Exploring Connections Between Proof Complexity and Practical Hardness of SAT

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Université d’Artois
Lens, France
February 2, 2015

Joint work with Matti Järvisalo, Massimo Lauria, Arie Matsliah, Marc Vinyals, and Stanislav Živný
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**Proof Complexity and SAT Solving**

**Proof complexity**

- Satisfiability fundamental problem in theoretical computer science
- SAT proven NP-complete by Stephen Cook in 1971
- Hence totally intractable in worst case (probably)
- One of the million dollar “Millennium Problems”
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**SAT solving**
- Enormous progress in performance last 15–20 years
- State-of-the-art solvers can deal with real-world instances with millions of variables
- But best solvers still based on methods from early 1960s
- Tiny formulas known that are totally beyond reach
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What makes formulas hard or easy in practice for SAT solvers?
What (if anything) can proof complexity say about this?
Outline

1 SAT solving and Proof Complexity
   - SAT solving and DPLL
   - Proof Complexity and Resolution
   - Our Results

2 Experiments
   - Benchmark Formulas
   - Set-up
   - Results

3 Directions for Future Research
The General Set-up

**Conjunctive normal form (CNF)**

ANDs of ORs of variables or negated variables (or conjunctions of disjunctive clauses)

Example:

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

**Proof complexity**

Find short certificate that CNF formula is unsatisfiable (i.e., always work in UNSAT regime)
Some Terminology

- **Literal** $a$: variable $x$ or its negation $\overline{x}$

- **Clause** $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)

- **CNF formula** $F = C_1 \land \cdots \land C_m$: conjunction of clauses

- **$k$-CNF formula**: CNF formula with clauses of size $\leq k$
  (where $k$ is some constant)

- All formulas assumed to be $k$-CNFs in this talk
  (for simplicity of exposition)
The DPLL Method

Based on [Davis & Putnam ’60] and [Davis, Logemann & Loveland ’62]

Somewhat simplified description:
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- Set $x = 0$ (FALSE), simplify $F$ and try to refute recursively
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- Set $x = 1$ (true), simplify $F$ and try to refute recursively
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- Otherwise pick some variable $x$ in $F$
  - Set $x = 0$ (FALSE), simplify $F$ and try to refute recursively
  - Set $x = 1$ (TRUE), simplify $F$ and try to refute recursively
- If result in both cases “unsatisfiable”, then report “unsatisfiable”
A DPLL Toy Example

\[ F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \]
\[ \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \]
A DPLL Toy Example

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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found
A DPLL Toy Example

\[
F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\
\land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})
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\[ F = (z) \land (y \lor \overline{z}) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \]

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\[ F = ( \overline{z} ) \land ( \overline{y} \lor u ) \land ( \overline{y} \lor \overline{u} ) \land ( u \lor v ) \land ( \overline{x} \lor \overline{v} ) \land ( \overline{u} \lor w ) \land ( \overline{x} \lor \overline{u} \lor \overline{w} ) \]

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\[ F = (z) \land (\quad) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \]

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found
A DPLL Toy Example

\[ F = ( \neg z) \land (y \lor \neg z) \land ( \neg u) \land (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w) \]

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found
A DPLL Toy Example

\[ F = (z) \land (y \lor \overline{z}) \land (u) \land (v) \land (x \lor \overline{v}) \land (u \lor w) \land (x \lor u \lor \overline{w}) \]

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found

\begin{center}
\begin{tikzpicture}
  \node (x) at (0,0) [circle, draw] {x}；\node (y) at (-2,-2) [circle, draw] {y}；\node (z) at (-4,-4) [circle, draw] {z}；\node (u) at (-6,-6) [circle, draw] {u}；\node (v) at (-8,-8) [circle, draw] {v}；\node (w) at (-10,-10) [circle, draw] {w}；
  \draw (x) -- (y) node[midway, fill=red] {0}；\draw (x) -- (z) node[midway, fill=red] {0}；\draw (x) -- (u) node[midway, fill=red] {1}；\draw (x) -- (v) node[midway, fill=red] {0}；\draw (x) -- (w) node[midway, fill=red] {1}；
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  \draw (w) -- (x) node[midway, fill=red] {1}；\draw (w) -- (y) node[midway, fill=red] {0}；\draw (w) -- (z) node[midway, fill=red] {1}；\draw (w) -- (u) node[midway, fill=red] {0}；\draw (w) -- (v) node[midway, fill=red] {1}；
\end{tikzpicture}
\end{center}
A DPLL Toy Example

\[ F = \left( z \right) \land \left( y \lor \overline{z} \right) \land \left( u \right) \land \left( \begin{array}{l}
\land \left( u \lor v \right) \land \left( \overline{x} \lor \overline{v} \right) \land \left( w \right) \land \left( \overline{x} \lor \overline{w} \right)
\end{array} \right) \]

