Narrow Proofs May Be Spacious: Separating Space and Width in Resolution

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Outline

1. Background
   - Resolution
   - Previous Work
2. Our Contribution
   - An Idea That Doesn't Work
   - An Idea That Does
   - Pebble Games and Resolution
3. Conclusion and Open Problems

Some Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \ldots \vee a_k$: disjunction of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \wedge \ldots \wedge C_m$: conjunction of clauses
  k-CNF formula: CNF formula consisting of k-clauses
  (assume k fixed)
- Refer to clauses of CNF formula as axioms
  (as opposed to derived clauses)

Resolution

- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Also used in many real-world automated theorem provers

Resolution Rule

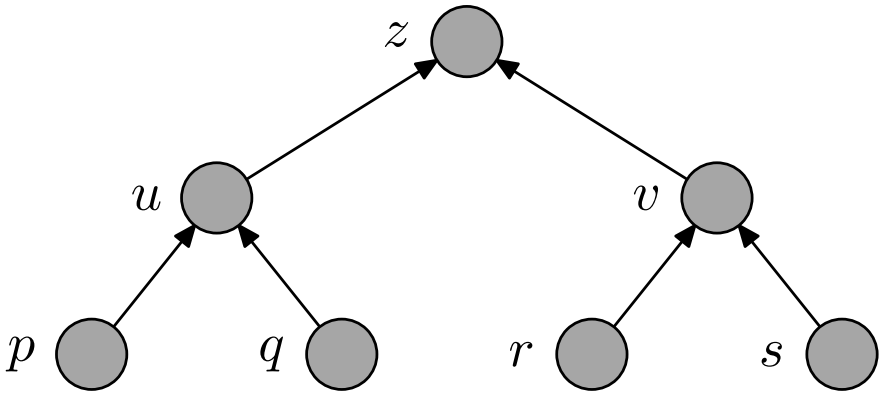
Resolution rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

Example CNF Formula

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



- source vertices true
- truth propagates upwards
- but target vertex is false

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7. $\overline{u} \vee \overline{v} \vee z$
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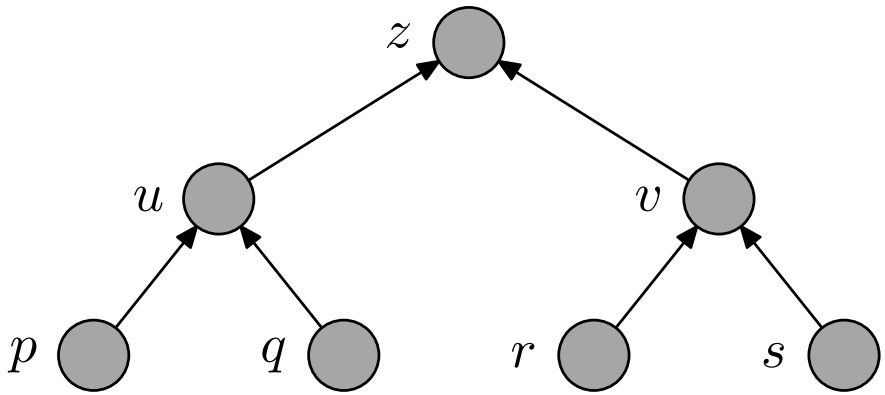

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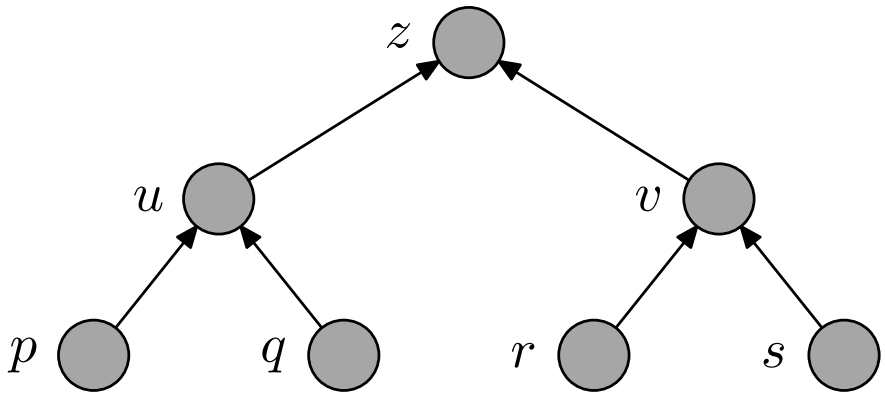

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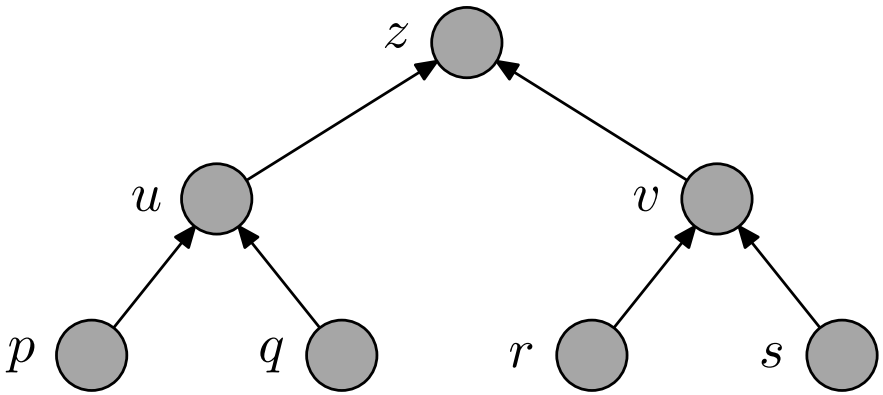
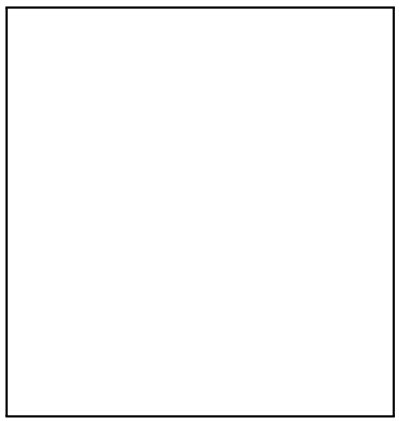


- source vertices true
- truth propagates upwards
- but target vertex is false

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

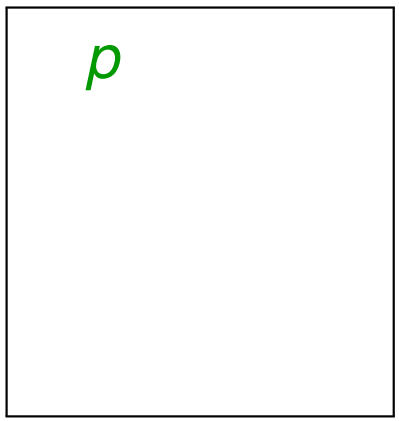


Blackboard bookkeeping	
# distinct clauses on board	0
# literals in largest clause	0
# lines on blackboard used	0

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	1
# literals in largest clause	1
# lines on blackboard used	1

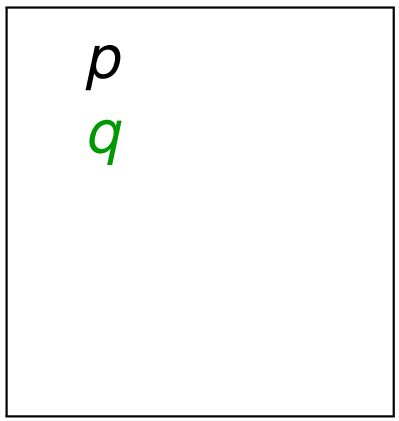


Write down axiom 1: p

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	2
# literals in largest clause	1
# lines on blackboard used	2



Write down axiom 2: q

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

p
q
$\overline{p} \vee \overline{q} \vee u$

Write down axiom 5: $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

p
q
$\overline{p} \vee \overline{q} \vee u$

Infer $\overline{q} \vee u$ from
p and $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

p
q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Infer $\overline{q} \vee u$ from
p and $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

p
q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause p

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause p

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{p} \vee \overline{q} \vee u$
$\overline{q} \vee u$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$

Infer u from
q and $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$
u

Infer u from
q and $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

q
$\overline{q} \vee u$
u

Erase clause q

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

$\overline{q} \vee u$
u

Erase clause q

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

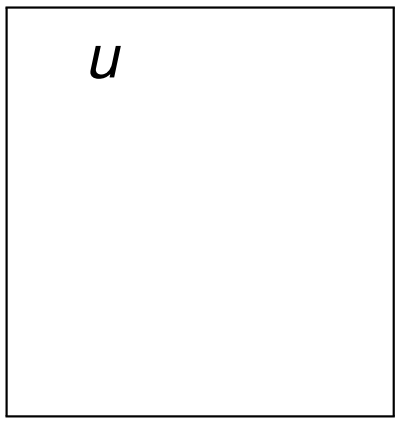
$\overline{q} \vee u$
u

Erase clause $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4



Erase clause $\overline{q} \vee u$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

u
r

Write down axiom 3: r

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

u
r
s

Write down axiom 4: s

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

u
r
s
$\overline{r} \vee \overline{s} \vee v$

Write down axiom 6: $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

u
r
s
$\overline{r} \vee \overline{s} \vee v$

Infer $\overline{s} \vee v$ from
r and $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
r
s
$\overline{r} \vee \overline{s} \vee v$
$\overline{s} \vee v$

Infer $\overline{s} \vee v$ from
r and $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
r
s
$\overline{r} \vee \overline{s} \vee v$
$\overline{s} \vee v$

Erase clause r

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{r} \vee \overline{s} \vee v$
$\overline{s} \vee v$

Erase clause r

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u

s

$\overline{r} \vee \overline{s} \vee v$

$\overline{s} \vee v$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \vee v$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \vee v$

Infer v from
s and $\overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \vee v$
v

Infer v from
s and $\overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
s
$\overline{s} \vee v$
v

