A (Biased) Survey of Space Complexity and Time-Space Trade-offs in Proof Complexity

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Topic of This Survey

Study of space in proof complexity initiated in late 1990s
Motivated by considerations of SAT solver memory usage
But also (and mainly?) intrinsically interesting for proof complexity

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Make most sense for relatively weak proof systems — focus on:

- resolution
- polynomial calculus
- cutting planes (only mention very briefly)

By necessity, selective coverage — apologies for omissions

Outline

1. Space Complexity
   - Preliminaries
   - Space Lower Bounds for Resolution
   - Space Lower Bounds for Polynomial Calculus
2. Size-Space Trade-offs
   - Trade-offs for Resolution
   - Trade-offs for Polynomial Calculus
   - Trade-offs for Superlinear Space
3. Open Problems
   - Open Problems for Resolution
   - Open Problems for Polynomial Calculus
   - Open Problems for Cutting Planes

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- Mostly assume formulas k-CNFs (for simplicity of exposition)
  Conversion to 3-CNF most often doesn't change much
  [except sometimes the difference is huge. . .]
- N denotes size of formula ($\#$ literals, which is $\approx \#$ clauses)

The Resolution Proof System

Goal: refute unsatisfiable CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- DAG

Tree-like resolution if DAG is tree

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\perp$	Res$(7, 8)$

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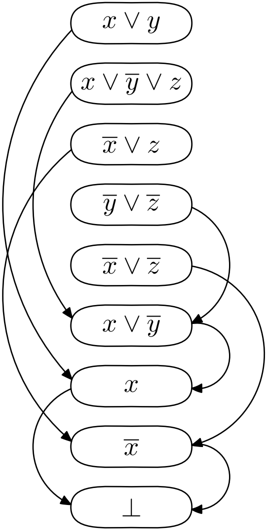

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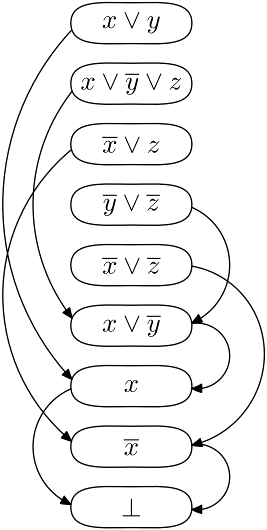

Resolution Size and Space

Size/length = total # clauses in refutation

Space = max # clauses in memory when performing refutation

(Exist other space measures also — focus here on most well-studied one)

Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

Example: Space at step 7 . . .

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
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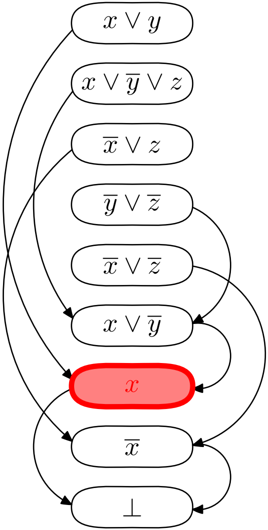

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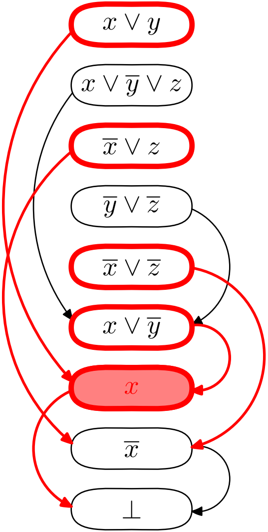
Example: Space at step 7 is 5



Upper Bounds on Resolution Size and Space

- **Size / space of refuting formula** defined by taking minimum over all resolution refutations
- Size always at most $\exp(\mathcal{O}(N))$
- Space always at most $N + \mathcal{O}(1)$
- Can be achieved simultaneously (even in tree-like resolution) [ET01]

Blackboard Definition of Resolution

Think of resolution refutation as being presented on blackboard:

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Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for axiom clause $C \in F$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{D\}$ inferred by resolution on clauses in $\mathbb{C}_{t-1}$

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Size $=$ $\#$ download & inference steps
Space $= \max_{0 \leq t \leq \tau} \{|\mathbb{C}_t|\}$

