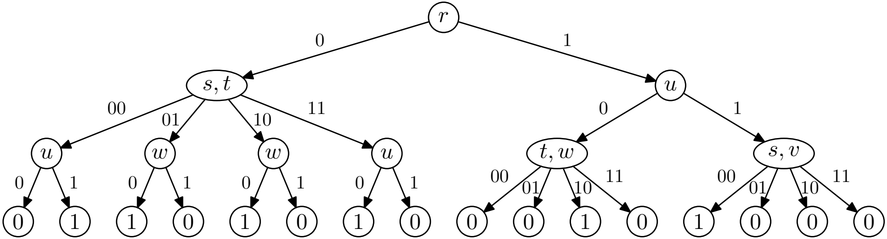
Building on ideas from e.g. [She08, BHP10]
Can encode lifted search problem for F as new CNF formula $Lift(F)$

Parallel Decision Trees

Relate lifted problem to parallel decision tree [Val75] for original problem



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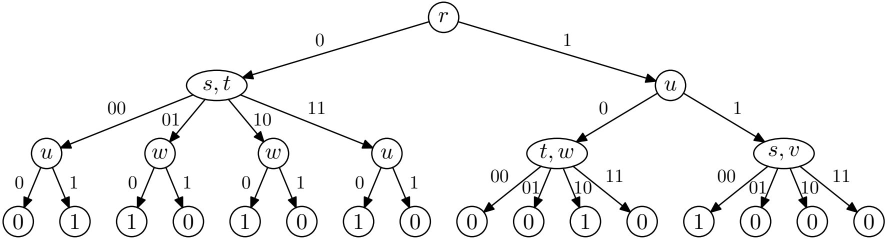

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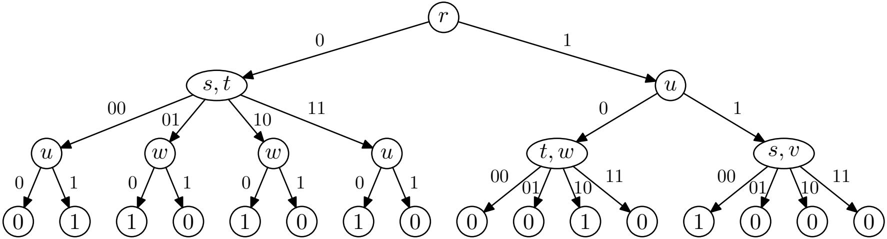

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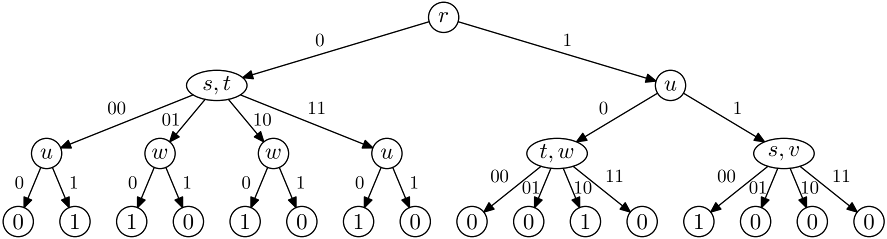

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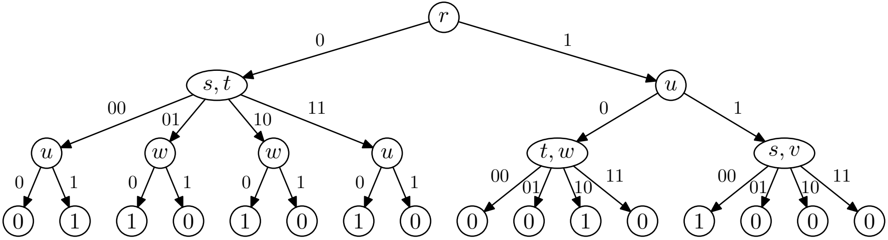