Erase clause s

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u
$\overline{s} \vee v$
v

Erase clause s

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5

u

$\overline{s} \vee v$

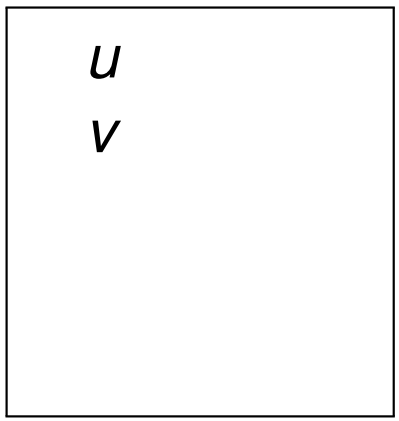
v

Erase clause $\overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	5



Erase clause $\overline{s} \vee v$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	11
# literals in largest clause	3
# lines on blackboard used	5

u

v

$\overline{u} \vee \overline{v} \vee z$

Write down axiom 7: $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	11
# literals in largest clause	3
# lines on blackboard used	5

u

v

$\overline{u} \vee \overline{v} \vee z$

Infer $\overline{v} \vee z$ from
u and $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

u
v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Infer $\overline{v} \vee z$ from
u and $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

u
v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Erase clause u

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v
$\overline{u} \vee \overline{v} \vee z$
$\overline{v} \vee z$

Erase clause u

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v

$\overline{u} \vee \overline{v} \vee z$

$\overline{v} \vee z$

Erase clause $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

v

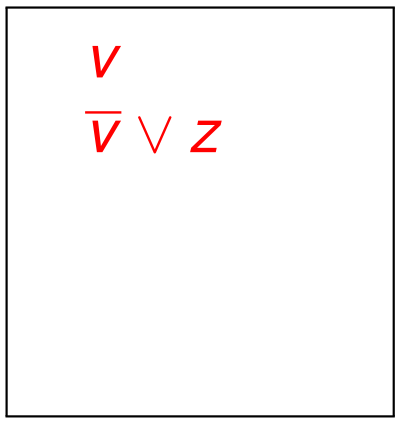
$\overline{v} \vee z$

Erase clause $\overline{u} \vee \overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5



Infer z from
v and $\overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

v

$\overline{v} \vee z$

z

Infer z from
v and $\overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

v

$\overline{v} \vee z$

z

Erase clause v

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \vee z$

z

Erase clause v

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \vee z$

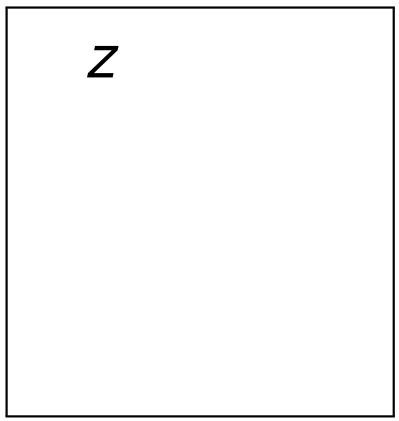
z

Erase clause $\overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5



Erase clause $\overline{v} \vee z$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

z

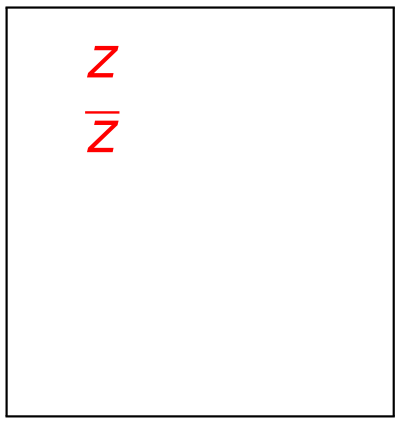
$\overline{z}$

Write down axiom 8: $\overline{z}$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5



Infer 0 from
z and $\overline{z}$

Example Resolution Refutation

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	15
# literals in largest clause	3
# lines on blackboard used	5

z
$\overline{z}$
0

Infer 0 from
z and $\overline{z}$

Length, Width and Space

- Length $L(\pi)$ of resolution refutation $\pi : F \vdash 0$
  # distinct clauses on blackboard
- Width $W(\pi)$ of resolution refutation $\pi : F \vdash 0$
  # literals in largest clause on blackboard
- Space $Sp(\pi)$ of resolution refutation $\pi : F \vdash 0$
  # lines used simultaneously on blackboard

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{L(\pi)\}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{W(\pi)\}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{Sp(\pi)\}$$

Results for Length and Width

Length

Haken (1985): polynomial-size CNF formula family with exponential lower bound on resolution refutation length (pigeonhole principle)

Width

- $W(F \vdash 0) \leq$ # variables in F
- Ben-Sasson & Wigderson (1999): strong correlation between length and width of refuting formula
- Proof search heuristic: search for narrow refutations

Results for Space

- Space introduced by Esteban & Torán (1999)
- Maximal # clauses in memory while verifying proof—related to performance of proof search algorithms
- $Sp(F \vdash 0) \leq$ size of F, or more precisely $\leq \min(\#\text{ variables in } F, \#\text{ clauses in } F) + \mathcal{O}(1)$
- Many lower bounds on space proven
- All turned out to match width lower bounds—true in general?

Connection between Space and Width

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F it holds that

$$Sp(F \vdash 0) \geq W(F \vdash 0) - \mathcal{O}(1).$$

But do space and width always coincide?
Are they in fact the same measure asymptotically?

Or can they be separated?
I.e., is there a k-CNF formula family $\{F_n\}_{n=1}^{\infty}$ such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?

A Separation of Space and Width in Resolution

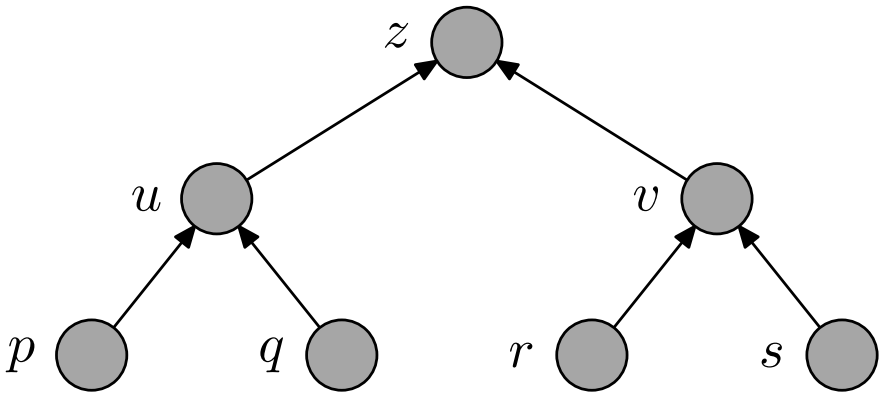
Our Main Theorem

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and
- refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.

Returning to Our Example Formula

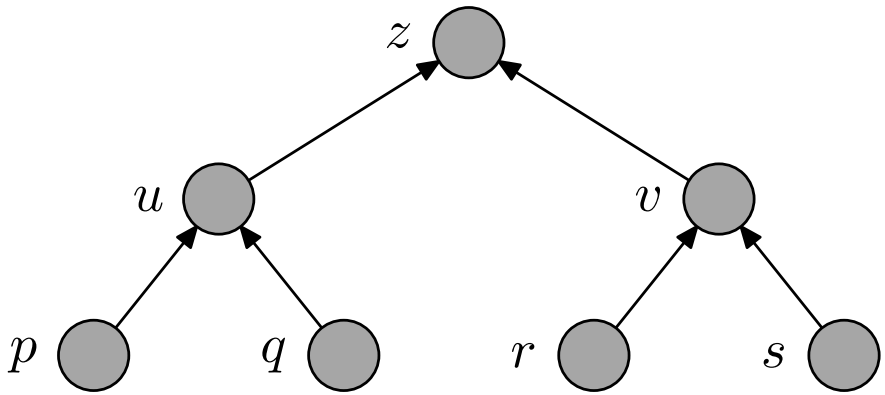
1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

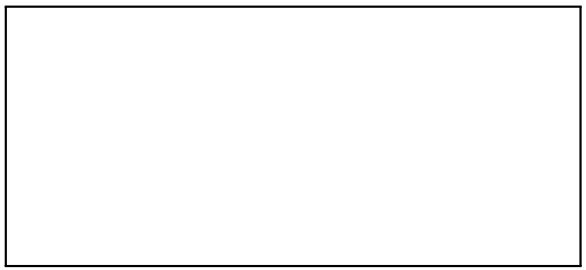


- Easily generalized to trees of height h
- Refutation width $=$ formula width $= \mathcal{O}(1)$
- Space seems to grow linearly with tree height h?

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

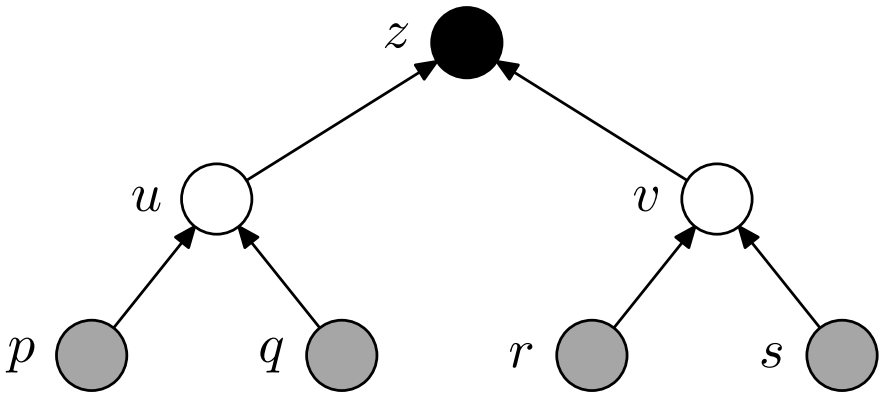




Refutation in space 3
by Ben-Sasson (2002)

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

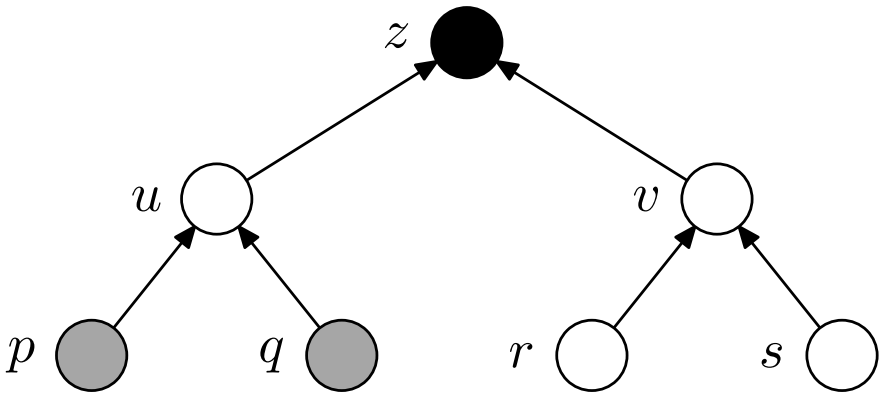


$\overline{u} \vee \overline{v} \vee z$

Write down axiom 7: $\overline{u} \vee \overline{v} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