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F requires space $s \Leftrightarrow$ all $\mathbb{C}_t$ derived from F in space $< s$ satisfiable

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Inference	Do nothing	Do nothing
Erasure	Pick α_t of size $\leq \lvert\mathbb{C}_t\rvert$	Shrink to $\alpha_t \subseteq \alpha_{t-1}$ of size $\leq \lvert\mathbb{C}_t\rvert$

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Space game exactly characterizes space (but hard to play)

Restricted lower bound game: can construct α_t inductively (but no guarantee this will work)

General Proof Strategy for Space Lower Bound

Hard to get a handle on structure of derived configuration $\mathbb{C}_t$

Construct auxiliary configuration $\mathbb{D}_t$ (view α_t as 1-CNF) that is easier to understand but still gives information about $\mathbb{C}_t$:

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If we can do this, clearly we get lower bound on space

Two observations:

- "On the safe side" of adversary ($\mathbb{D}_t$ stronger than $\mathbb{C}_t$)
- History-dependent (can get different $\mathbb{D}_t$ for same $\mathbb{C}_t$)

Resolution Space Lower Bound for Random k-CNFs (1/2)

Random k-CNF formulas

Δn randomly sampled k-clauses over n variables

Resolution space lower bound $\Omega(n)$ [BG03]

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Graph $G(F)$ of CNF F

- Bipartite graph $G(U \dot\cup V, E)$
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(d, δ, s)-bipartite expander

- Bipartite graph $G(U \dot\cup V, E)$ with left degree d
- Every $A \subseteq U$ s.t. $|A| \leq s$ has neighbourhood $|N_G(A)| \geq \delta |A|$

Resolution Space Lower Bound for Random k-CNFs (2/2)

Theorem ([BG03])

If F is random k-CNF for $k \geq 3$ over n variables with Δn clauses then F requires space $\Omega(n)$ almost surely

Proof sketch.

Given small-space derivation $(\mathbb{C}_0, \mathbb{C}_1, \mathbb{C}_2, \ldots)$ from F, inductively construct 1-CNF $\mathbb{D}_t$ implying $\mathbb{C}_t$ and satisfying $|\mathbb{D}_t| \leq |\mathbb{C}_t|$:

1. Download of $C \in F$: Since $G(F)$ has expansion $1 + \epsilon$, can find variable in C not in $\mathbb{D}_{t-1}$ [needs an argument, of course]
2. Inference: Set $\mathbb{D}_t = \mathbb{D}_{t-1}$
3. Erasure: Pick $\mathbb{D}_t \subseteq \mathbb{D}_{t-1}$ of size $|\mathbb{D}_t| \leq |\mathbb{C}_t|$ implying $\mathbb{C}_t$ □

Taking Care of Erasures by Locality Lemma

Lemma (Locality lemma for resolution)

Suppose $\mathbb{D}$ *1-CNF;* $\mathbb{C}$ *clause configuration;* $\mathbb{D}$ *implies* $\mathbb{C}$

Then $\exists$ *1-CNF* $\mathbb{D}'$ *of size* $|\mathbb{D}'| \leq |\mathbb{C}|$ *s.t.* $\mathbb{D}'$ *implies* $\mathbb{C}$

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Then by construction:

- $|\mathbb{D}'| \leq |\mathbb{C}|$
- $\mathbb{D}' \vDash \mathbb{C}$ $\square$

Space Lower Bounds from Width Lower Bounds

Tight space lower bound obtained in this way also for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]

Matching width lower bounds (min size of largest clause in proof)
Under the hood proofs of very similar flavour... What is going on?

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With hindsight, almost all space lower bounds obtainable this way

But not quite — get back to this later

Polynomial Calculus (or Actually PCR)

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over (fixed) field in variables $x, \overline{x}, y, \overline{y}, z, \overline{z}, \ldots$ (where x and $\overline{x}$ distinct variables)

Example: $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

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Size — analogue of resolution size
total # monomials in refutation (counted with repetitions)
[Can also define length measure — might be much smaller]

Degree — analogue of resolution width
largest degree of monomial in refutation

(Monomial) space — analogue of resolution (clause) space
max # monomials in memory during refutation (with repetitions)

PCR Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently with respect to length/size, width/degree, and space simultaneously