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Pick variables in internal nodes; terminate in leaves when falsified clause found
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F = (x \lor z) \land (y \lor \bar{z}) \land (x \lor \bar{y} \lor u) \land (\bar{y} \lor \bar{u}) \\
\land (u \lor v) \land (\bar{x} \lor \bar{v}) \land (\bar{u} \lor w) \land (\bar{x} \lor \bar{u} \lor \bar{w})
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State-of-the-art DPLL SAT solvers

Many more ingredients in modern SAT solvers, for instance:

- Choice of pivot variables crucial
- In particular, always do unit propagation on sole remaining variable in a clause [which our toy example didn’t]
- When reaching falsified clause, compute why partial assignment failed — add this info to formula as new clause Conflict-driven clause learning (CDCL)
- Can’t keep everything learned — prune clause database when it gets too large (but which clauses should be removed?)
- Every once in a while, restart (but save computed info)
Proof Complexity

Proof search algorithm: defines proof system with derivation rules

Proof complexity: study of proofs in such systems

- **Lower bounds**: no algorithm can do better (even optimal one always guessing the right move)
- **Upper bounds**: gives hope for good algorithms if we can search for proofs in system efficiently
Resolution

Resolution rule:

\[
\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}
\]
Resolution

Resolution rule:

\[
\begin{array}{c}
B \lor x \\
C \lor \overline{x}
\end{array}
\quad \frac{}{B \lor C}
\]

Observation

If \( F \) is a satisfiable CNF formula and \( D \) is derived from clauses \( C_1, C_2 \in F \) by the resolution rule, then \( F \land D \) is satisfiable.
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Observation

If \( F \) is a satisfiable CNF formula and \( D \) is derived from clauses \( C_1, C_2 \subseteq F \) by the resolution rule, then \( F \land D \) is satisfiable.

Prove \( F \) unsatisfiable by deriving the unsatisfiable empty clause \( \bot \) from \( F \) by resolution.
CDCL Solvers Generate Resolution Proofs

Simple example for DPLL:

\[ x \lor z \quad y \lor \overline{z} \quad x \lor \overline{y} \lor u \quad \overline{y} \lor \overline{u} \quad u \lor v \quad \overline{x} \lor \overline{v} \quad \overline{u} \lor w \quad \overline{x} \lor \overline{u} \lor \overline{w} \]
CDCL Solvers Generate Resolution Proofs

Simple example for DPLL:

\[
\begin{align*}
x & \lor y \\
x & \lor z \\
y & \lor \overline{z} \\
x & \lor y & \lor u & \lor \overline{y} & \lor \overline{u} \\
y & \lor v & \lor \overline{x} & \lor \overline{v} & \lor \overline{u} & \lor \overline{v} & \lor \overline{w} & \lor \overline{w}
\end{align*}
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CDCL Solvers Generate Resolution Proofs

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    y &\lor \overline{z} \\
    x &\lor \overline{y} &\lor u \\
    \overline{y} &\lor \overline{u} \\
    u &\lor v \\
    \overline{x} &\lor \overline{v} \\
    \overline{u} &\lor w \\
    \overline{x} &\lor \overline{u} &\lor \overline{w}
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x \lor z \quad y \lor \overline{z} \quad x \lor \overline{y} \lor u \quad \overline{y} \lor \overline{u} \quad u \lor v \quad \overline{x} \lor v \quad \overline{u} \lor w \quad \overline{x} \lor \overline{u} \lor w
\]
Simple example for DPLL:

- Conflict-driven clause learning adds “shortcut edges” in tree
- But still yields resolution proof
- True also for (most) preprocessing techniques
The Theoretical Model

- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)
- Proof system operates with disjunctive clauses
- Proof/refutation is “presented on blackboard”
- Derivation steps:
  - Write down clauses of CNF formula being refuted (axiom clauses)
  - Infer new clauses by resolution rule
  - Erase clauses that are not currently needed (to save space on blackboard)
- Refutation ends when empty clause $\bot$ is derived
Example CNF Formula

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Defined in terms of directed acyclic graph (DAG):
- source vertices true
- truth propagates upwards
- but sink vertex is false
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Example Resolution Refutation

1. \( u \)
2. \( v \)
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4. \( \overline{u} \lor \overline{v} \lor x \)
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**Blackboard bookkeeping**