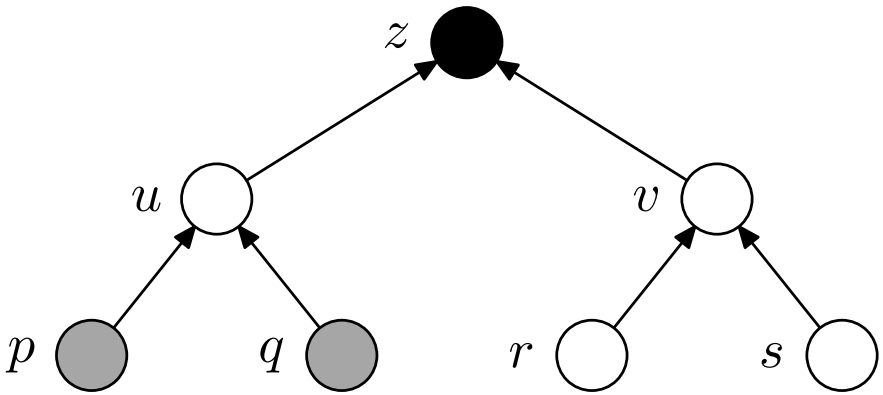


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$

Write down axiom 6: $\overline{r} \vee \overline{s} \vee v$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

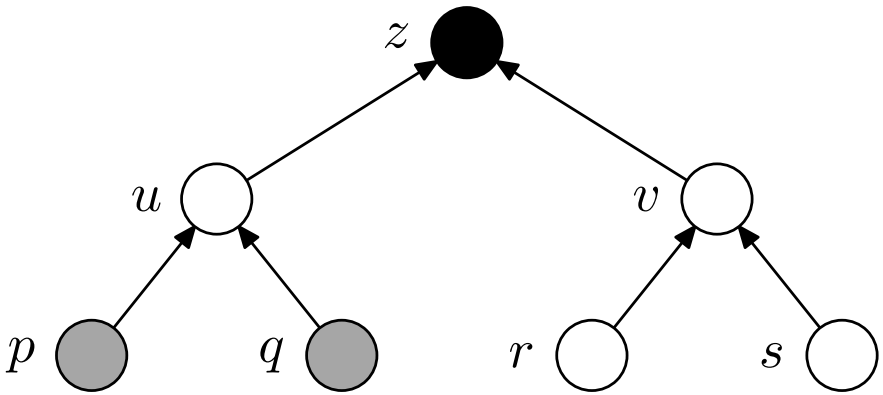


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$

Infer $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$ from
$\overline{r} \vee \overline{s} \vee v$ and $\overline{u} \vee \overline{v} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

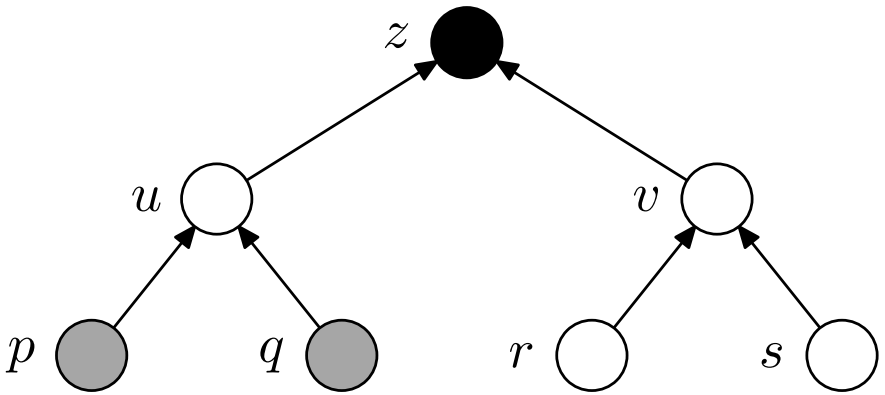


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Infer $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$ from $\overline{r} \vee \overline{s} \vee v$ and $\overline{u} \vee \overline{v} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

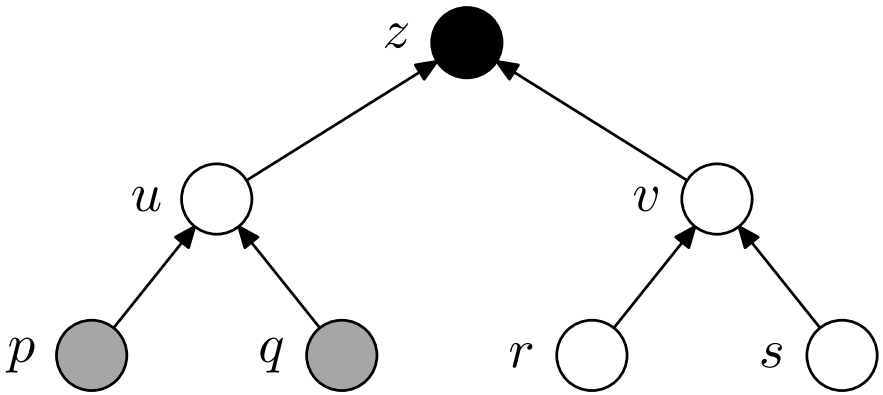


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee v$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

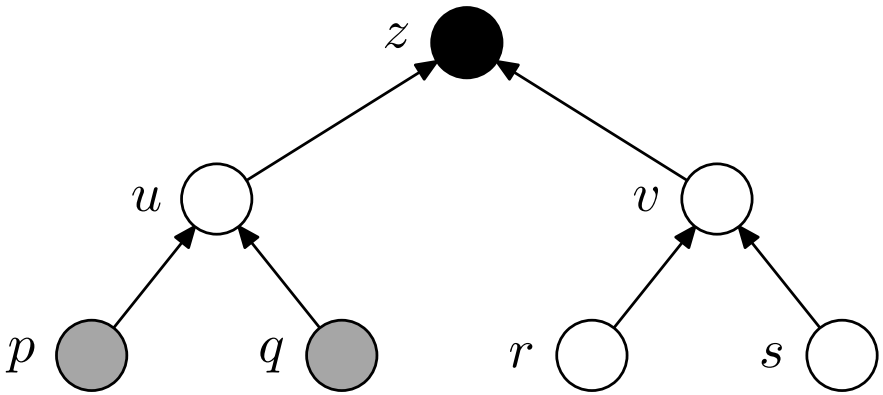


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee v$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

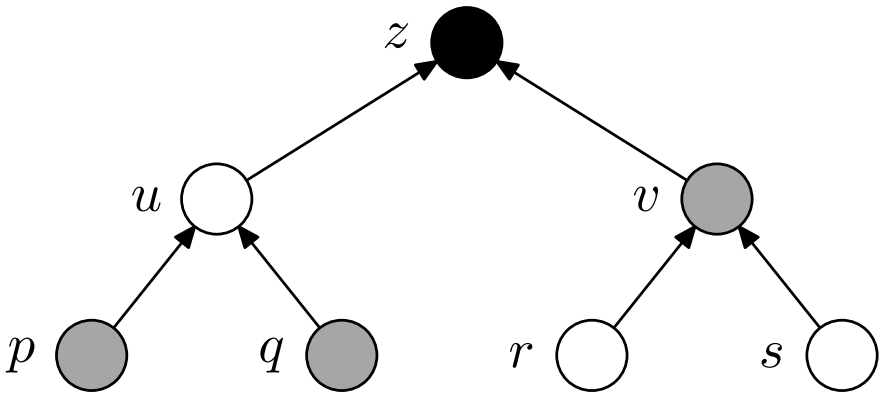


$\overline{u} \vee \overline{v} \vee z$
$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{u} \vee \overline{v} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

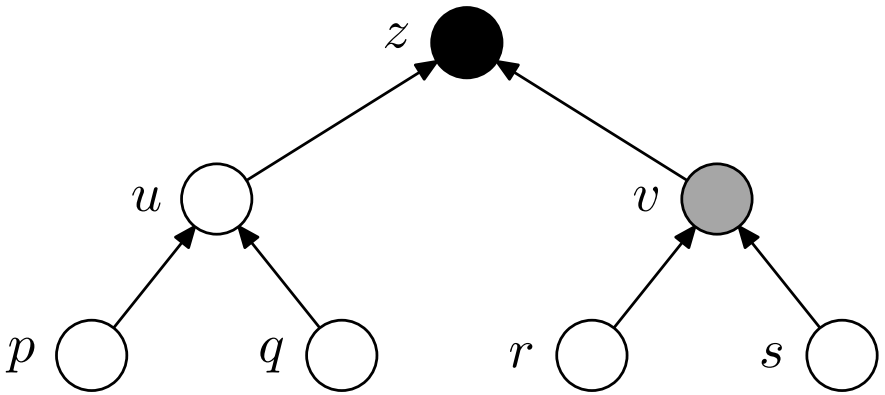


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Erase clause $\overline{u} \vee \overline{v} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

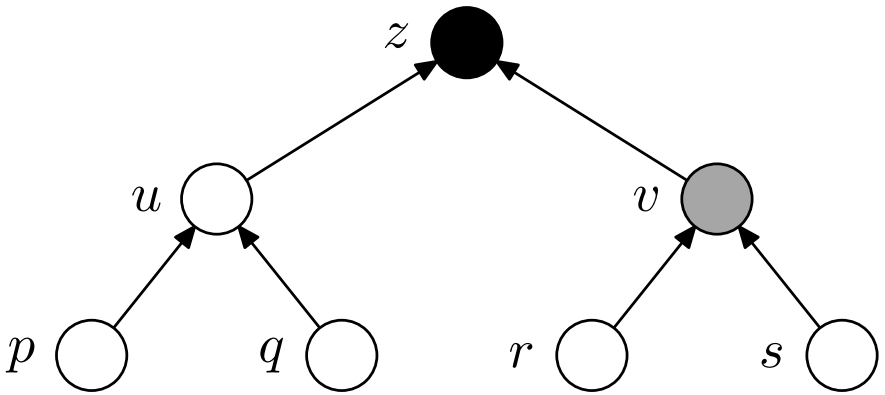


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$

Write down axiom 5: $\overline{p} \vee \overline{q} \vee u$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