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Open Problem

Show that PCR is strictly stronger than resolution w.r.t. space

Lower Bounds on PCR Space

Lower bound for PHP with wide clauses [ABRW02]

k-CNFs much trickier — sequence of lower bounds for

- Obfuscated 4-CNF versions of PHP [FLN$^+$12]
- Random 4-CNFs + general technique [BG13]
- Tseitin formulas on (some) expanders [FLM$^+$13]

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Open Problem

- *Prove tight space lower bounds for Tseitin on any expander*
- *Prove tight space lower bounds for ordering principle formulas*
- *Prove any space lower bound on random 3-CNFs*
- *Prove any space lower bound for any 3-CNF!?*

What We Want (Recap of Lower Bound Proof Strategy)

Given PCR derivation $(\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots)$ in small space

Want to construct "auxiliary configurations" $\mathbb{D}_0, \mathbb{D}_1, \mathbb{D}_2, \ldots$ s.t.

- $\mathbb{D}_t$ highly structured, so easier to understand than $\mathbb{P}_t$
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If so, small-space derivation cannot derive contradiction

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Not true for polynomials!

Example

- Consider polynomial equation $-1 + xyzuvw = 0$
- Monomial space 2
- But have to set 6 variables to satisfy
- Obviously generalizes to arbitrary number of variables

Cannot use 1-CNFs / assignments as auxiliary configurations!

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Example

- Consider polynomial equation $-1 + xyzuvw = 0$
- Monomial space 2
- But have to set 6 variables to satisfy
- Obviously generalizes to arbitrary number of variables

Cannot use 1-CNFs / assignments as auxiliary configurations!

But miraculously, 2-**CNFs sometimes work!** [ABRW02]

PCR Space Lower Bound for Random k-CNFs

Theorem ([BG13])

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(Straightforward to verify that any such $\mathbb{D}_t$ is satisfiable)

Inductive Proof: Invariants and Inference

Proof invariants:

- $\mathbb{D}_t = \mathcal{A}_t \wedge \mathcal{B}_t$ structured auxiliary configuration
- $\mathbb{D}_t$ implies $\mathbb{P}_t$
- $|\mathbb{D}_t| \leq 6 \cdot (\#$ [distinct] monomials in $\mathbb{P}_t)$

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Proof is by case analysis over derivation step

1. Inference $\mathbb{P}_t = \mathbb{P}_t \cup \{Q\}$ for polynomial Q derived from $\mathbb{P}_{t-1}$

- Set $\mathbb{D}_t := \mathbb{D}_{t-1}$
- $\mathbb{D}_t = \mathbb{D}_{t-1}$ implies Q by soundness
- Space of $\mathbb{D}_t$ stays the same
- Space of $\mathbb{P}_t$ goes up

Inductive Proof: Axiom Download

2. **Download** $\mathbb{P}_t = \mathbb{P}_t \cup \{C\}$ for $C \in F$
   - For simplicity, assume extra download of $C' \in F$
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- $\mathcal{A}_t := \mathcal{A}_{t-1} \cup \{A, A'\}$
- $\mathcal{B}_t := \mathcal{B}_{t-1} \cup \{B\}$
- Space of $\mathbb{D}_t = \mathcal{A}_t \wedge \mathcal{B}_t$ up by 3
- Space of $\mathbb{P}_t$ up by 1

Inductive Proof: Erasure

3. **Erasure** $\mathbb{P}_t = \mathbb{P}_{t-1} \setminus \{Q\}$ for $Q \in \mathbb{P}_{t-1}$
   - Know $\mathbb{D}_{t-1}$ implies $\mathbb{P}_t \subseteq \mathbb{P}_{t-1}$
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Lemma (Locality lemma for PCR [ABRW02, BG13])

Suppose

- $\mathbb{D} = \mathcal{A} \wedge \mathcal{B}$ *structured auxiliary configuration*
- $\mathbb{P}$ *PCR-configuration*
- $\mathbb{D}$ *implies* $\mathbb{P}$

Then

$\exists\, \mathbb{D}^* = \mathcal{A}^* \wedge \mathcal{B}^*$ *with* $|\mathbb{D}^*| \leq 6 \cdot (\#$ *monomials in* $\mathbb{P})$ *s.t.* $\mathbb{D}^*$ *implies* $\mathbb{P}$

