How Limited Interaction Hinders Real Communication
(and What It Means for Proof and Circuit Complexity)

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Workshop on Algorithms in Communication Complexity,
Property Testing and Combinatorics
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Joint work with Susanna F. de Rezende and Marc Vinyals
SAT in Theory

The Satisfiability Problem (SAT)

Given a formula $F$ in conjunctive normal form (CNF), can the variables be assigned so as to satisfy all constraints?

- Has played leading role in TCS ever since discovery of NP-completeness in [Coo71, Lev73]
- Conventional wisdom: this is a very hard problem indeed (Exponential Time Hypothesis [IP01] standard assumption)
- Yet essentially no nontrivial time complexity lower bounds
- More limited goal of time-space trade-offs also not very successful: E.g. SAT cannot be decided in time $n^{1.8}$ and space $n^{o(1)}$ [Wil08]
- Not only a sign of our weakness — there is a formidable adversary...
Enormous progress on applied SAT algorithms last 15-20 years

Current state-of-the-art SAT solvers can deal with real-world instances containing millions of variables

Use methods such as
  - conflict-driven clause learning (CDCL)
  - Gaussian elimination
  - pseudo-Boolean reasoning

Only known rigorous analysis approach: use proof complexity [CR79] to study underlying methods of reasoning

Requires lower-bounding optimal, nondeterministic algorithms — yet here we can prove strong (and sometimes tight!) trade-offs between size/time and space for resolution and polynomial calculus

This work: First such strong trade-offs capturing also cutting planes
... and in Practice

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Informal Statement of Results

Theorem (Main)

First time-space trade-offs holding uniformly for resolution, polynomial calculus, and cutting planes for formulas such that:
- ∃ proofs in small size
- ∃ proofs in small total space
- ∀ proofs few formulas in memory ⇒ length exponential

Theorem (By-product)

Exponential separation in monotone-AC$^i$ hierarchy

However, this is not a workshop on proof or circuit complexity. . .

But we need communication complexity to attack cutting planes.
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- ∃ proofs in *small size*
- ∃ proofs in *small total space*
- ∀ proofs *few formulas in memory ⇒ length exponential*

**Theorem (By-product)**

*Exponential separation in monotone-AC^i hierarchy*

However, this is **not** a workshop on proof or circuit complexity…

But we need **communication complexity** to attack cutting planes
Outline

1 Proof Complexity
   - Preliminaries
   - Previous Work
   - Our Results

2 Tools and Techniques
   - Communication Complexity
   - Pebbling Formulas
   - Lifting/Composition of Search Problems
   - Dymond–Tompa Game

3 Open Problems
Some Terminology and Notation

- **Literal** $a$: variable $x$ or its negation $\overline{x}$

- **Clause** $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)

- **CNF formula** $F = C_1 \land \cdots \land C_m$: conjunction of clauses

- **$k$-CNF formula**: all clauses of size $\leq k = \mathcal{O}(1)$

- **Goal**: **Refute** given CNF formula (i.e., prove it is unsatisfiable)
Proof Complexity

Preliminaries

The Theoretical Model

- Proof system operates with formulas of some syntactic form
- Proof/refutation is “presented on blackboard”
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
- Refutation ends when (explicit) contradiction is derived
Cutting Planes (CP)

Clauses interpreted as linear inequalities

E.g., \( x \lor y \lor \overline{z} \leadsto x + y + (1 - z) \geq 1 \leadsto x + y - z \geq 0 \)
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Works for any system of linear inequalities with integer coefficients
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Works for any system of linear inequalities with integer coefficients

Variable axioms

\[
0 \leq x \leq 1
\]

Addition

\[
\sum a_i x_i \geq A \quad \sum b_i x_i \geq B \\
\sum (a_i + b_i) x_i \geq A + B
\]

Multiplication

\[
\sum a_i x_i \geq A \\
\sum c a_i x_i \geq cA
\]