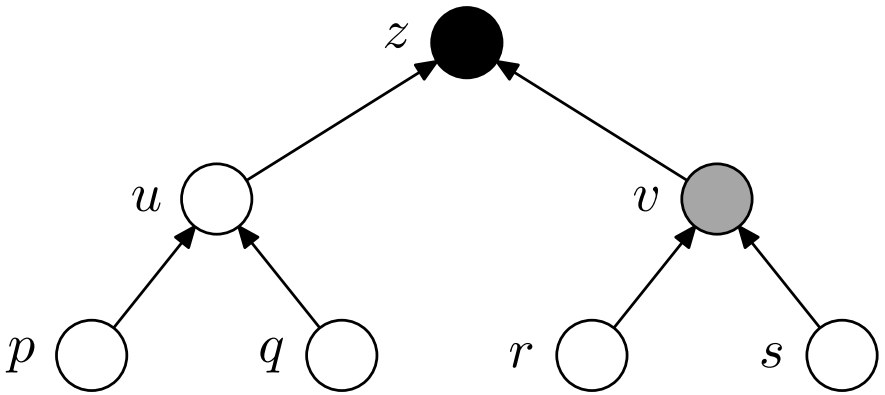


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$

Infer $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$ from $\overline{p} \vee \overline{q} \vee u$ and $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

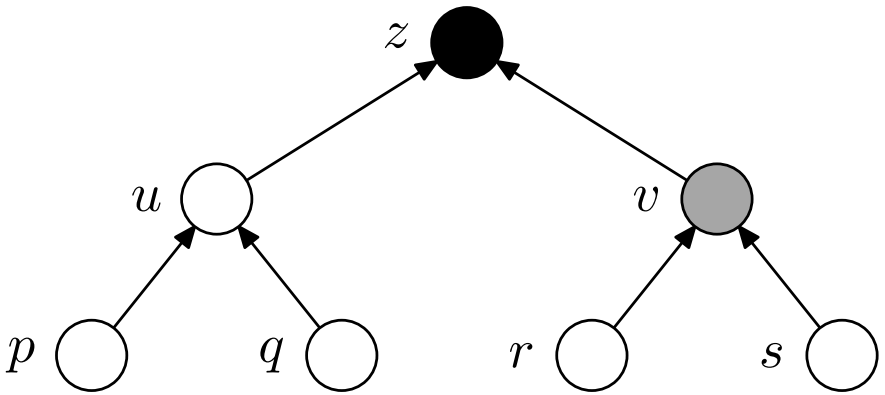


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$
$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Infer $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$ from $\overline{p} \vee \overline{q} \vee u$ and $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

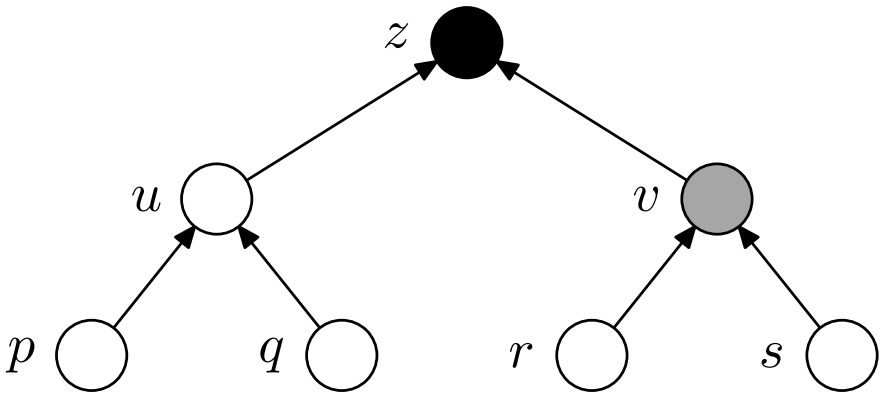


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee u$
$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

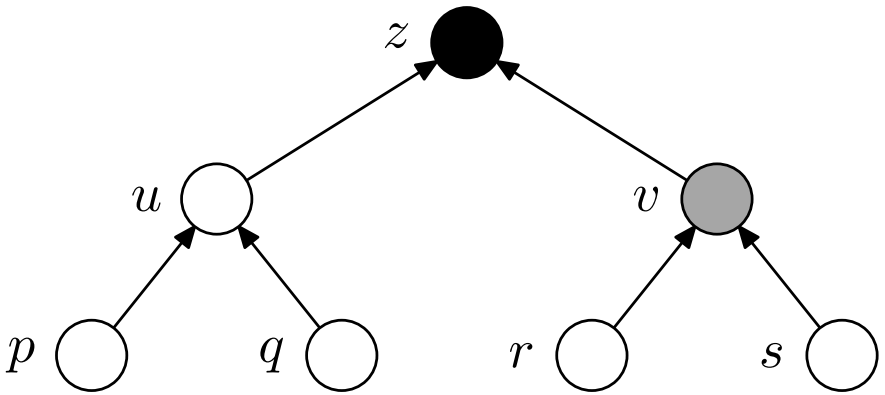


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee u$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

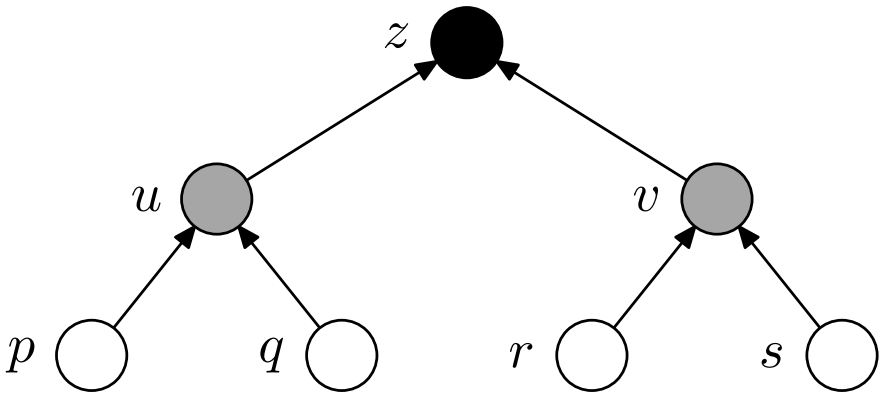


$\overline{r} \vee \overline{s} \vee \overline{u} \vee z$
$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

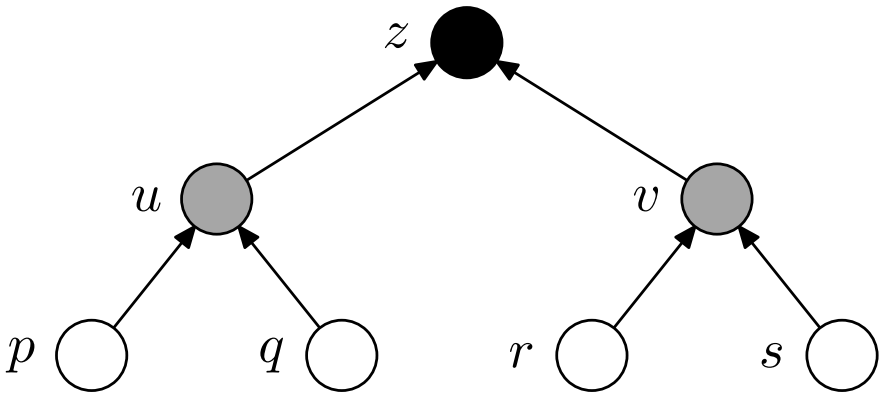


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee \overline{u} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

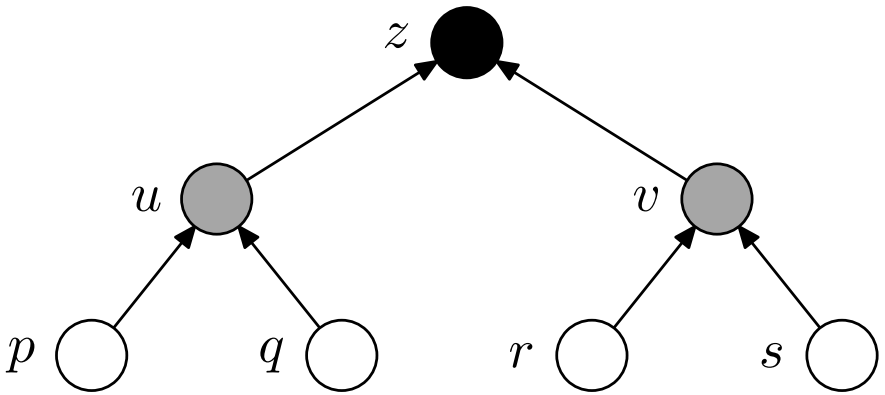


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
p

Write down axiom 1: p

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

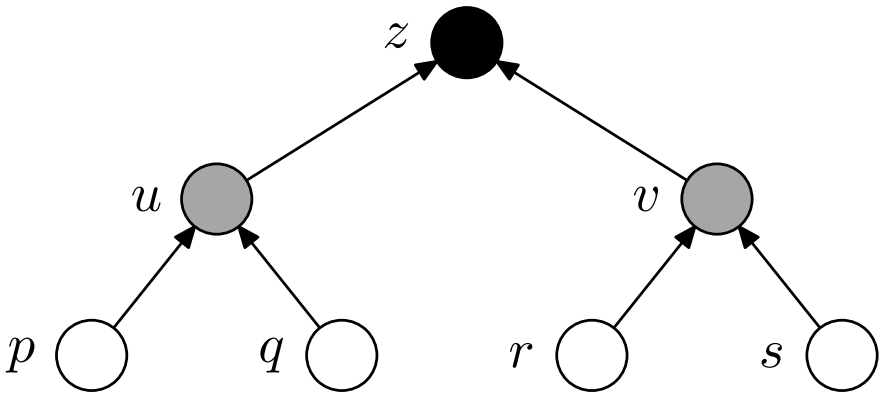


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
p

Infer $\overline{q} \vee \overline{r} \vee \overline{s} \vee z$ from p and $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