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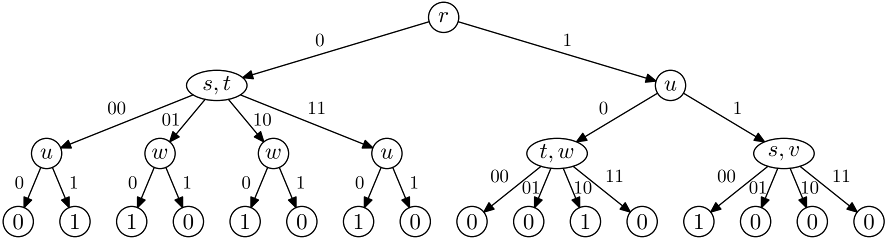

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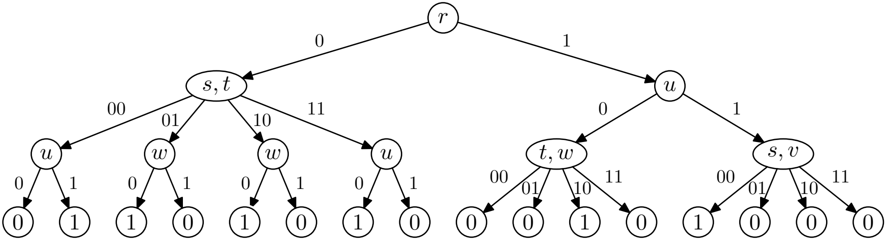
- Easy for Alice & Bob to simulate decision tree to solve lifted problem

Simulation of Decision Trees by Protocols



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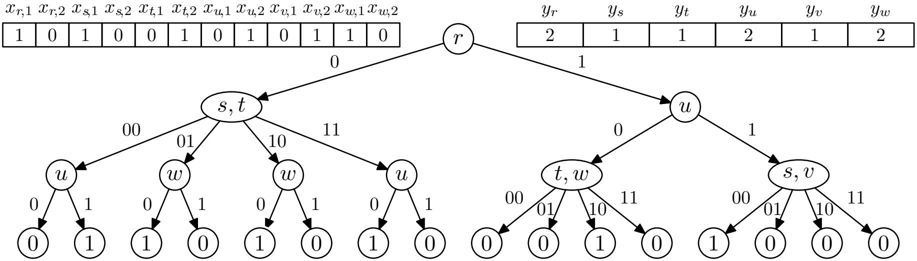

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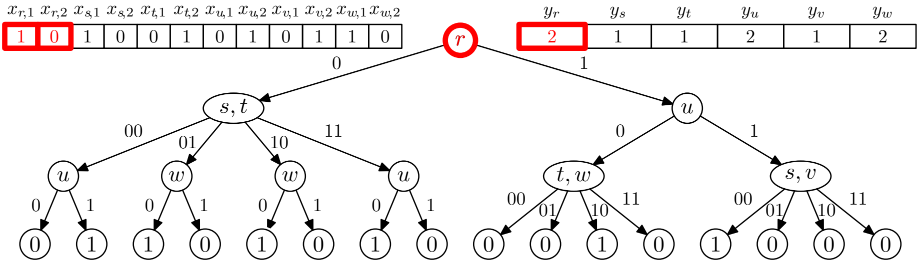

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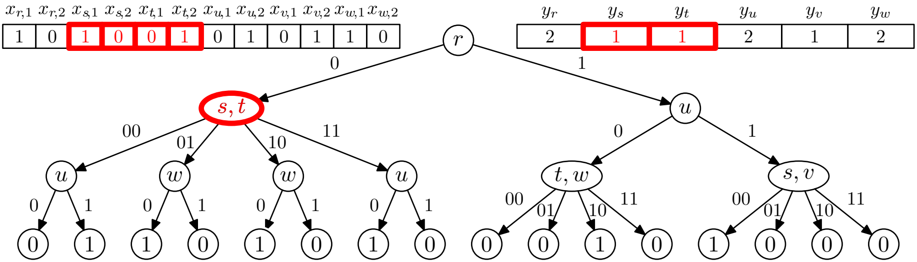