Proof sketch for Locality Lemma for PCR (1/4)

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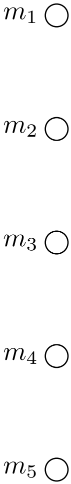

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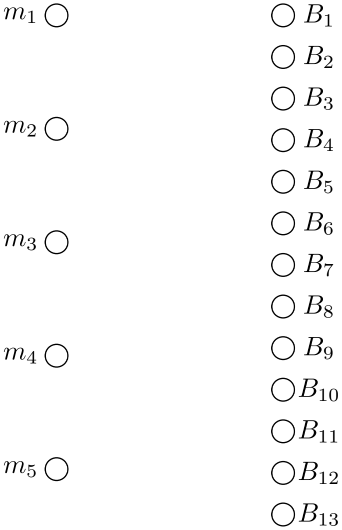

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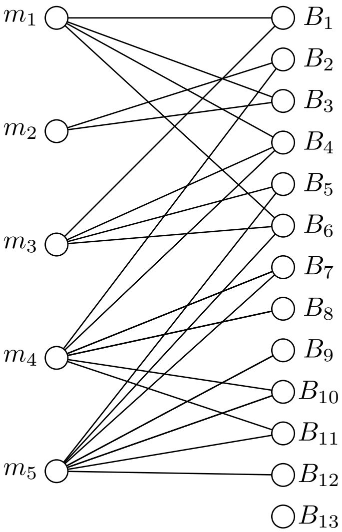

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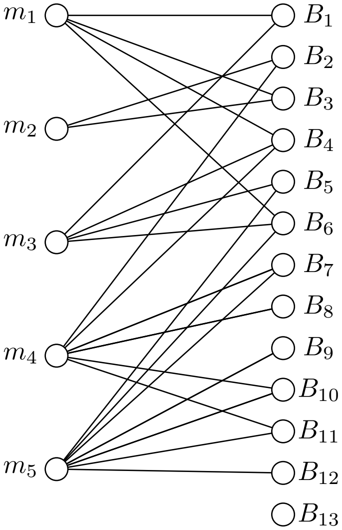

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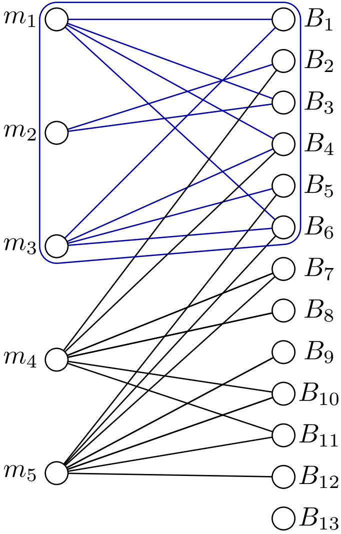

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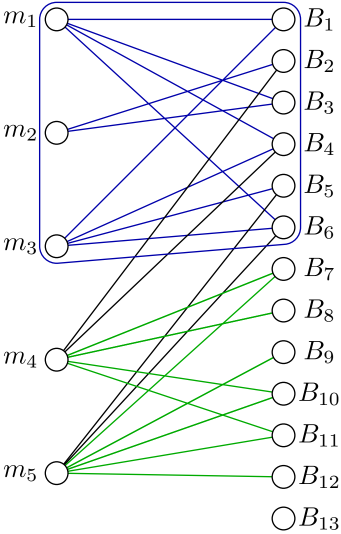

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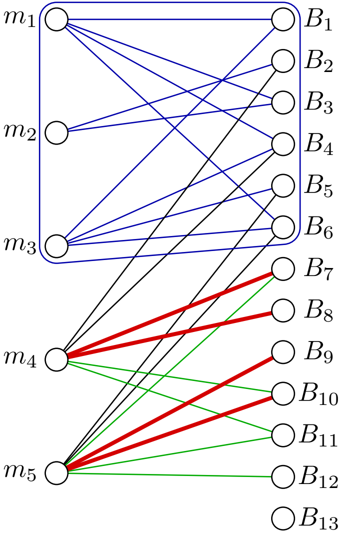