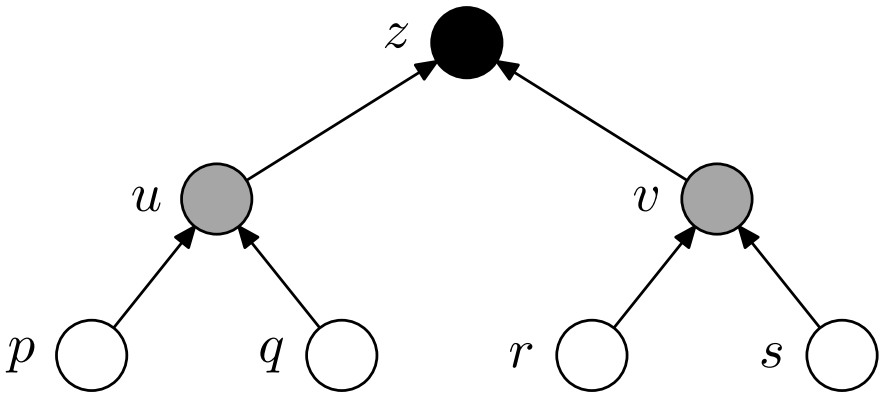


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
p
$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Infer $\overline{q} \vee \overline{r} \vee \overline{s} \vee z$ from
p and $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

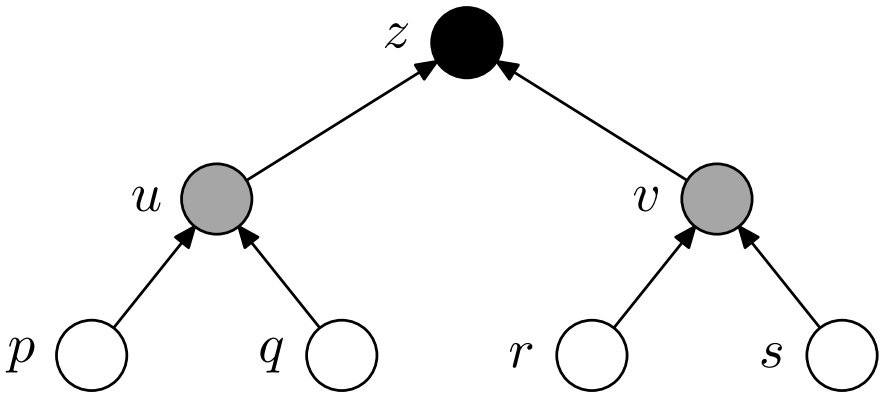


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
p
$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause p

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

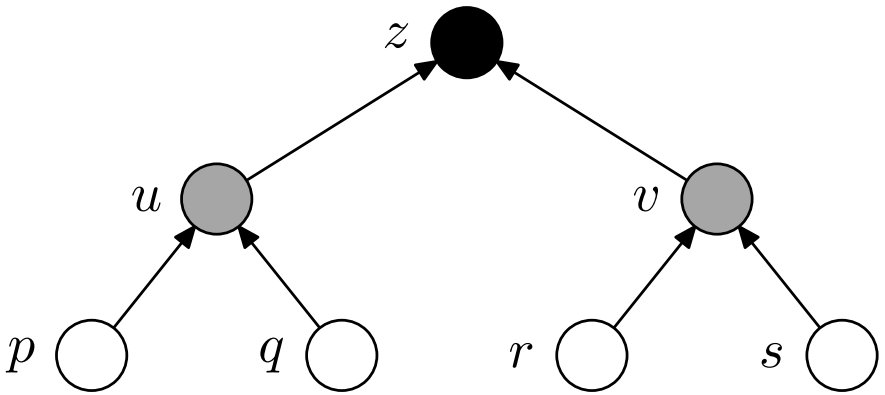


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause p

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

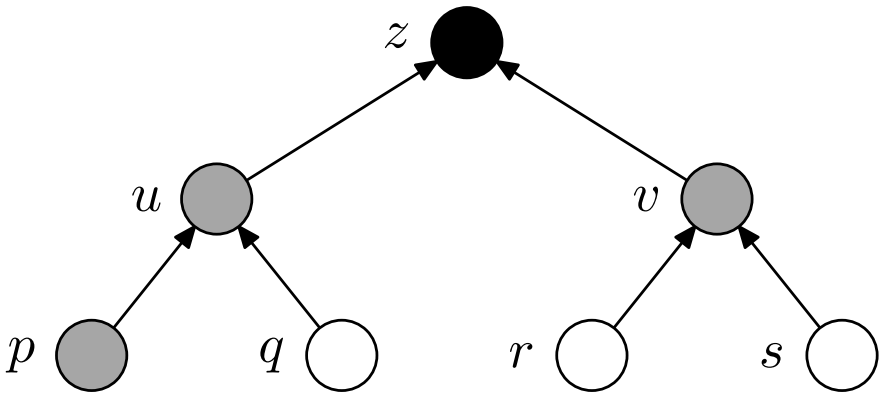


$\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

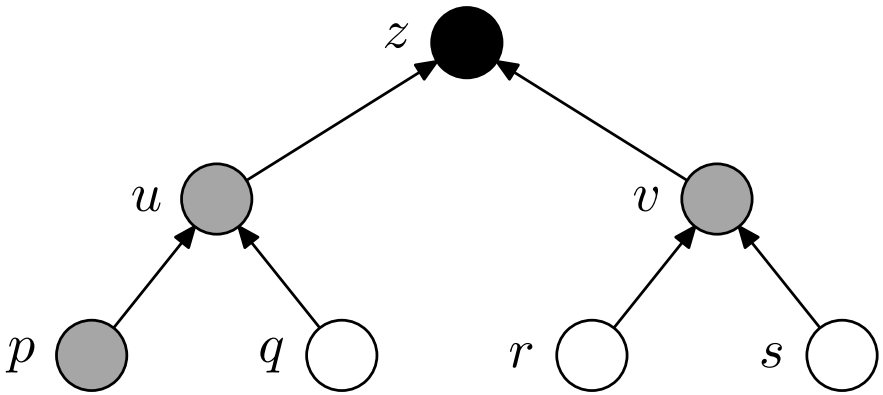


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{p} \vee \overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

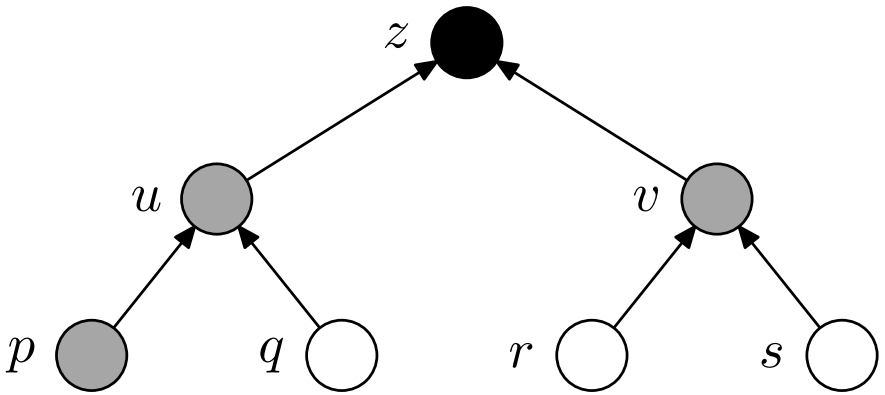


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
q

Write down axiom 2: q

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

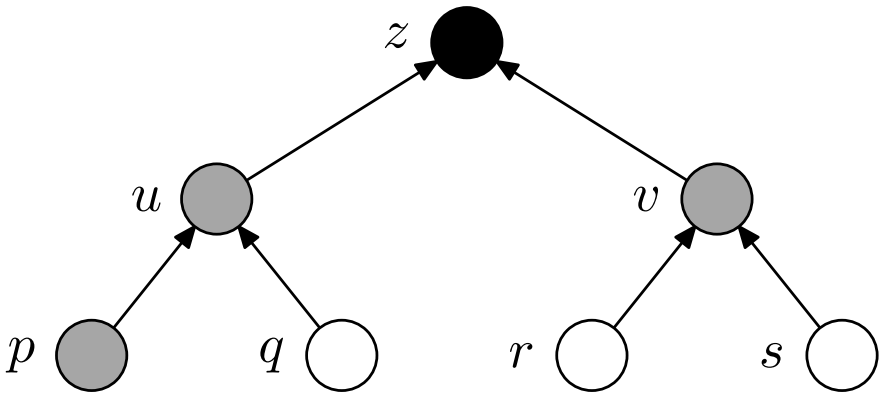


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
q

Infer $\overline{r} \vee \overline{s} \vee z$ from
q and $\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

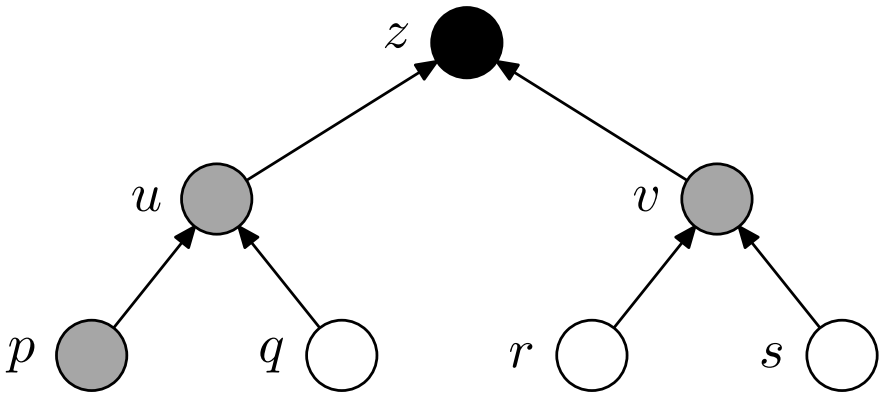


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
q
$\overline{r} \vee \overline{s} \vee z$

Infer $\overline{r} \vee \overline{s} \vee z$ from
q and $\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

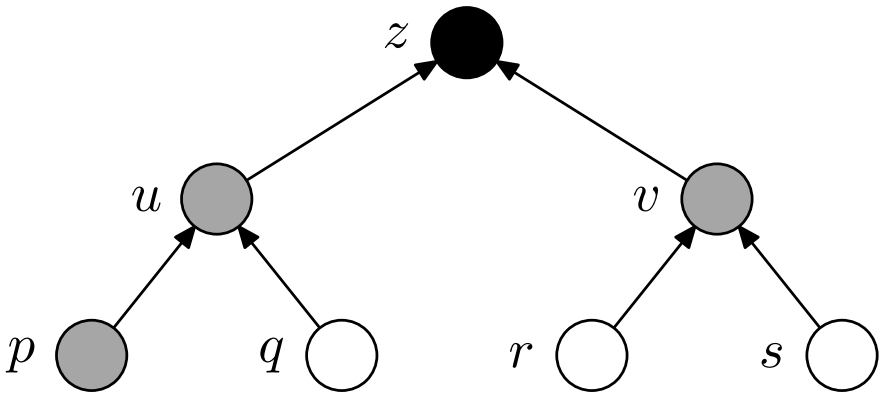


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
q
$\overline{r} \vee \overline{s} \vee z$