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Can **write down axioms**, **erase used clauses** or **infer new clauses** by resolution rule (but only from clauses currently on the board!)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
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Write down axiom 1: $u$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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Write down axiom 1: \( u \)
Write down axiom 2: \( v \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
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Write down axiom 1: $u$
Write down axiom 2: $v$
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Example Resolution Refutation

1. $u$
2. $v$
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4. $\overline{u} \lor \overline{v} \lor x$
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Write down axiom 1: $u$
Write down axiom 2: $v$
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from $u$ and $\overline{u} \lor \overline{v} \lor x$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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Write down axiom 1: \( u \)
Write down axiom 2: \( v \)
Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)

Infer \( \overline{v} \lor x \) from
\( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Example Resolution Refutation

1. $u$
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3. $w$
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Write down axiom 2: $v$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from $u$ and $\overline{u} \lor \overline{v} \lor x$

Erase the clause $\overline{u} \lor \overline{v} \lor x$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
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Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)
Infer \( \overline{v} \lor x \) from \( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( \overline{u} \lor \overline{v} \lor x \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from $u$ and $\overline{u} \lor \overline{v} \lor x$

Erase the clause $\overline{u} \lor \overline{v} \lor x$

Erase the clause $u$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( z \)

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Write down axiom 4: \( \overline{u} \lor \overline{v} \lor x \)
Infer \( \overline{v} \lor x \) from
\( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( \overline{u} \lor \overline{v} \lor x \)
**Erase** the clause \( u \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[ \begin{align*}
& v \\
& \overline{v} \lor x \\
\end{align*} \]

\( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( u \)
Infer \( x \) from
\( v \) and \( \overline{v} \lor x \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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\( \begin{array}{l}
\text{v} \\
\text{\overline{v} \lor x} \\
\text{x}
\end{array} \)

\( u \) and \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( u \)
Infer \( x \) from
\( \begin{array}{l}
\text{v} \\
\text{\overline{v} \lor x}
\end{array} \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause $u$
Infer $x$ from $v$ and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\( v \\
\hline
x \)

Erase the clause \( \overline{u} \lor \overline{v} \lor x \)
Erase the clause \( u \)
Infer \( x \) from \( v \) and \( \overline{v} \lor x \)
Erase the clause \( \overline{v} \lor x \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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Erase the clause $u$
Infer $x$ from
$\overline{v}$ and $\overline{u} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause $v$
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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Erase the clause $u$
Infer $x$ from $v$ and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause $v$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Infer \( x \) from \( v \) and \( \overline{v} \lor x \)

Erase the clause \( \overline{v} \lor x \)

Erase the clause \( v \)

Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[ x \]
\[ \overline{x} \lor \overline{y} \lor z \]

Erase the clause \( \overline{v} \lor x \)
Erase the clause \( v \)
Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
Infer \( \overline{y} \lor z \) from
\( x \) and \( \overline{x} \lor \overline{y} \lor z \)
**Example Resolution Refutation**