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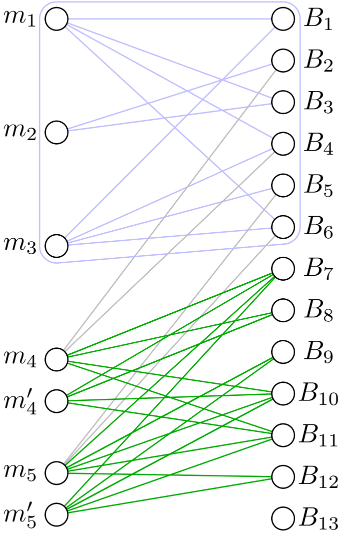

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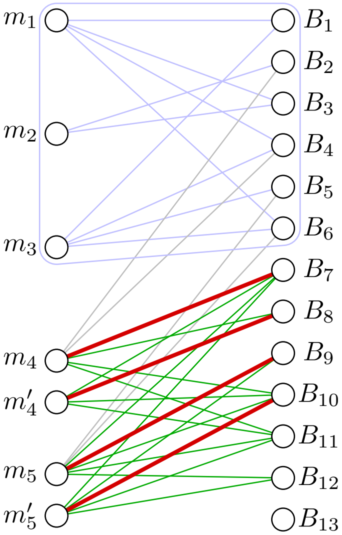

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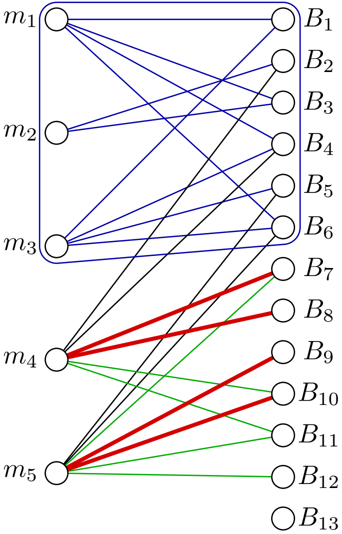

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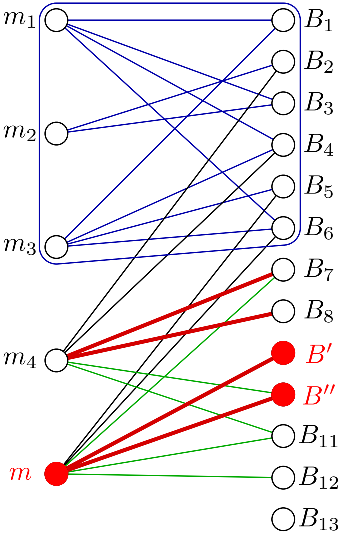
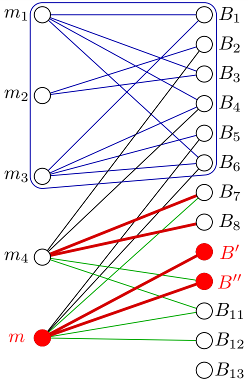


Proof sketch for Locality Lemma for PCR (2/4)

Look at $m \in M \setminus \Gamma$ — suppose matched to $B' = \overline{x} \vee \overline{y}$ and $B'' = \overline{z} \vee \overline{w}$



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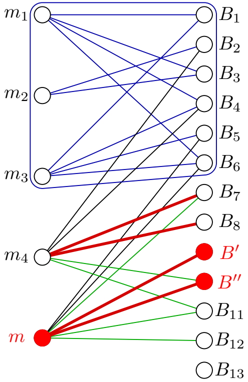

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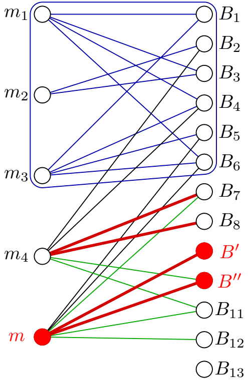

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- $B^* = x \vee z$ [common variables with signs as in m]
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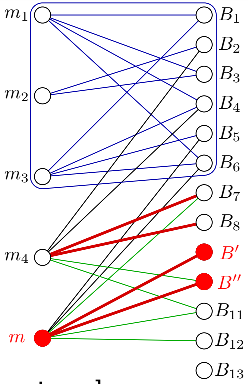