Erase clause q

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

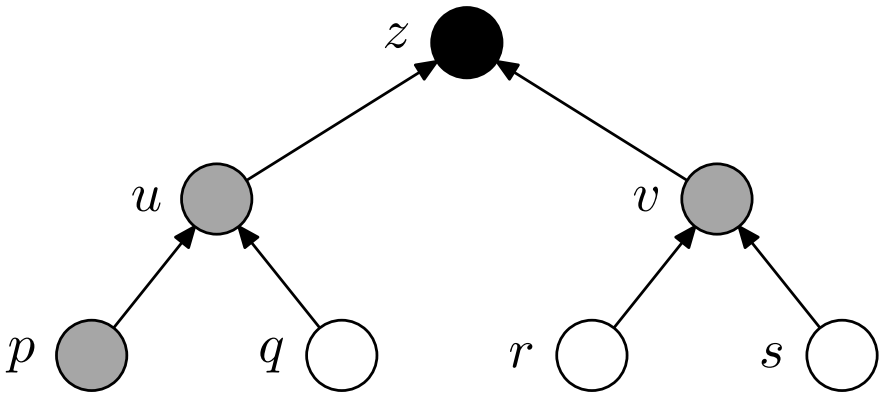


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{r} \vee \overline{s} \vee z$

Erase clause q

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

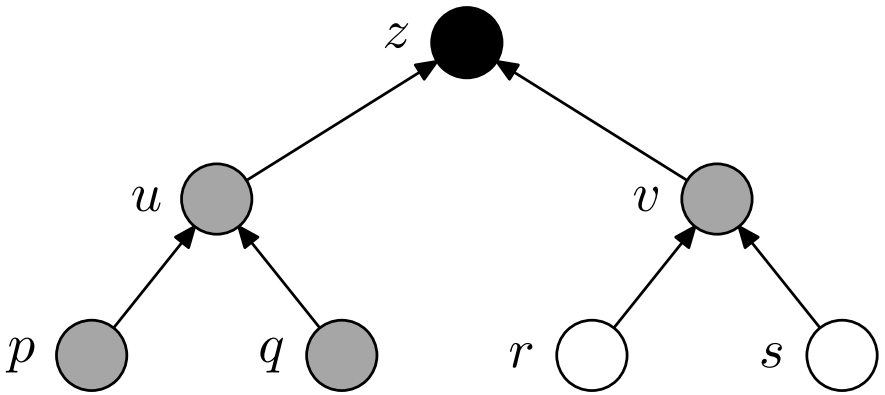


$\overline{q} \vee \overline{r} \vee \overline{s} \vee z$
$\overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

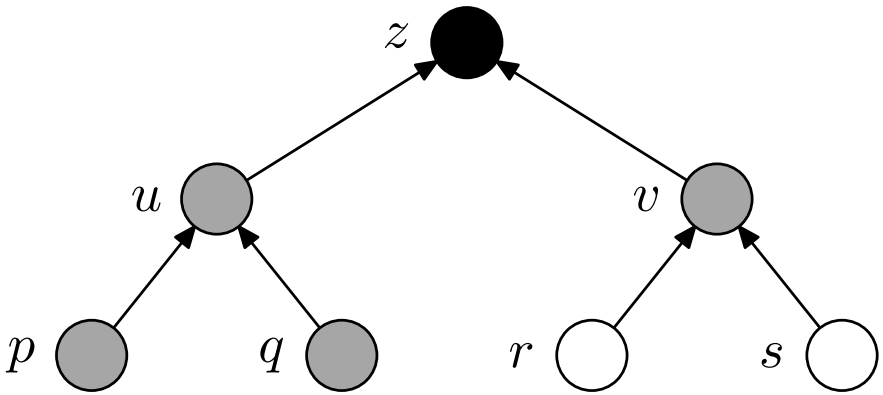


$\overline{r} \vee \overline{s} \vee z$

Erase clause $\overline{q} \vee \overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



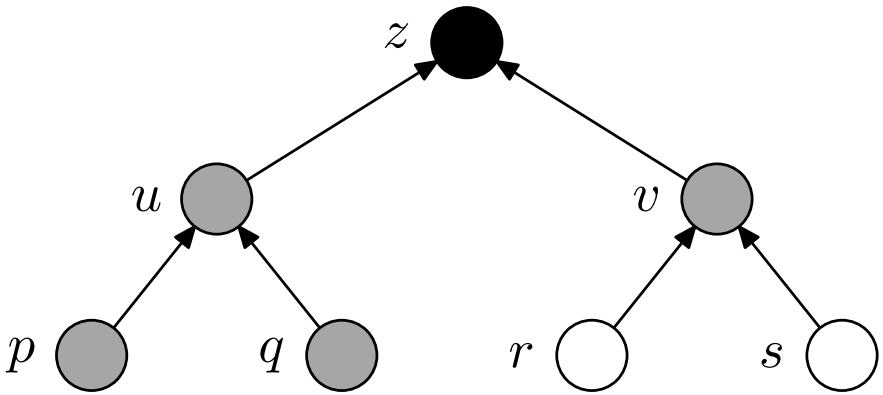
$\overline{r} \vee \overline{s} \vee z$

r

Write down axiom 3: r

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

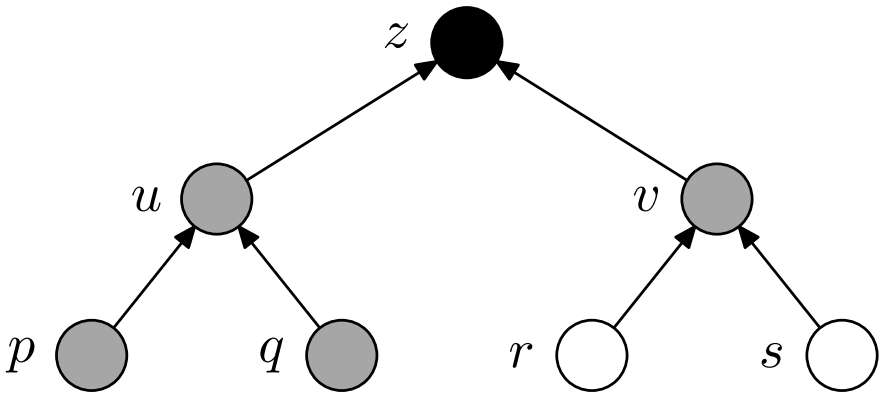


$\overline{r} \vee \overline{s} \vee z$
r

Infer $\overline{s} \vee z$ from
r and $\overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

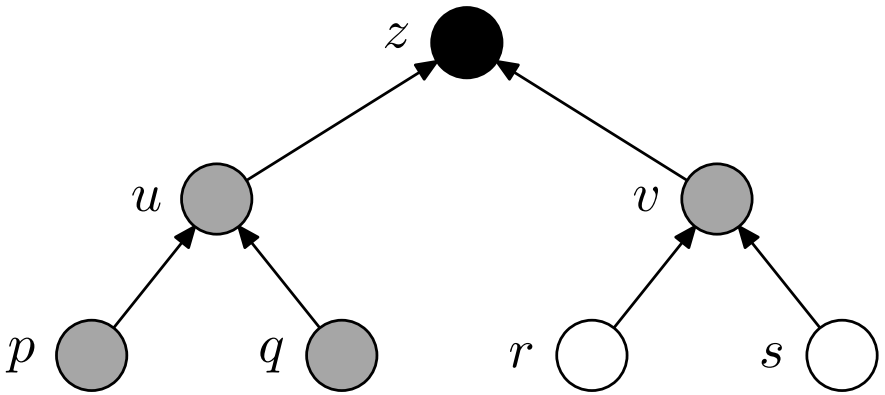


$\overline{r} \vee \overline{s} \vee z$
r
$\overline{s} \vee z$

Infer $\overline{s} \vee z$ from
r and $\overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



$\overline{r} \vee \overline{s} \vee z$

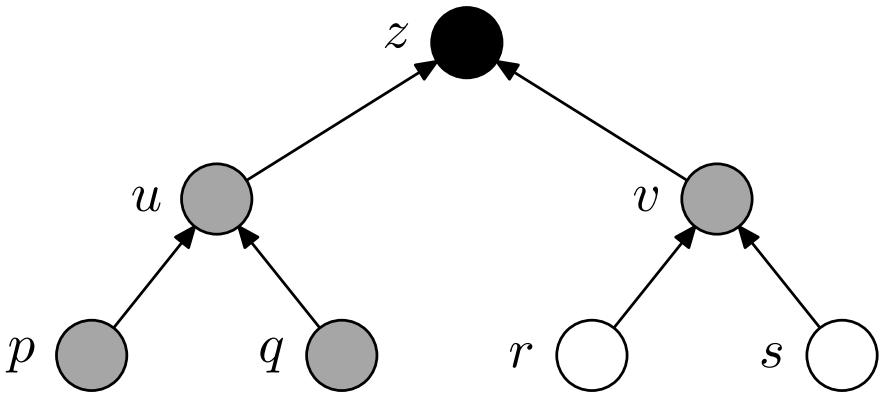
r

$\overline{s} \vee z$

Erase clause r

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

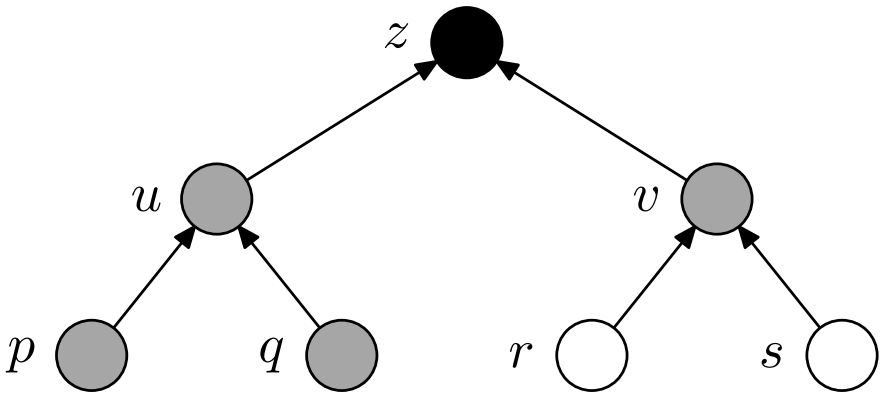


$\overline{r} \vee \overline{s} \vee z$
$\overline{s} \vee z$

Erase clause r

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

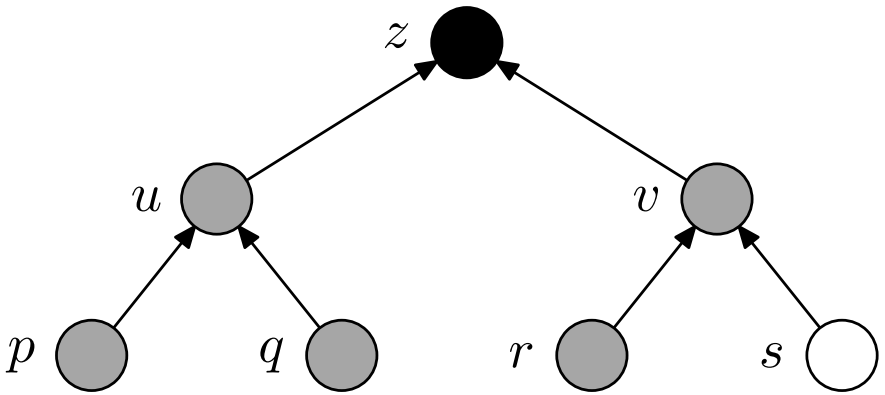


$\overline{r} \vee \overline{s} \vee z$
$\overline{s} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