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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- Erase the clause \( \overline{u} \lor x \)
- Erase the clause \( v \)
- Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
- Infer \( \overline{y} \lor z \) from \( x \) and \( \overline{x} \lor \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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```
x
\overline{x} \lor \overline{y} \lor z
\overline{y} \lor z
```

Erase the clause \( v \)

Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer \( \overline{y} \lor z \) from

\( x \) and \( \overline{x} \lor \overline{y} \lor z \)

**Erase** the clause \( \overline{x} \lor \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Erase the clause \( u \)
Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)
Infer \( \overline{y} \lor z \) from
\( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase the clause \( \overline{x} \lor \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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Write down axiom 6: \( \overline{x} \lor \overline{y} \lor z \)

Infer \( \overline{y} \lor z \) from

\( x \) and \( \overline{x} \lor \overline{y} \lor z \)

Erase the clause \( \overline{x} \lor \overline{y} \lor z \)

Erase the clause \( x \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( u \lor v \lor x \)
5. \( v \lor w \lor y \)
6. \( x \lor y \lor z \)
7. \( \neg z \)

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Write down axiom 6: \( x \lor \neg y \lor z \)
Infer \( \neg y \lor z \) from
\( x \) and \( x \lor \neg y \lor z \)
Erase the clause \( x \lor \neg y \lor z \)
Erase the clause \( x \)
# Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

## Blackboard bookkeeping

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\[
\overline{y} \lor z \\
\overline{v} \lor \overline{w} \lor y
\]

Infer \( \overline{y} \lor z \) from 
\( x \) and \( \overline{x} \lor \overline{y} \lor z \)
Erase the clause \( \overline{x} \lor \overline{y} \lor z \)
Erase the clause \( x \)
Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[ \overline{y} \lor z \]
\[ \overline{v} \lor \overline{w} \lor y \]

Erase the clause \( \overline{x} \lor \overline{y} \lor z \)
Erase the clause \( x \)
Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer \( \overline{v} \lor \overline{w} \lor z \) from \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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| $\overline{y} \lor z$     |
| $\overline{v} \lor \overline{w} \lor y$ |
| $\overline{v} \lor \overline{w} \lor z$ |

Erase the clause $\overline{x} \lor \overline{y} \lor z$

Erase the clause $x$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
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\[ \overline{y} \lor z \]
\[ \overline{v} \lor \overline{w} \lor y \]
\[ \overline{v} \lor \overline{w} \lor z \]

Erase the clause \( x \)
Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer \( \overline{v} \lor \overline{w} \lor z \) from \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase the clause \( \overline{v} \lor \overline{w} \lor y \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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- $\overline{y} \lor z$
- $\overline{u} \lor \overline{w} \lor z$

Erase the clause $x$
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
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\[ \overline{y} \lor z \]
\[ \overline{u} \lor \overline{w} \lor z \]

Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)

Infer \( \overline{v} \lor \overline{w} \lor z \) from

\[ \overline{y} \lor z \text{ and } \overline{v} \lor \overline{w} \lor y \]

Erase the clause \( \overline{v} \lor \overline{w} \lor y \)

Erase the clause \( \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
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6. \( \overline{x} \lor \overline{y} \lor z \)
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Write down axiom 5: \( \overline{v} \lor \overline{w} \lor y \)
Infer \( \overline{v} \lor \overline{w} \lor z \) from
\( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)
Erase the clause \( \overline{v} \lor \overline{w} \lor y \)
Erase the clause \( \overline{y} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
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<td>max # lines on board</td>
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</tbody>
</table>

Infer \( \overline{v} \lor \overline{w} \lor z \) from \( \overline{y} \lor z \) and \( \overline{v} \lor \overline{w} \lor y \)

Erase the clause \( \overline{v} \lor \overline{w} \lor y \)

Erase the clause \( \overline{y} \lor z \)

Write down axiom 2: \( v \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>total # clauses on board</td>
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</tr>
<tr>
<td>max # lines on board</td>
<td>4</td>
</tr>
</tbody>
</table>

$v \lor w \lor z$

$v$

$w$

$v \lor w \lor y$

Erase the clause $\overline{v} \lor \overline{w} \lor y$

$\overline{y} \lor z$

Erase the clause $\overline{y} \lor z$

Write down axiom 2: $v$

Write down axiom 3: $w$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

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\[
\begin{align*}
\overline{v} \lor \overline{w} \lor z \\
v \\
w \\
\overline{z}
\end{align*}
\]

Erase the clause \( \overline{v} \lor \overline{w} \lor y \)
Erase the clause \( \overline{y} \lor z \)
Write down axiom 2: \( v \)
Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
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<tr>
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<tbody>
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<tr>