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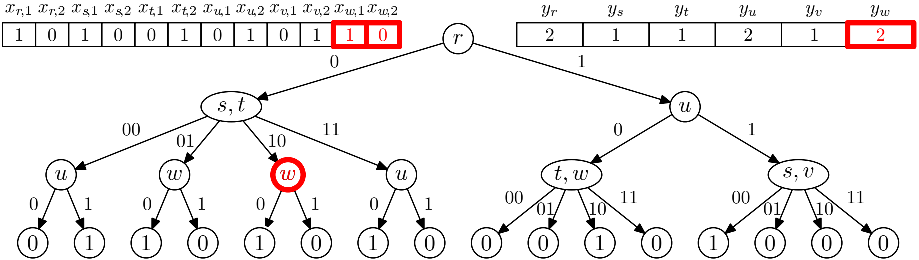

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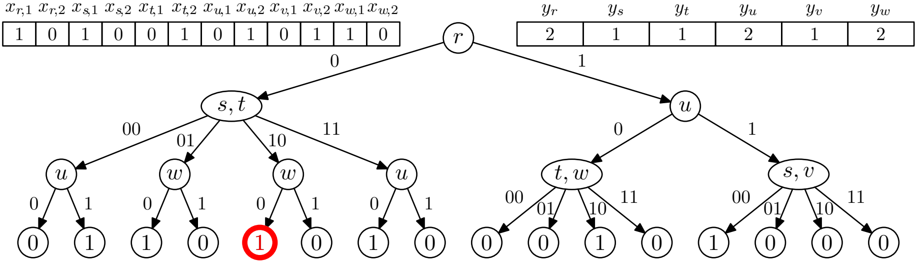

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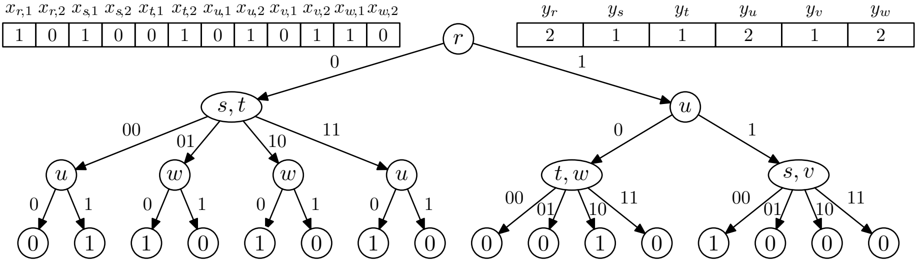
Simulation of Decision Trees by Protocols (and Vice Versa)



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Simulation theorem of protocol by decision tree (hard direction)

Let S search problem with domain $\{0,1\}^m$ and let $\ell = m^{3+\epsilon}$, $\epsilon > 0$. Then:
$\exists$ r-round real communication protocol in cost c solving $Lift_\ell(S)$
$\Rightarrow$ $\exists$ depth-r parallel decision tree solving S width $\mathcal{O}(c/\log \ell)$ queries

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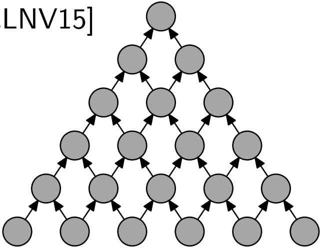

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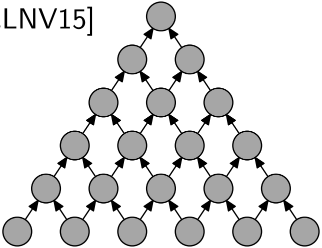

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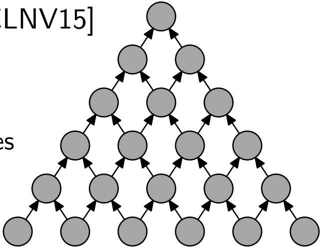

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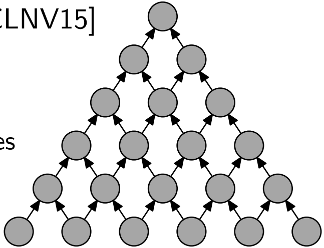