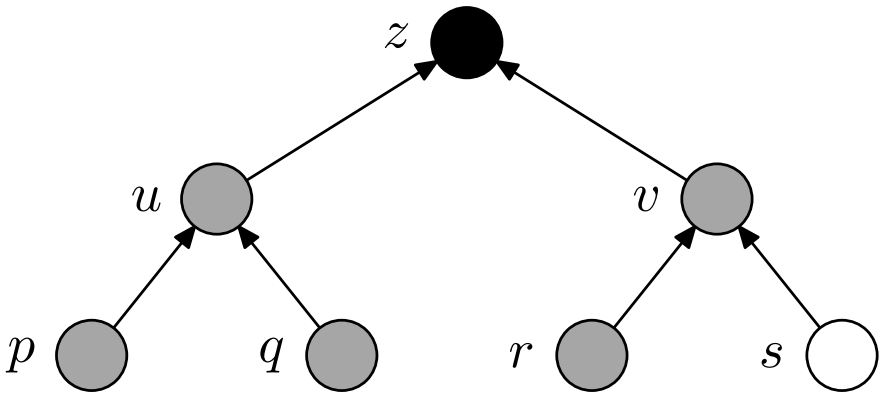


$\overline{s} \vee z$

Erase clause $\overline{r} \vee \overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



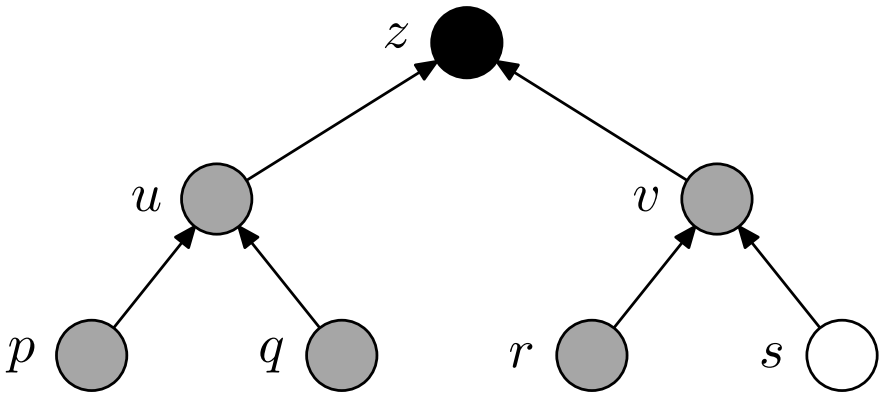
$\overline{s} \vee z$

s

Write down axiom 4: s

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



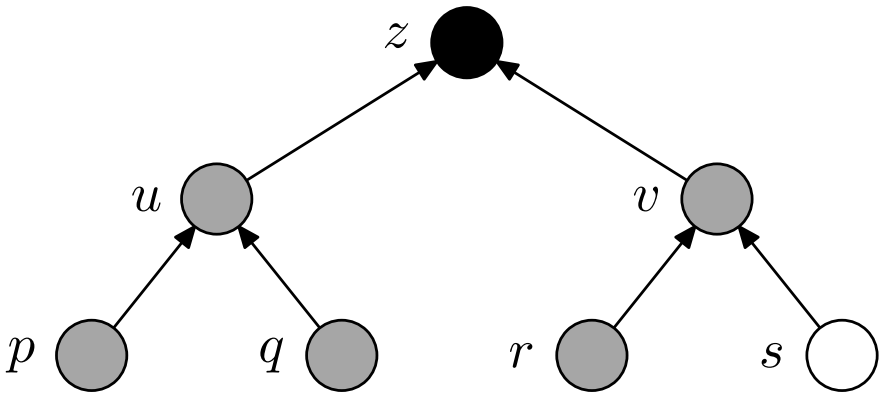
$\overline{s} \vee z$

s

Infer z from
s and $\overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$

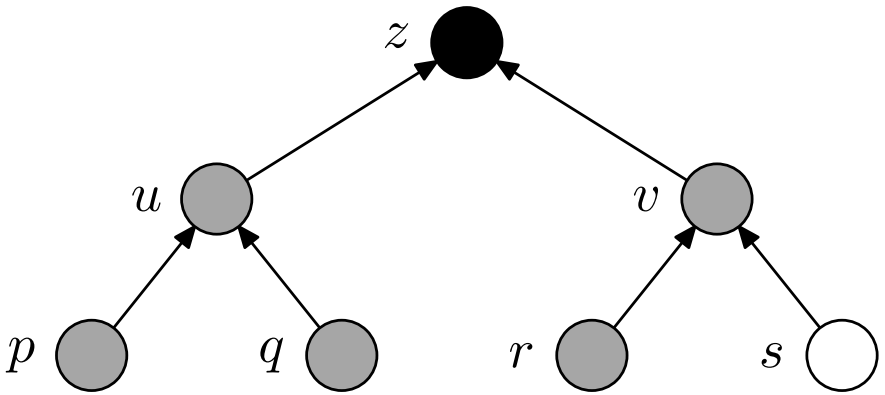


$\overline{s} \vee z$
s
z

Infer z from
s and $\overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



$\overline{s} \vee z$

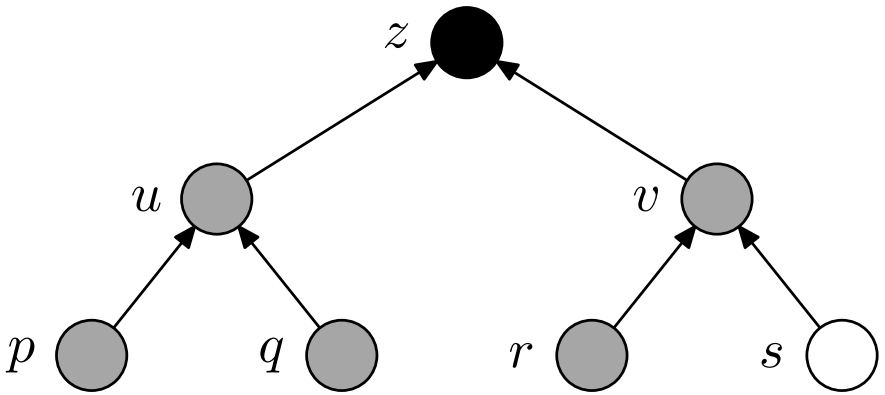
s

z

Erase clause s

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



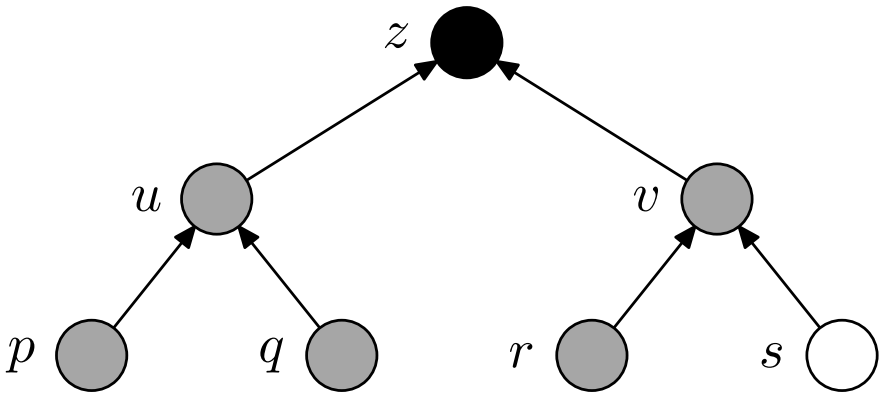
$\overline{s} \vee z$

z

Erase clause s

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



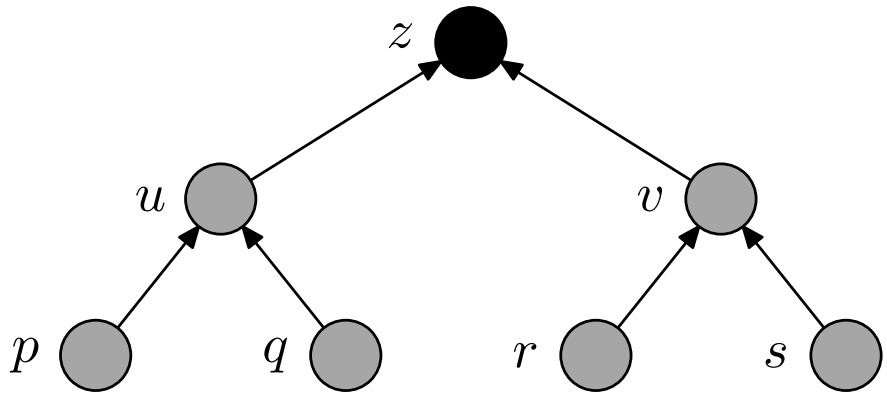
$\overline{s} \vee z$

z

Erase clause $\overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



z

Erase clause $\overline{s} \vee z$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
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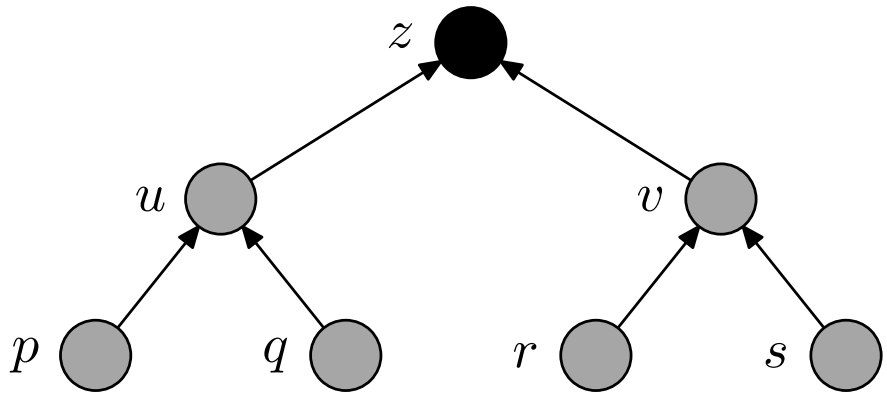