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<td>3</td>
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<tr>
<td>max # lines on board</td>
<td>4</td>
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</table>

Write down axiom 2: \( v \)
Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)

**Infer** \( \overline{w} \lor z \) from
\( v \) and \( \overline{v} \lor \overline{w} \lor z \)
### Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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<tbody>
<tr>
<td>total $#$ clauses on board</td>
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<tr>
<td>largest clause seen on board</td>
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<tr>
<td>max $#$ lines on board</td>
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</table>

Write down axiom 2: $v$
Write down axiom 3: $w$
Write down axiom 7: $\overline{z}$

Infer $\overline{w} \lor z$ from $v$ and $\overline{v} \lor \overline{w} \lor z$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
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</table>

\[ \overline{v} \lor \overline{w} \lor z \]
\[ v \]
\[ w \]
\[ \overline{z} \]
\[ \overline{w} \lor z \]

Write down axiom \( 3: w \)

Write down axiom \( 7: \overline{z} \)

Infer \( \overline{w} \lor z \) from \( v \) and \( \overline{v} \lor \overline{w} \lor z \)

Erase the clause \( v \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
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<thead>
<tr>
<th>Description</th>
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<tbody>
<tr>
<td>total # clauses on board</td>
<td>13</td>
</tr>
<tr>
<td>largest clause seen on board</td>
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<tr>
<td>max # lines on board</td>
<td>5</td>
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</table>

Write down axiom 3: \( w \)
Write down axiom 7: \( \overline{z} \)
Infer \( \overline{w} \lor z \) from \( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Erase the clause \( v \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

**Blackboard bookkeeping**

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Write down axiom 7: \( \overline{z} \)

Infer \( \overline{w} \lor z \) from

\( v \lor \overline{u} \lor \overline{w} \lor z \)

Erase the clause \( v \)

Erase the clause \( \overline{u} \lor \overline{w} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
7. \( \overline{z} \)

Blackboard bookkeeping

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Write down axiom 7: \( \overline{z} \)
Infer \( \overline{w} \lor z \) from
\( v \) and \( \overline{v} \lor \overline{w} \lor z \)
Erase the clause \( v \)
Erase the clause \( \overline{u} \lor \overline{w} \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
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\( w \)
\( \overline{z} \)
\( \overline{w} \lor z \)

\( v \) and \( \overline{v} \lor \overline{w} \lor z \)

Erase the clause \( v \)

Erase the clause \( \overline{v} \lor \overline{w} \lor z \)

Infer \( z \) from

\( w \) and \( \overline{w} \lor z \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

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$w$
$\overline{z}$
$\overline{w} \lor z$
$z$

$v$ and $\overline{v} \lor \overline{w} \lor z$

Erase the clause $v$

Erase the clause $\overline{v} \lor \overline{w} \lor z$

Infer $z$ from

$w$ and $\overline{w} \lor z$
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
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- Erase the clause \( v \)
- Erase the clause \( \overline{v} \lor \overline{w} \lor z \)
- Infer \( z \) from \( w \) and \( \overline{w} \lor z \)
- Erase the clause \( w \)
Example Resolution Refutation

1. \(u\)
2. \(v\)
3. \(w\)
4. \(\overline{u} \lor \overline{v} \lor x\)
5. \(\overline{v} \lor \overline{w} \lor y\)
6. \(\overline{x} \lor \overline{y} \lor z\)
7. \(\overline{z}\)

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- Erase the clause \(v\)
- Erase the clause \(\overline{v} \lor \overline{w} \lor z\)
- Infer \(z\) from \(w\) and \(\overline{w} \lor z\)
- Erase the clause \(w\)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
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Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer $z$ from $w$ and $\overline{w} \lor z$
Erase the clause $w$
Erase the clause $\overline{w} \lor z$
### Example Resolution Refutation

1. \( \neg u \)
2. \( \neg v \)
3. \( w \)
4. \( \neg u \lor \neg v \lor x \)
5. \( \neg v \lor \neg w \lor y \)
6. \( \neg x \lor \neg y \lor z \)
7. \( z \)

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</table>

Erase the clause \( \neg u \lor \neg v \lor z \)

Infer \( z \) from

\( w \) and \( \neg w \lor z \)

Erase the clause \( w \)

**Erase** the clause \( \neg w \lor z \)
Example Resolution Refutation

1. \( u \)
2. \( v \)
3. \( w \)
4. \( \overline{u} \lor \overline{v} \lor x \)
5. \( \overline{v} \lor \overline{w} \lor y \)
6. \( \overline{x} \lor \overline{y} \lor z \)
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\( w \) and \( \overline{w} \lor z \)

Erase the clause \( w \)
Erase the clause \( \overline{w} \lor z \)

\textbf{Infer } \bot \textbf{ from}

\( \overline{z} \) and \( z \)
Example Resolution Refutation

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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$z$
$\neg z$
$z$
$\perp$

$w$ and $\overline{w} \lor z$
Erase the clause $w$
Erase the clause $\overline{w} \lor z$
Infer $\perp$ from $\overline{z}$ and $z$
Let $n = \text{size of formula}$

**Length/size**

# clauses in refutation — at most $\exp(n)$  
[in our example: 15]

**Width**

Size of largest clause in refutation — at most $n$  