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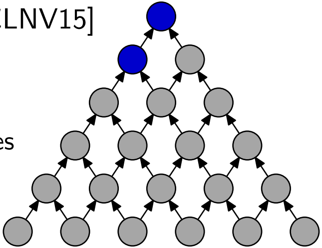

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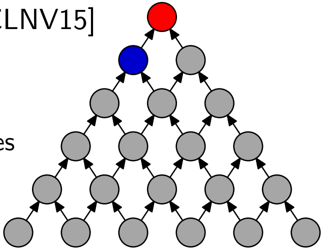

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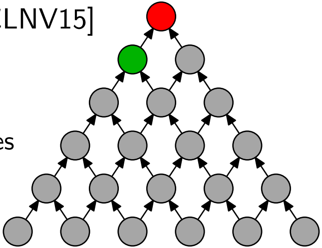

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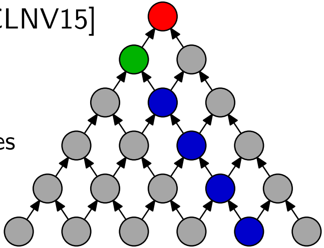

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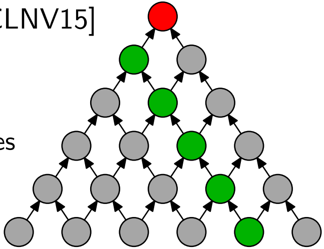

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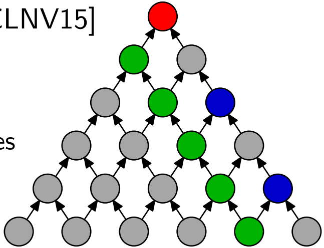

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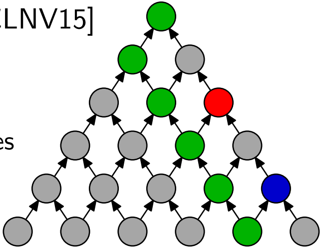

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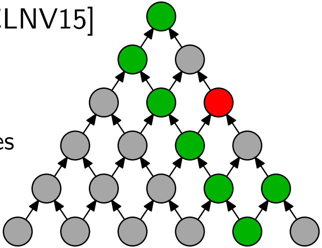

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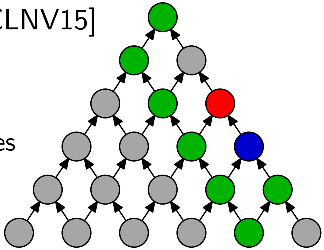

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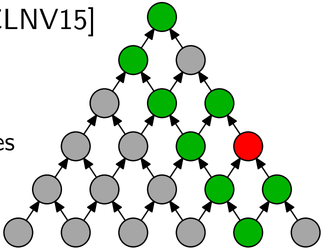

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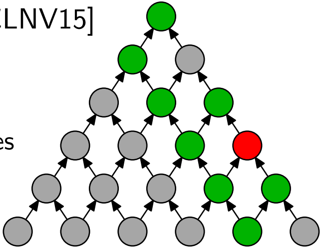

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Lemma

$\exists$ *depth-r parallel decision tree for pebbling formula Peb_G with $\leq c$ queries*
$\Rightarrow$ *Pebbler wins r-round Dymond–Tompa game on G in cost $\leq c+1$*

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$\Downarrow$

Cutting planes length-space trade-offs for lifted CNF formulas $Lift(Peb_G)$

Some Remaining Open Questions

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- Smaller lifting gadget? ($\Rightarrow$ stronger trade-offs)
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Proof complexity

- Better Dymond–Tompa trade-offs?
- Size-space trade-offs for Tseitin formulas à la [BBI12, BNT13]?
- Line space lower bounds for CP with bounded coefficients (strengthening [GPT15])

Take-Home Message

Summary of results

- Modern SAT solvers enormously successful in practice — key issue is to minimize time and memory consumption
- Modelled by proof size and space in proof complexity
- We show uniform trade-offs indicating that simultaneous optimization impossible for (essentially all) state-of-the-art techniques

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Thank you for your attention!

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