Division

\[
\sum c a_i x_i \geq A \\
\sum a_i x_i \geq \lceil A/c \rceil
\]

Goal: Derive $0 \geq 1 \iff$ formula/system of inequalities unsatisfiable
Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} \lor x_{1,2}$
2. $x_{2,1} \lor x_{2,2}$
3. $x_{3,1} \lor x_{3,2}$
4. $\overline{x}_{1,1} \lor \overline{x}_{2,1}$
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**Pigeonhole principle (PHP)**

“$n + 1$ pigeons don’t fit into $n$ holes”

Variables $x_{i,j} =$ “pigeon $i$ goes into hole $j$”

$x_{i,1} \lor x_{i,2} \lor \cdots \lor x_{i,n}$  

every pigeon $i$ gets a hole

$\overline{x}_{i,j} \lor \overline{x}_{i',j}$  

no hole $j$ gets two pigeons $i \neq i'$
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8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Write down axiom 4: \( -x_{1,1} - x_{2,1} \geq -1 \)
Write down axiom 5: \( -x_{1,1} - x_{3,1} \geq -1 \)
Add to get \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)
Erase the line \( -x_{1,1} - x_{3,1} \geq -1 \)
Erase the line \( -x_{1,1} - x_{2,1} \geq -1 \)
Write down axiom 6: \( -x_{2,1} - x_{3,1} \geq -1 \)
Add to get \( -2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3 \)
Erase the line \( -x_{2,1} - x_{3,1} \geq -1 \)
Erase the line \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)

\[
\begin{align*}
-2x_{1,1} - x_{2,1} - x_{3,1} & \geq -2 \\
-2x_{1,1} - 2x_{2,1} - 2x_{3,1} & \geq -3
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
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**History of derivation steps**

Write down axiom 4: \(-x_{1,1} - x_{2,1} \geq -1\)
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Add to get \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
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**Erase** the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)

\[-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\]
Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
2. \(x_{2,1} + x_{2,2} \geq 1\)
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4. \(-x_{1,1} - x_{2,1} \geq -1\)
5. \(-x_{1,1} - x_{3,1} \geq -1\)
6. \(-x_{2,1} - x_{3,1} \geq -1\)
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History of derivation steps

Write down axiom 4: \(-x_{1,1} - x_{2,1} \geq -1\)
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Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)

\[-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
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**History of derivation steps**

Write down axiom 4: \( -x_{1,1} - x_{2,1} \geq -1 \)
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Erase the line \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)

Divide to get \( -x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
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**History of derivation steps**

Write down axiom 5: \(-x_{1,1} - x_{3,1} \geq -1\)
Add to get \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
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Write down axiom 6: \(-x_{2,1} - x_{3,1} \geq -1\)
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Erase the line \(-x_{2,1} - x_{3,1} \geq -1\)
Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)

**Erase** the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

### History of derivation steps

Write down axiom 5: \( -x_{1,1} - x_{3,1} \geq -1 \)

Add to get \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)

Erase the line \( -x_{1,1} - x_{3,1} \geq -1 \)

Erase the line \( -x_{1,1} - x_{2,1} \geq -1 \)

Write down axiom 6: \( -x_{2,1} - x_{3,1} \geq -1 \)

Add to get \( -2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3 \)

Erase the line \( -x_{2,1} - x_{3,1} \geq -1 \)

Erase the line \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)

Divide to get \( -x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \)

Erase the line \( -2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3 \)

\[-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Add to get \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
Erase the line \(-x_{1,1} - x_{3,1} \geq -1\)
Erase the line \(-x_{1,1} - x_{2,1} \geq -1\)
Write down axiom 6: \(-x_{2,1} - x_{3,1} \geq -1\)
Add to get \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
Erase the line \(-x_{2,1} - x_{3,1} \geq -1\)
Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