[in our example: 3]

**Space**

Max # clauses one needs to remember when “verifying correctness of refutation on blackboard” — at most $n$ (!)  
[in our example: 5]
Length

- Clearly lower bound on running time for any CDCL algorithm (except for preprocessing techniques going beyond resolution)
Length

- Clearly lower bound on running time for any CDCL algorithm (except for preprocessing techniques going beyond resolution)
- But if there is a short refutation, not clear how to find it
Length

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- So small length upper bound might be much too optimistic

- Not the right measure of “hardness in practice”
Length vs. Width

- Searching for small width refutations known heuristic in AI community

Small width $\Rightarrow$ small length (by counting)

But small length does not necessarily imply small width — can have $\sqrt{n}$ width and linear length [Bonet & Galesi '99]

So width stricter hardness measure than length

Small width $\Rightarrow$ CDCL solver will provably be fast [Atserias, Fichte & Thurley '09]

(but slightly idealized theoretical model)

Better practical hardness measure?
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  - (but slightly idealized theoretical model)
- Better practical hardness measure?
In practice, memory consumption is a very important bottleneck for SAT solvers.
Width vs. Space

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- So maybe space complexity can be relevant hardness measure?
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So maybe space complexity can be relevant hardness measure?

Space ≥ width [Atserias & Dalmau ’03]
Width vs. Space

- In practice, memory consumption is a very important bottleneck for SAT solvers.

- So maybe space complexity can be relevant hardness measure?

- Space $\geq$ width [Atserias & Dalmau ’03]

- But small width does not imply anything about space [N. ’06], [N. & Håstad ’08], [Ben-Sasson & N. ’08]
Width vs. Space

- In practice, memory consumption is a very important bottleneck for SAT solvers.
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- Space ≥ width [Atserias & Dalmau ’03]
- But small width does not imply anything about space [N. ’06], [N. & Håstad ’08], [Ben-Sasson & N. ’08]
- So space stricter hardness measure than width
Space vs. Tree-like Space

- **Tree-like resolution**: Only allowed to use each clause once.
  - Have to rederive from scratch if needed again.

Proposed as practical measure of hardness of SAT instances in [Ansótegui, Bonet, Levy & Manyà '08].
Clearly tree-like space \( \geq \) space but not known to be different.
Space vs. Tree-like Space

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- **Tree-like space**: Usual space measure but restricted to such proofs.
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- Clearly tree-like space $\geq$ space but not known to be different.

This work can be viewed as implementing program outlined in [ABLM08]
Result 1: Separation of Space and Tree-like Space

We don’t believe in tree-like space as hardness measure

- Tree-like space tightly connected with tree-like length
- Corresponds to DPLL without clause learning
- Would suggest CDCL doesn’t buy you anything
Result 1: Separation of Space and Tree-like Space

We don’t believe in tree-like space as hardness measure

- Tree-like space tightly connected with tree-like length
- Corresponds to DPLL without clause learning
- Would suggest CDCL doesn’t buy you anything

We prove first asymptotic separation of space and tree-like space

Theorem

There are formulas requiring space $O(1)$ for which tree-like space grows like $\Omega(\log n)$

Only constant-factor separation known before [Esteban & Torán ’03]
Result 2: Small Backdoor Sets Imply Small Space

- **Backdoor sets**: practically motivated hardness measure
- First studied in [Williams, Gomes & Selman ’03]
- Real-world SAT instances often have small backdoors
Result 2: Small Backdoor Sets Imply Small Space

- **Backdoor sets**: practically motivated hardness measure
- First studied in [Williams, Gomes & Selman ’03]
- Real-world SAT instances often have small backdoors

We show connections between (strong) backdoors and space complexity (elaborating on [ABLM08])

**Theorem (Informal)**

*If a formula has a small backdoor set (for some common flavours of backdoors), then it requires small space*
Result 3: Correlation Between Hardness and Space?

Recall

$$\log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space}$$
Result 3: Correlation Between Hardness and Space?

Recall

\[ \log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space} \]

Width and space seem like most promising hardness candidates
Result 3: Correlation Between Hardness and Space?

Recall

\[ \log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space} \]

Width and space seem like most promising hardness candidates.

Run experiments on formulas with fixed complexity w.r.t. width (and length) but varying space

- Is running time essentially the same?
- Or does it increase with increasing space?
Result 3: Correlation Between Hardness and Space?

Recall

\[ \log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space} \]

Width and space seem like most promising hardness candidates

Run experiments on formulas with fixed complexity w.r.t. width (and length) but varying space*

- Is running time essentially the same?
- Or does it increase with increasing space?

Experimental results

[Järvisalo et al., CP ’12]: Hardness somewhat correlated with space
Result 3: Correlation Between Hardness and Space?

Recall

\[ \log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space} \]

Width and space seem like most promising hardness candidates