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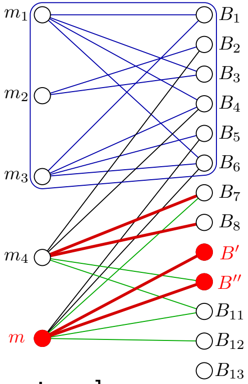

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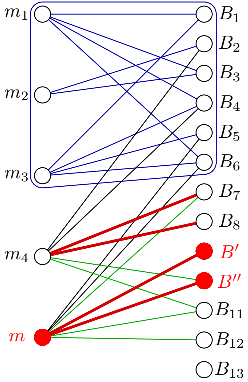
Proof sketch for Locality Lemma for PCR (3/4)

Need to prove three things:



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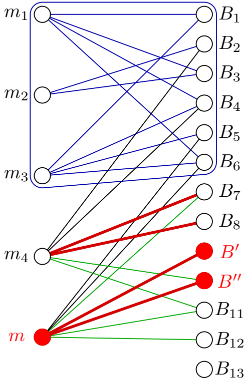

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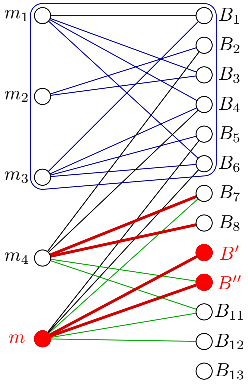

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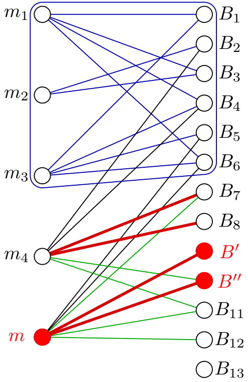

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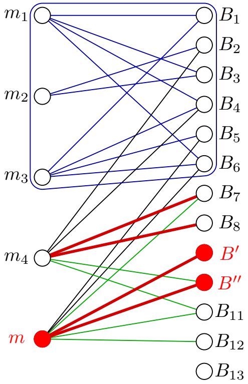

Prove implication in slightly roundabout way:
Given any β satisfying $\mathbb{D}^*$, find α such that

- $\mathbb{P}(\alpha) = \mathbb{P}(\beta)$
- α satisfies $\mathbb{D}$

Proof sketch for Locality Lemma for PCR (4/4)

Look at our example monomial

- $m = xz \cdot m' \in M \setminus \Gamma$

with new clauses in $\mathbb{D}^*$ [satisfied by β]

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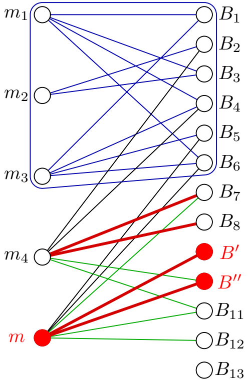

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Old clauses in $\mathbb{D}$ [to be satisfied by α] are:

- $B' = \overline{x} \vee \overline{y} \leftrightarrow A_1' = x \vee x',\ A_2' = y \vee y'$
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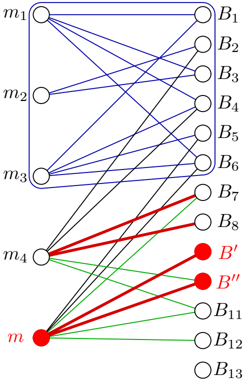

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Old clauses in $\mathbb{D}$ [to be satisfied by α] are:

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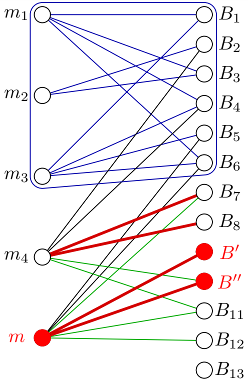
Let $\alpha = \beta$ except that for $m \in M \setminus \Gamma$ we set $y = w = \mathit{false}$ and $x' = y' = z' = w' = \mathit{true}$



Proof sketch for Locality Lemma for PCR (4/4)

Look at our example monomial

- $m = xz \cdot m' \in M \setminus \Gamma$

with new clauses in $\mathbb{D}^*$ [satisfied by β]