- Erase the line \(-x_{1,1} - x_{3,1} \geq -1\)
- Erase the line \(-x_{1,1} - x_{2,1} \geq -1\)
- Write down axiom 6: \(-x_{2,1} - x_{3,1} \geq -1\)
- Add to get \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
- Erase the line \(-x_{2,1} - x_{3,1} \geq -1\)
- Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
- Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
- Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
- Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Erase the line \( -x_{1,1} - x_{2,1} \geq -1 \)
Write down axiom 6: \( -x_{2,1} - x_{3,1} \geq -1 \)
Add to get \( -2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3 \)
Erase the line \( -x_{2,1} - x_{3,1} \geq -1 \)
Erase the line \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)
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Write down axiom 8: \( -x_{1,2} - x_{3,2} \geq -1 \)
Add to get \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)

\[\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} & \geq -1 \\
-x_{1,2} - x_{2,2} & \geq -1 \\
-x_{1,2} - x_{3,2} & \geq -1
\end{align*}\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
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4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
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**History of derivation steps**

Erase the line \(-x_{1,1} - x_{2,1} \geq -1\)
Write down axiom 6: \(-x_{2,1} - x_{3,1} \geq -1\)
Add to get \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
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Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
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Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
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4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
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**History of derivation steps**

Write down axiom 6: \(-x_{2,1} - x_{3,1} \geq -1\)
Add to get \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
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Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
2. \(x_{2,1} + x_{2,2} \geq 1\)
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4. \(-x_{1,1} - x_{2,1} \geq -1\)
5. \(-x_{1,1} - x_{3,1} \geq -1\)
6. \(-x_{2,1} - x_{3,1} \geq -1\)
7. \(-x_{1,2} - x_{2,2} \geq -1\)
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**History of derivation steps**

Write down axiom 6: \(-x_{2,1} - x_{3,1} \geq -1\)
Add to get \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
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Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
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Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
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Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)

Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
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Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
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4. \( -x_{1,1} - x_{2,1} \geq -1 \)
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7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Add to get \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
Erase the line \(-x_{2,1} - x_{3,1} \geq -1\)
Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
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Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} & \geq -1 \\
-2x_{1,2} - x_{2,2} - x_{3,2} & \geq -2
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
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4. \( -x_{1,1} - x_{2,1} \geq -1 \)
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6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
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**History of derivation steps**

- Erase the line \( -x_{2,1} - x_{3,1} \geq -1 \)
- Erase the line \( -2x_{1,1} - x_{2,1} - x_{3,1} \geq -2 \)
- Divide to get \( -x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \)
- Erase the line \( -2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3 \)
- Write down axiom 7: \( -x_{1,2} - x_{2,2} \geq -1 \)
- Write down axiom 8: \( -x_{1,2} - x_{3,2} \geq -1 \)
- Add to get \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
- Erase the line \( -x_{1,2} - x_{3,2} \geq -1 \)
- Erase the line \( -x_{1,2} - x_{2,2} \geq -1 \)

Write down axiom 9: \( -x_{2,2} - x_{3,2} \geq -1 \)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

- Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
- Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
- Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
- Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \(-x_{1,1} - x_{2,1} \geq -1 \)
5. \(-x_{1,1} - x_{3,1} \geq -1 \)
6. \(-x_{2,1} - x_{3,1} \geq -1 \)
7. \(-x_{1,2} - x_{2,2} \geq -1 \)
8. \(-x_{1,2} - x_{3,2} \geq -1 \)
9. \(-x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Erase the line \(-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2\)
Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

**History of derivation steps**

Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Erase the line $-x_{2,2} - x_{3,2} \geq -1$

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} & \geq -1 \\
-2x_{1,2} - x_{2,2} - x_{3,2} & \geq -2 \\
-x_{2,2} - x_{3,2} & \geq -1 \\
-2x_{1,2} - 2x_{2,2} - 2x_{3,2} & \geq -3
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
2. \(x_{2,1} + x_{2,2} \geq 1\)
3. \(x_{3,1} + x_{3,2} \geq 1\)
4. \(-x_{1,1} - x_{2,1} \geq -1\)
5. \(-x_{1,1} - x_{3,1} \geq -1\)
6. \(-x_{2,1} - x_{3,1} \geq -1\)
7. \(-x_{1,2} - x_{2,2} \geq -1\)
8. \(-x_{1,2} - x_{3,2} \geq -1\)
9. \(-x_{2,2} - x_{3,2} \geq -1\)