Run experiments on formulas with fixed complexity w.r.t. width (and length) but varying space*

- Is running time essentially the same?
- Or does it increase with increasing space?

Experimental results

[Järvisalo et al., CP ’12]: Hardness somewhat correlated with space except that some results seem a bit funky...

(*) But such formulas are nontrivial to find
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How to Get Hold of Good Benchmark Formulas?

Questions about space complexity and time-space trade-offs fundamental in theoretical computer science
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In particular, well-studied (and well-understood) for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and others)

- **Time** needed for calculation: \# pebbling moves
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Some quick graph terminology

- DAGs consist of **vertices** with directed **edges** between them
- vertices with no incoming edges: **sources**
- vertices with no outgoing edges: **sinks**
The Black-White Pebble Game

Goal: get single black pebble on sink vertex $z$ of $G$

<table>
<thead>
<tr>
<th># moves</th>
<th>0</th>
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<tbody>
<tr>
<td>Current # pebbles</td>
<td>0</td>
</tr>
<tr>
<td>Max # pebbles so far</td>
<td>0</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th># moves</th>
<th>1</th>
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<tbody>
<tr>
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<table>
<thead>
<tr>
<th># moves</th>
<th>2</th>
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<tr>
<td>Current # pebbles</td>
<td>2</td>
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<tr>
<td>Current # pebbles</td>
<td>3</td>
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2. Can always **remove black pebble** from vertex

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<table>
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<tr>
<th># moves</th>
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<tbody>
<tr>
<td>Current # pebbles</td>
<td>2</td>
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1. Can place black pebble on (empty) vertex $v$ if all predecessors (vertices with edges to $v$) have pebbles on them
2. Can always remove black pebble from vertex

| # moves | 5 |
| Current # pebbles | 1 |
| Max # pebbles so far | 3 |
The Black-White Pebble Game

**Goal:** get single black pebble on sink vertex $z$ of $G$

1. Can **place black pebble** on (empty) vertex $v$ if all predecessors (vertices with edges to $v$) have pebbles on them
2. Can always **remove black pebble** from vertex
3. Can always **place white pebble** on (empty) vertex

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<td># moves</td>
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<td>Current # pebbles</td>
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<tr>
<td>Max # pebbles so far</td>
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<td># moves</td>
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<td>Current # pebbles</td>
<td>2</td>
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<tr>
<td>Max # pebbles so far</td>
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4. Can remove white pebble if all predecessors have pebbles

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<table>
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<th># moves</th>
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<tr>
<td>Current # pebbles</td>
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- Can always place white pebble on (empty) vertex
- Can remove white pebble if all predecessors have pebbles

| # moves | 9 |
| Current # pebbles | 3 |
| Max # pebbles so far | 3 |
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# moves | 11
---|---
Current # pebbles | 3
Max # pebbles so far | 4
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<td>2</td>
</tr>
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Use Pebbling Formulas...

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

- sources are true
- truth propagates upwards
- but sink is false
Use Pebbling Formulas...  

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Use Pebbling Formulas... 

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

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Extensive literature on pebbling time-space trade-offs from 1970s and 80s

Pebbling formulas studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas.
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...with Functions Substituted for Variables...

Won’t work — pebbling formulas solved by unit propagation, so supereasy

Make formula harder by substitution of Boolean functions for variables
... with Functions Substituted for Variables...

Won’t work — pebbling formulas solved by unit propagation, so supereasy

Make formula harder by substitution of Boolean functions for variables

Example 1: Exclusive or

\[ x \leftarrow (x_1 \oplus x_2) = (x_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2) \]

\[ \overline{x} \leftarrow \neg(x_1 \oplus x_2) = (x_1 \lor \overline{x}_2) \land (\overline{x}_1 \lor x_2) \]

Example 2: Not-all-equal

\[ x \leftarrow \text{NAE}(x_1, x_2, x_3) = (x_1 \lor x_2 \lor x_3) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_3) \]

\[ \overline{x} \leftarrow \neg\text{NAE}(x_1, x_2, x_3) = (x_1 \lor \overline{x}_2) \land (\overline{x}_1 \lor x_2) \land (x_1 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_3) \land (x_2 \lor \overline{x}_3) \land (\overline{x}_2 \lor x_3) \]
... and Expand to Get New CNF Formula

Example with substituting every variable $x$ with $x_1 \oplus x_2$:

$$
\overline{y} \lor z \\
\downarrow \\
\neg(y_1 \oplus y_2) \lor (z_1 \oplus z_2) \\