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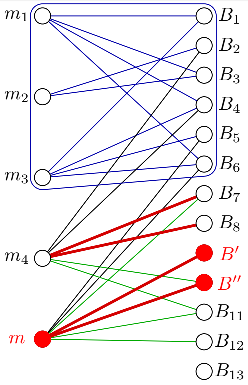

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- $\alpha(m) = \beta(m)$ for all $m \in \Gamma$ [didn't touch those variables]
- $\alpha(m) = \beta(m) = 0$ for all $m \in M \setminus \Gamma$ [by construction of $\mathbb{D}^*$]
- α satisfies $\mathbb{D}$ and hence $\mathbb{P}$
- But then β must also satisfy $\mathbb{P}$, **Q.E.D.**

Another Intriguing Problem: Space vs. Degree

Open Problem (analogue of [AD08])

Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?

Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM^{+}13]

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Optimal separation of space and degree in [FLM+13] by flavour of Tseitin formulas which

- can be refuted in degree $\mathcal{O}(1)$
- require space $\Omega(N)$
- but separating formulas depend on characteristic of field

Comparing Size and Space

Some "rescaling" needed to get meaningful comparisons of size/length and space

- Size exponential in formula size in worst case
- Space at most linear in worst case
- So natural to compare space to logarithm of size

Size-Space Correlations and/or Trade-offs?

$\exists$ constant space refutation $\Rightarrow$ $\exists$ polynomial size refutation [AD03]

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So essentially no trade-offs for tree-like resolution

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So essentially no trade-offs for tree-like resolution

Does short size imply small space for general resolution?

Are there size-space trade-offs for general resolution?
(Some trade-off results in restricted settings in [Ben02, Nor09])

An Optimal Size-Space Separation

Size and space in resolution are "completely uncorrelated"

Theorem ([BN08])

There are k-CNF formula families of size N with

- *refutation size* $\mathcal{O}(N)$
- *refutation space* $\Omega(N/\log N)$

Optimal separation of size and space — given size $\mathcal{O}(N)$, always possible to get clause space $\mathcal{O}(N/\log N)$

Size-Space Trade-offs

There is a rich collection of size-space trade-offs

Results hold for

- resolution
- even k-DNF resolution (which we won't go into here)

Different trade-offs covering (almost) whole range of space from constant to linear

Simple, explicit formulas

One Example: Robust Trade-offs for Small Space

Theorem ([BN11] (informal))

For any arbitrarily slowly growing function g there exist explicit k-CNF formulas of size N

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And an open problem:

Open Problem

Seems likely that $\sqrt[3]{N}$ above should be possible to improve to $\sqrt{N}$, but don't know how to prove this...

Proof Strategy for Size-Space Separations and Trade-offs

- Both of these theorems proved in the same way
- Want to sketch intuition and main ideas in proofs
- For details, see survey [Nor13]
- To prove the theorems, need to go back to the early days of computer science. . .

A Detour into Combinatorial Games

Want to find formulas that

- can be quickly refuted but require large space
- have space-efficient refutations requiring much time

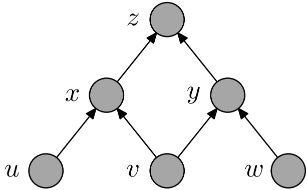
Such time-space trade-off questions well-studied for pebble games modelling calculations described by DAGs ([CS76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

Pebbling Formulas: Vanilla Version

CNF formulas encoding pebble games on DAGs

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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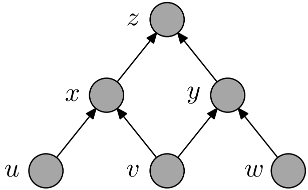

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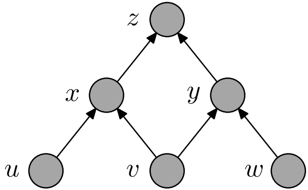

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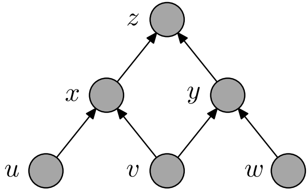

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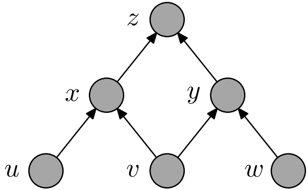