History of derivation steps

Divide to get \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

- Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
- Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)

\[
-x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \\
-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \\
-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3
\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \(-x_{1,1} - x_{2,1} \geq -1 \)
5. \(-x_{1,1} - x_{3,1} \geq -1 \)
6. \(-x_{2,1} - x_{3,1} \geq -1 \)
7. \(-x_{1,2} - x_{2,2} \geq -1 \)
8. \(-x_{1,2} - x_{3,2} \geq -1 \)
9. \(-x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

- Erase the line \(-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3\)
- Write down axiom 7: \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)

\[-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\]
\[-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Write down axiom 7: \( -x_{1,2} - x_{2,2} \geq -1 \)
Write down axiom 8: \( -x_{1,2} - x_{3,2} \geq -1 \)
Add to get \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Erase the line \( -x_{1,2} - x_{3,2} \geq -1 \)
Erase the line \( -x_{1,2} - x_{2,2} \geq -1 \)
Write down axiom 9: \( -x_{2,2} - x_{3,2} \geq -1 \)
Add to get \( -2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3 \)
Erase the line \( -x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Divide to get \( -x_{1,2} - x_{2,2} - x_{3,2} \geq -1 \)
Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

**Erase** the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Write down axiom 8: \(-x_{1,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)

**Erase** the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Add to get \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Erase the line \( -x_{1,2} - x_{3,2} \geq -1 \)
Erase the line \( -x_{1,2} - x_{2,2} \geq -1 \)
Write down axiom 9: \( -x_{2,2} - x_{3,2} \geq -1 \)
Add to get \( -2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3 \)
Erase the line \( -x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Divide to get \( -x_{1,2} - x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3 \)
Add to get \( -x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Add to get \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
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4. \( -x_{1,1} - x_{2,1} \geq -1 \)
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7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

- Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
- Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
- Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
2. \(x_{2,1} + x_{2,2} \geq 1\)
3. \(x_{3,1} + x_{3,2} \geq 1\)
4. \(-x_{1,1} - x_{2,1} \geq -1\)
5. \(-x_{1,1} - x_{3,1} \geq -1\)
6. \(-x_{2,1} - x_{3,1} \geq -1\)
7. \(-x_{1,2} - x_{2,2} \geq -1\)
8. \(-x_{1,2} - x_{3,2} \geq -1\)
9. \(-x_{2,2} - x_{3,2} \geq -1\)

**History of derivation steps**

Erase the line \(-x_{1,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,2} - x_{2,2} \geq -1\)
Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)

Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Erase the line \( -x_{1,2} - x_{2,2} \geq -1 \)
Write down axiom 9: \( -x_{2,2} - x_{3,2} \geq -1 \)
Add to get \( -2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3 \)
Erase the line \( -x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -2x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Divide to get \( -x_{1,2} - x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3 \)
Add to get \( -x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Erase the line \( -x_{1,2} - x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \)

\[
-x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \\
-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2
\]
Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{2,2} \geq -1$

Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$

Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Erase the line $-x_{2,2} - x_{3,2} \geq -1$

Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Write down axiom 9: \(-x_{2,2} - x_{3,2} \geq -1\)
Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Write down axiom 1: \(x_{1,1} + x_{1,2} \geq 1\)

\[-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\]

\[x_{1,1} + x_{1,2} \geq 1\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Add to get \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Write down axiom 1: \( x_{1,1} + x_{1,2} \geq 1 \)
Write down axiom 2: \( x_{2,1} + x_{2,2} \geq 1 \)

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} & \geq -2 \\
x_{1,1} + x_{1,2} & \geq 1 \\
x_{2,1} + x_{2,2} & \geq 1
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