\downarrow \\
((y_1 \lor \overline{y}_2) \land (\overline{y}_1 \lor y_2)) \lor ((z_1 \lor z_2) \land (\overline{z}_1 \lor \overline{z}_2)) \\
\downarrow \\
(y_1 \lor \overline{y}_2 \lor z_1 \lor z_2) \\
\land (y_1 \lor \overline{y}_2 \lor \overline{z}_1 \lor \overline{z}_2) \\
\land (\overline{y}_1 \lor y_2 \lor z_1 \lor z_2) \\
\land (\overline{y}_1 \lor y_2 \lor \overline{z}_1 \lor \overline{z}_2)
$$

Now CNF formula inherits pebbling graph properties!

(Also works for other functions with "right" properties, like NAE)
...and Expand to Get New CNF Formula

Example with substituting every variable $x$ with $x_1 \oplus x_2$:

$$\overline{y} \lor z$$

$$\Downarrow$$

$$\neg(y_1 \oplus y_2) \lor (z_1 \oplus z_2)$$

$$\Downarrow$$

$$( (y_1 \lor \overline{y}_2) \land (\overline{y}_1 \lor y_2) ) \lor ( (z_1 \lor z_2) \land (\overline{z}_1 \lor \overline{z}_2) )$$

$$\Downarrow$$

$$(y_1 \lor \overline{y}_2 \lor z_1 \lor z_2)$$

$$\land (y_1 \lor \overline{y}_2 \lor \overline{z}_1 \lor \overline{z}_2)$$

$$\land (\overline{y}_1 \lor y_2 \lor z_1 \lor z_2)$$

$$\land (\overline{y}_1 \lor y_2 \lor \overline{z}_1 \lor \overline{z}_2)$$

Now CNF formula inherits pebbling graph properties!
(Also works for other functions with “right” properties, like $NAE$)
About the Experiments

- 12 graph families with varying space complexity
- 12 different functions used to obtain CNF formulas from graphs
- Total of 144 formula families with around 50 instances per family
- CDCL solvers Minisat 2.2, Glucose 2.2, and Lingeling ala
- Experiments
  - with and without preprocessing
  - with and without random shuffling of formulas
- AMD Opteron 2.2 GHz CPU (2374 HE) with 16 GB of memory
- Time-out 1 hour per instance
- Massive amounts of data...
Example Results for MiniSat Without Preprocessing

Looks nice... “Easy” formulas solved faster than “hard” ones
Experiments

Results

Example Results for MiniSat with Preprocessing

Preprocessing makes formulas much easier; order still mostly right
Example Results for Lingelng with Preprocessing

And sometimes clear differences even after preprocessing
Less Nice Example for Lingeling Without Preprocessing

Hardness of formulas is in opposite order of that expected...
"Easy" formulas are too hard and running time oscillates?!
What is going on with formulas generated from pyramids?!
For Some Functions Preprocessing Really Doesn’t Help…

For the $\text{NAE}_3$ substitution function the MiniSat preprocessor doesn’t seem to help at all [left: without preprocessing; right: with preprocessing]

For other functions, though, preprocessing can decrease running times by orders of magnitude, as we just saw
... Or Sometimes Even Hurts

Same experiments for $NAE_3$ substitution function but run on Lingeling
[left: without preprocessing; right: with preprocessing]

Note that all of these formulas have very short resolution proofs

But some of them seem totally infeasible for Lingeling!
Discussion (1/3): Theory vs. Practice

**Dependence on substitution functions**
- In theory all functions equal — clearly not the case in practice
- Sometimes we can understand why, but more often not
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The problem of easy benchmarks
- All formulas easy by design — very short proofs in small width
- By design: Want to isolate space complexity as the relevant parameter
- But means SAT solvers can “get lucky”
Discussion (2/3): Behaviour of Different SAT Solvers

MiniSAT and Glucose

- Similar behaviour
- Fairly well-behaved/regular
Discussion (2/3): Behaviour of Different SAT Solvers

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- Exhibits much “wilder” behaviour in two ways:
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  - Sometimes larger formulas easier than smaller ones from same family
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  - Less correlation between running time and space complexity
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**Effects of preprocessing**
- Always improves running time, but much more significantly for MiniSAT/Glucose (and dampens correlation with space complexity)
- Not surprising — formulas amenable to preprocessing by construction
- Also, space measure doesn’t capture what happens during preprocessing
Discussion (3/3): Criticism of Benchmarks

**Artificial benchmarks**

- True, but the only formulas where we know how to control space
- In general, computing space complexity probably PSPACE-complete
- And computing width complexity EXPTIME-complete [Berkholz ’12]
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- What we can hope to determine is whether space is “more fine-grained” hardness indicator
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... Or finding a better measure than both width and space...
- Maybe some other property of formulas captures hardness better?
- Is there even a clean mathematical measure that can get close to capturing messy real-world hardness?
Some Open Problems

- Is width complexity a better measure of hardness in practice?
- Or is there some other mathematical measure that can explain practical CDCL hardness?
- Do theoretical time-space trade-offs turn up in practice for CDCL solvers?
- Can we build better SAT solvers based on algebra or geometry?
Summing up

- Modern CDCL SAT solvers amazingly successful in practice
- But poorly understood which formulas are easy or hard
- We study space as candidate measure of hardness in practice
- We see no conclusive evidence, but present some intriguing results.
- We believe that connections between proof complexity and SAT solving would be worth further exploration
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Thank you for your attention!