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Extensive literature on pebbling space and time-space trade-offs from 1970s and 80s

Have been useful in proof complexity before in various contexts

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas

Pebbling Formula Trade-offs

- Reduction from resolution to pebbling [Ben02]
- Pebbling time-space trade-offs $\Rightarrow$ size-variable space trade-offs in resolution [BN11]
- In fact, size-variable space trade-offs for any "semantic" proof system [BNT13]
- But we want trade-offs for stronger space measures!
- And pebbling formulas supereasy — can do constant (clause) space and linear size simultaneously

Key New (Old?) Idea: Variable Substitution

Make formula harder by substituting exclusive or $x_1 \oplus x_2$ of two new variables x_1 and x_2 for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \lor y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \lor (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \lor \overline{x}_2 \lor y_1 \lor y_2) \\ \land\ & (x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor y_1 \lor y_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2) \end{aligned}$$

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for refuting $F[\oplus]$: mimic refutation of F

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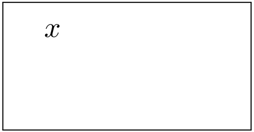

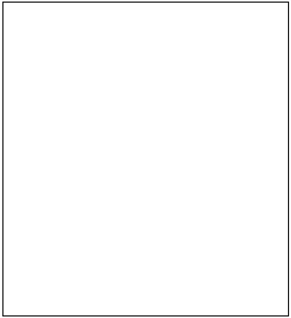

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- size $\geq$ size for F
- clause space $\geq$ # variables on board in proof for F

Prove that this is (sort of) best one can do for $F[\oplus]$!

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F

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... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Putting the Pieces Together

Making variable substitutions in pebbling formulas

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Get our results by

- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results [Nor12]

Some Philosophical Notes

- Projections "on the wrong side" of adversary (we throw away info and get weaker configuration)
- Independent of history (always same projection from same configuration)
- Only technique for proving space lower bounds without dependence on width lower bounds (pebbling formulas refutable in constant width)
- Is there a "safe side of adversary," history-dependent space lower bound proof for pebbling formulas?

Projections v.s. Restrictions for Polynomial Calculus

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Obtain similar trade-offs as for resolution but with some loss in parameters [BNT13]

No unconditional space lower bounds — inherent limitation due to random restriction argument

Going Beyond Linear Space...

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- Don't have time to go into any details — topic for a separate talk, probably...

Some Open Problems for Resolution

Resolution arguably fairly well-understood by now, but several good open questions remain

For instance:

- Can we get (much) sharper trade-offs for superlinear space than in [BBI12, BNT13]?
- Are there trade-offs between proof size and proof width? Or can both measures be minimized simultaneously?

Some Open Problems for Polynomial Calculus/PCR

Long list of open problems — mentioned in this talk:

- Show that PCR is strictly stronger than resolution w.r.t. space
- Prove PCR space lower bounds for
  - Tseitin on any expander
  - ordering principle formulas
  - random 3-CNFs
  - Or any 3-CNF, really...
- Is it true for PCR that space $\geq$ degree $+ \mathcal{O}(1)$?

Definition of Cutting Planes [CCT87]

Clauses interpreted as linear inequalities over the reals with integer coefficients

Example: $x \lor y \lor \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Space = max # lines in memory during refutation

No (useful) analogue of width/degree

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Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously
- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]

Open Problem

Show cutting planes strictly stronger than resolution w.r.t. space

Hard Formulas w.r.t Cutting Planes Space?

No space lower bounds known except conditional ones

All short cutting planes refutations of

- Tseitin formulas on expanders require large space [GP14]
  (But such short refutations probably don't exist anyway)
- (some) pebbling formulas require large space [HN12, GP14]
  (and such short refutations do exist; hard to see how exponential length could help bring down space)

Above results obtained via communication complexity

No (true) length-space trade-off results known
Although results above can also be phrased as trade-offs

Summing up

- Survey of space complexity and size-space trade-offs
- Focus on resolution and polynomial calculus/PCR
- Resolution fairly well understood
- Polynomial calculus less so — several nice open problems
- And cutting planes almost not at all understood!

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Thank you for your attention!

References I

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