### History of derivation steps

- Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
- Write down axiom 1: \( x_{1,1} + x_{1,2} \geq 1 \)
- Write down axiom 2: \( x_{2,1} + x_{2,2} \geq 1 \)
- Add to get \( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2 \)
Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
2. \(x_{2,1} + x_{2,2} \geq 1\)
3. \(x_{3,1} + x_{3,2} \geq 1\)
4. \(-x_{1,1} - x_{2,1} \geq -1\)
5. \(-x_{1,1} - x_{3,1} \geq -1\)
6. \(-x_{2,1} - x_{3,1} \geq -1\)
7. \(-x_{1,2} - x_{2,2} \geq -1\)
8. \(-x_{1,2} - x_{3,2} \geq -1\)
9. \(-x_{2,2} - x_{3,2} \geq -1\)

History of derivation steps

- Erase the line \(-x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
- Write down axiom 1: \(x_{1,1} + x_{1,2} \geq 1\)
- Write down axiom 2: \(x_{2,1} + x_{2,2} \geq 1\)
- Add to get \(x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Write down axiom 1: \( x_{1,1} + x_{1,2} \geq 1 \)
Write down axiom 2: \( x_{2,1} + x_{2,2} \geq 1 \)
Add to get \( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2 \)
Erase the line \( x_{2,1} + x_{2,2} \geq 1 \)

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} & \geq -2 \\
x_{1,1} + x_{1,2} & \geq 1 \\
x_{2,1} + x_{2,2} & \geq 1 \\
x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} & \geq 2
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Erase the line \(-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Divide to get \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Write down axiom 1: \( x_{1,1} + x_{1,2} \geq 1 \)
Write down axiom 2: \( x_{2,1} + x_{2,2} \geq 1 \)
Add to get \( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2\)

Erase the line \( x_{2,1} + x_{2,2} \geq 1\)

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} & \geq -2 \\
x_{1,1} + x_{1,2} & \geq 1 \\
x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} & \geq 2
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

History of derivation steps

Divide to get \( -x_{1,2} - x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3 \)
Add to get \( -x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2 \)
Erase the line \( -x_{1,2} - x_{2,2} - x_{3,2} \geq -1 \)
Erase the line \( -x_{1,1} - x_{2,1} - x_{3,1} \geq -1 \)
Write down axiom 1: \( x_{1,1} + x_{1,2} \geq 1 \)
Write down axiom 2: \( x_{2,1} + x_{2,2} \geq 1 \)
Add to get \( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2 \)
Erase the line \( x_{2,1} + x_{2,2} \geq 1 \)

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} & \geq -2 \\
x_{1,1} + x_{1,2} & \geq 1 \\
x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} & \geq 2
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

History of derivation steps

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$

Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$

Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Erase the line $x_{2,1} + x_{2,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

- Erase the line \(-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3\)
- Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
- Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
- Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
- Write down axiom 1: \( x_{1,1} + x_{1,2} \geq 1 \)
- Write down axiom 2: \( x_{2,1} + x_{2,2} \geq 1 \)
- Add to get \( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2\)
- Erase the line \( x_{2,1} + x_{2,2} \geq 1\)
- Erase the line \( x_{1,1} + x_{1,2} \geq 1\)
- **Write down** axiom 3: \( x_{3,1} + x_{3,2} \geq 1\)

\[
\begin{align*}
-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} & \geq -2 \\
x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} & \geq 2 \\
x_{3,1} + x_{3,2} & \geq 1
\end{align*}
\]
Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
2. \(x_{2,1} + x_{2,2} \geq 1\)
3. \(x_{3,1} + x_{3,2} \geq 1\)
4. \(-x_{1,1} - x_{2,1} \geq -1\)
5. \(-x_{1,1} - x_{3,1} \geq -1\)
6. \(-x_{2,1} - x_{3,1} \geq -1\)
7. \(-x_{1,2} - x_{2,2} \geq -1\)
8. \(-x_{1,2} - x_{3,2} \geq -1\)
9. \(-x_{2,2} - x_{3,2} \geq -1\)

**History of derivation steps**

Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)
Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
Erase the line \(-x_{1,1} - x_{2,1} - x_{3,1} \geq -1\)
Write down axiom 1: \(x_{1,1} + x_{1,2} \geq 1\)
Write down axiom 2: \(x_{2,1} + x_{2,2} \geq 1\)
Add to get \(x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2\)
Erase the line \(x_{2,1} + x_{2,2} \geq 1\)
Erase the line \(x_{1,1} + x_{1,2} \geq 1\)
Write down axiom 3: \(x_{3,1} + x_{3,2} \geq 1\)
Add to get \(x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2} \geq 3\)
Example: CP Refutation of Pigeonhole Principle

1. \( x_{1,1} + x_{1,2} \geq 1 \)
2. \( x_{2,1} + x_{2,2} \geq 1 \)
3. \( x_{3,1} + x_{3,2} \geq 1 \)
4. \( -x_{1,1} - x_{2,1} \geq -1 \)
5. \( -x_{1,1} - x_{3,1} \geq -1 \)
6. \( -x_{2,1} - x_{3,1} \geq -1 \)
7. \( -x_{1,2} - x_{2,2} \geq -1 \)
8. \( -x_{1,2} - x_{3,2} \geq -1 \)
9. \( -x_{2,2} - x_{3,2} \geq -1 \)

**History of derivation steps**

Add to get \(-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\)

Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)

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Write down axiom 1: \(x_{1,1} + x_{1,2} \geq 1\)

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Add to get \(x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2\)

Erase the line \(x_{2,1} + x_{2,2} \geq 1\)

Erase the line \(x_{1,1} + x_{1,2} \geq 1\)

Write down axiom 3: \(x_{3,1} + x_{3,2} \geq 1\)

Add to get \(x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2} \geq 3\)
Example: CP Refutation of Pigeonhole Principle

1. \(x_{1,1} + x_{1,2} \geq 1\)
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**History of derivation steps**

- Erase the line \(-x_{1,2} - x_{2,2} - x_{3,2} \geq -1\)
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\begin{align*}
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\( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2 \)

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- Erase the line \( x_{3,1} + x_{3,2} \geq 1 \)
- **Erase** the line \( x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2 \)

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Erase the line $x_{3,1} + x_{3,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Add to get $0 \geq 1$

\[-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2\]

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Add to get \[ 0 \geq 1 \]
Complexity Measures for Cutting Planes

**Length** = total # lines/inequalities in refutation

**Size** = sum also size of coefficients

**Line space** = max # lines in memory during refutation

**Total space** = max # bits in memory (sum also size of coefficients)
Clique-coclique formulas
“A graph with an $m$-clique is not $(m-1)$-colourable”

Exponential lower bound via interpolation and circuit complexity [Pud97]

Technique very specifically tied to structure of formula
Hardness Results for Cutting Planes

**Clique-coclique formulas**

“A graph with an $m$-clique is not $(m-1)$-colourable”

Exponential lower bound via interpolation and circuit complexity [Pud97]

Technique very specifically tied to structure of formula

**Tseitin formulas**

“Sum of degrees of vertices in graph is even”

Short refutations of (lifted) Tseitin formulas on expanders must have large space [GP14]

Long-standing open problems to show such refutations don’t exist
Size-Space Trade-offs for Cutting Planes?

- Short refutations of some so-called pebbling formulas must have large space [HN12, GP14] (and such refutations do exist)

Jakob Nordström (KTH)

How Limited Interaction Hinders Real Communication

Skoltech Apr ’16 12/25
Size-Space Trade-offs for Cutting Planes?

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- Recent surprise: CP can refute any CNF in line space 5 (!) [GPT15] (But coefficients will be exponentially large)
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Size-Space Trade-offs for Cutting Planes?

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- But “constant-space” proofs with exponential-size coefficients somehow doesn’t feel quite right...

What about “true” trade-offs?

Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)
Our Main Result

**Theorem (Informal sample)**

There are families of 6-CNF formulas \( \{F_N\}_{N=1}^{\infty} \) of size \( \Theta(N) \) such that:

1. \( F_N \) can be refuted by cutting planes with constant-size coefficients in size \( O(N) \) and total space \( O(N^{2/5}) \).
2. \( F_N \) can be refuted by cutting planes with constant-size coefficients in total space \( O(N^{1/40}) \) and size \( 2^{O(N^{1/40})} \).
3. Any cutting planes refutation even with coefficients of unbounded size in line space less than \( N^{1/20 - \epsilon} \) requires length \( 2^{\Omega(N^{1/40})} \).

Remarks:

Upper bounds for # bits; lower bounds for # formulas/lines
Hold uniformly for resolution, polynomial calculus, and cutting planes
Even for semantic versions where anything implied by blackboard can be inferred in just one step.

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Outline of Proof

Proof is by carefully constructed chain of delicate reductions
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1. Short, space-efficient proof $\implies$ efficient communication protocol for falsified clause search problem [HN12]
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1. **Short, space-efficient proof** $\implies$ **efficient communication protocol** for falsified clause search problem [HN12]

2. **Crucial twists:**
   - Study real communication model [Kra98, BEGJ00]
   - Consider round efficiency of protocols
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3. Protocol for composed search problem $\implies$ parallel decision tree via simulation theorem à la [RM99, GPW15]

4. Parallel decision tree for pebbling formulas $Peb_G$ $\implies$ pebbling strategy for Dymond–Tompa game on $G$ [DT85]
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4. Parallel decision tree for pebbling formulas $Peb_G$ $\Rightarrow$ pebbling strategy for Dymond–Tompa game on $G$ [DT85]

5. Construct graphs $G$ with strong round-cost trade-offs for Dymond–Tompa pebbling
Real Communication [Kra98]

- **Main players:**
  - Alice with private input $x$
  - Bob with private input $y$
  - Both deterministic but have unbounded computational powers

Function $f$ solved by $r$-round real communication in cost $c$ if:
- # rounds $\leq r$
- total # comparisons made by referee $\leq c$

Strictly stronger than standard deterministic communication
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- Task: compute $f(x, y)$ by sending messages to referee
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Method: In each round $v$

- Alice sends $a_{v,1}(x), \ldots, a_{v,c_v}(x) \in \mathbb{R}^{c_v}$
- Bob sends $b_{v,1}(y), \ldots, b_{v,c_v}(y) \in \mathbb{R}^{c_v}$
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- **Function $f$ solved by $r$-round real communication in cost $c$** if $\exists$ protocol such that
  - $\# \text{ rounds} \leq r$
  - total $\# \text{ comparisons made by referee} \leq c$

- Strictly stronger than standard deterministic communication
Falsified Clause Search Problem

Fix:
- unsatisfiable CNF formula $F$
- (devious) partition of $\text{Vars}(F)$ between Alice and Bob

Falsified clause search problem $\text{Search}(F')$

**Input:** Assignment $\alpha$ to $\text{Vars}(F)$ split between Alice and Bob

**Output:** Clause $C \in F$ such that $\alpha$ falsifies $C$

Actually, computing not function but relation — will mostly ignore this for simplicity
Succinct Refutations Yield Efficient Protocols

Evaluate blackboard configurations of a refutation of $F$ under $\alpha$
Succinct Refutations Yield Efficient Protocols

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(Alice and Bob simply evaluate their parts of each inequality and ask referee to compare)
Where to Get Formulas with Trade-off Properties?

Questions about time-space trade-offs fundamental in theoretical computer science

Well-studied (and well-understood) for pebble games modelling calculations described by DAGs

In particular, for black-white pebble game investigated by [CS76] and many others
Pebbling Contradiction

CNF formulas encoding black-white pebble game played on DAG $G$

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

- sources are true
- truth propagates upwards
- but sink is false
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but sink is false

Appeared in various contexts in e.g. [RM99, BEGJ00, BW01]

Used in [Nor06, NH08, BN08, BN11, BNT13] to study space and size-space trade-offs in resolution and polynomial calculus

Inherit some DAG properties, but not enough — make formulas harder!
Lifting of Functions

Construct hard communication problems by “hardness amplification” using lifting or composition
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Start with function $f : \{0, 1\}^m \rightarrow \{0, 1\}$
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Construct new function on inputs $x \in \{0, 1\}^{\ell m}$ and $y \in [\ell]^m$
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Bob’s $y$-variables determine…

$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} & x_{3,1} & x_{3,2} \\ y_1 & y_2 & y_3 \end{bmatrix}$

$Lift_{\ell}(f)(x,y) := f(x_1, y_1, ..., x_m, y_m)$
Lifting of Functions

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Length-\( \ell \) lifting of \( f \) defined as \( \text{Lift}_\ell(f)(x, y) := f(x_1,y_1, \ldots, x_m,y_m) \)
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Building on ideas from e.g. [She08, BHP10]
Simulation of Protocols by Parallel Decision Trees [Val75]

Each node $t$ in tree labelled by variables $V_t$; has $2^{|V_t|}$ outgoing edges

$$
\begin{array}{c}
\text{node } x \\
\text{child at } y, z \\
\text{children at } \{u, w, w, u, w, z, v, y\}
\end{array}
$$

$$
\begin{array}{c}
\text{leaf values at } \{0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0\}
\end{array}
$$
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Parallel decision tree:
- uses $\#$ queries $= \max \sum |V_t|$ along any path
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Simulation theorem of protocol by decision tree (hard direction)

Let $S$ search problem with domain $\{0, 1\}^m$ and let $\ell = m^{3+\epsilon}$, $\epsilon > 0$. Then:
- $\exists r$-round real communication protocol in cost $c$ solving $\text{Lift}_\ell(S)$
- $\Rightarrow \exists$ depth-$r$ parallel decision tree solving $S$ width $\mathcal{O}(c/\log \ell)$ queries.
From Parallel Decision Trees to Dymond–Tompa Games

- From [DT85]; recently studied in [Cha13, CLNV15]
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  - Pebbler places pebbles on subset of vertices (including sink in 1st round)
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Lemma

$$\exists \text{ depth-} r \text{ parallel decision tree for } \text{Search}(\text{Peb} \ G) \Rightarrow \text{Pebbler wins} \ r \text{-round Dymond–Tompa game on } G \text{ in cost} \leq c + 1.$$
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Clinching the Argument

Prove round-cost trade-offs for Dymond–Tompa games on graphs $G$ (hacking graph constructions from [CS82, LT82, Nor12])
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Depth-query trade-offs for parallel decision trees for $\text{Search}(\text{Peb}_G)$
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Cutting planes length-space trade-off for $\text{Lift}(\text{Peb}_G)$
Some Remaining Open Questions

**Communication complexity**

- Smaller length of lift?
- Simulation theorems for stronger communication models (randomized, multi-party)?
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- Simulation theorems for stronger communication models (randomized, multi-party)?

**Proof complexity**
- Better Dymond–Tompa trade-offs?
- Reduction to black-white pebbling instead of Dymond–Tompa?
- Size-space trade-offs for Tseitin formulas à la [BBI12, BNT13]?
- Line space lower bounds for CP with bounded coefficients (strengthening [GPT15])
Take-Home Message

Summary of results

- Modern SAT solvers **enormously successful in practice** — key issue is to **minimize time and memory consumption**
- Modelled by **proof size and space** in proof complexity
- We show **uniform trade-offs** indicating that **simultaneous optimization impossible** for (essentially all) state-of-the-art techniques
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Future directions

- **Proof complexity:** Understand size and space in cutting planes better
- **Communication complexity:** Tighter reductions and/or lower bounds in stronger models
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Thank you for your attention!
References I


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References V


References VI