z
$\overline{z}$

Write down axiom 8: $\overline{z}$

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
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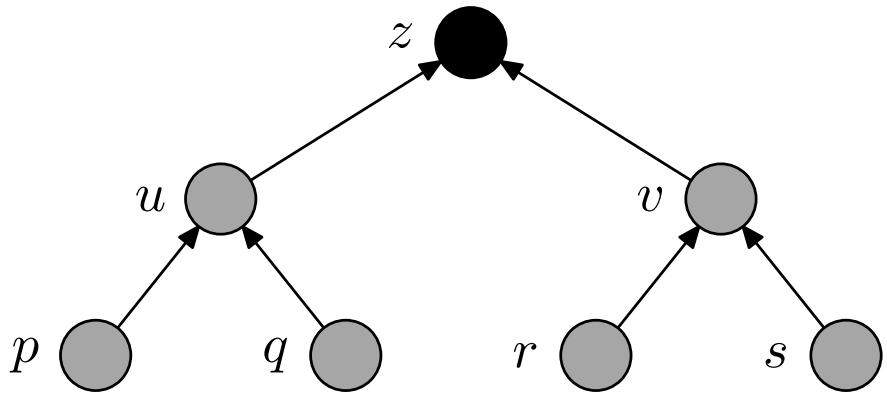
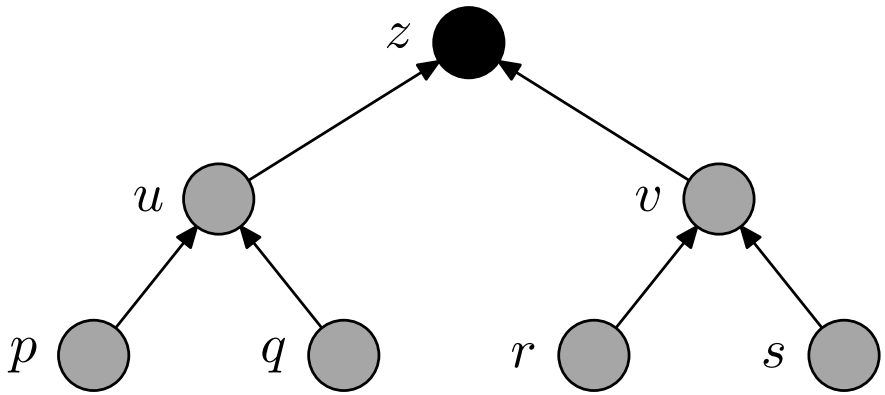


z
$\overline{z}$

Infer 0 from
$\overline{z}$ and z

Example Formula Refutable in Constant Space

1. p
2. q
3. r
4. s
5. $\overline{p} \vee \overline{q} \vee u$
6. $\overline{r} \vee \overline{s} \vee v$
7. $\overline{u} \vee \overline{v} \vee z$
8. $\overline{z}$



z
$\overline{z}$
0

Infer 0 from $\overline{z}$ and z

Expand Formula to Two Variables Per Vertex

Replace variable v with disjunction $v_1 \vee v_2$:

$$
\begin{array}{lcl}
p & \Rightarrow & p_1 \vee p_2 \\
\\
\overline{p} \vee \overline{q} \vee u & \Rightarrow & \overline{p}_1 \vee \overline{q}_1 \vee u_1 \vee u_2 \\
 & & \overline{p}_1 \vee \overline{q}_2 \vee u_1 \vee u_2 \\
 & & \overline{p}_2 \vee \overline{q}_1 \vee u_1 \vee u_2 \\
 & & \overline{p}_2 \vee \overline{q}_2 \vee u_1 \vee u_2 \\
\\
\overline{z} & \Rightarrow & \overline{z}_1 \\
 & & \overline{z}_2
\end{array}
$$

Expanded Formula Yields Separation

- Proof in constant width still works:
  Refutation width $= \mathcal{O}(\text{formula width}) = \mathcal{O}(1)$
- But proof in constant space breaks down:
  Refutation space $= \Omega(\text{tree height } h)$

Let us have a look at ideas behind proof

Black-White Pebble Game on Directed Acyclic Graphs

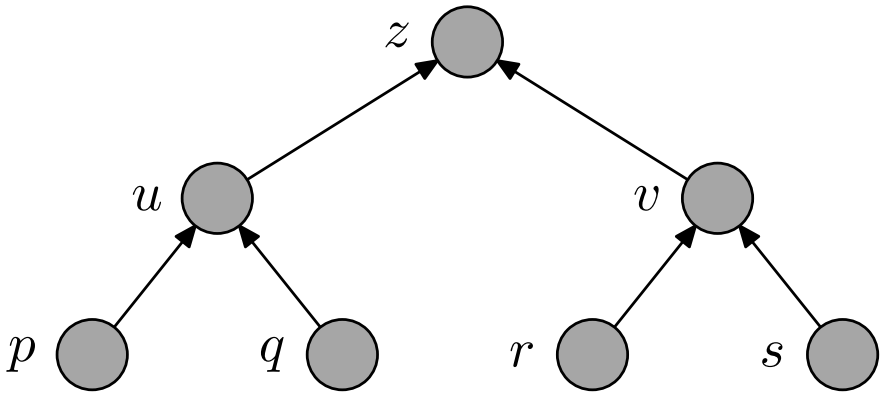
Start with all vertices of DAG G empty

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

Goal: get black pebble on target vertex of G with no other pebbles in G, using as few pebbles as possible

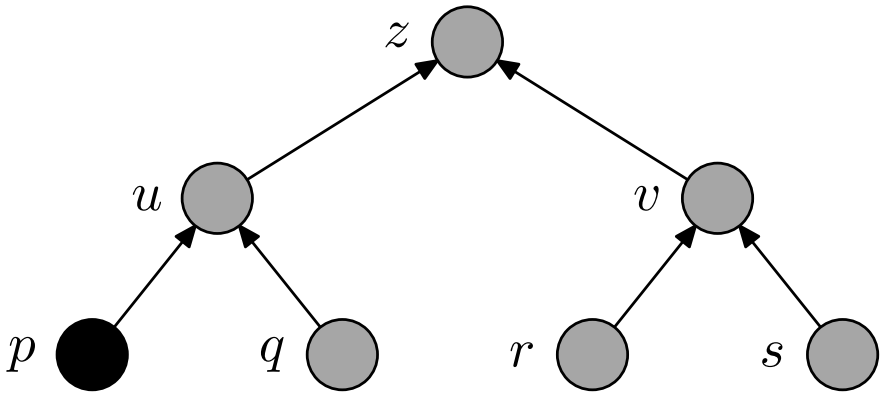
Studied by Cook & Sethi (1976) and many others

Example Pebbling and Pebbling Price



- Cost of pebbling:
  max # pebbles simultaneously in G
- Black-white pebbling price $\mathit{BW\text{-}Peb}(G)$ of DAG G:
  minimal cost of any pebbling

Example Pebbling and Pebbling Price



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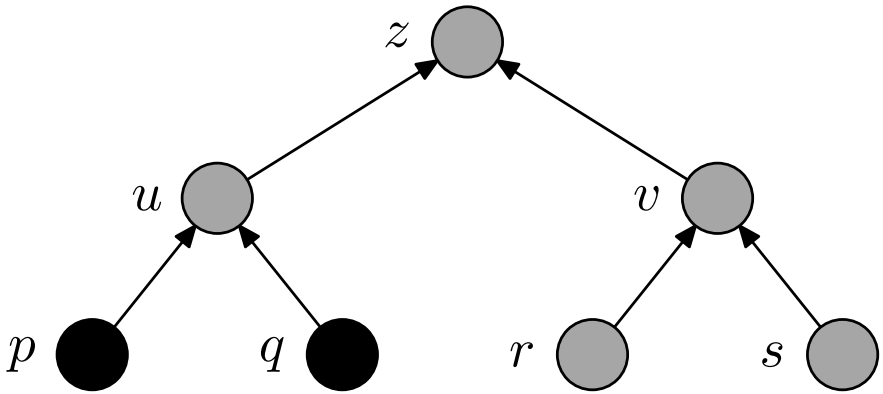

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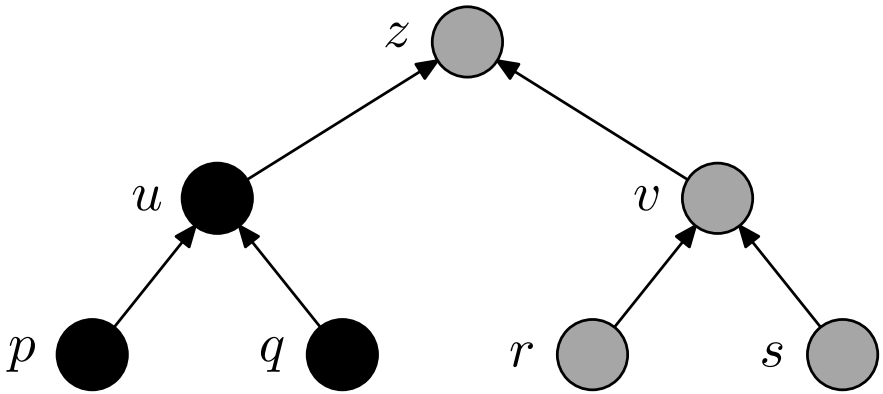

- Cost of pebbling:
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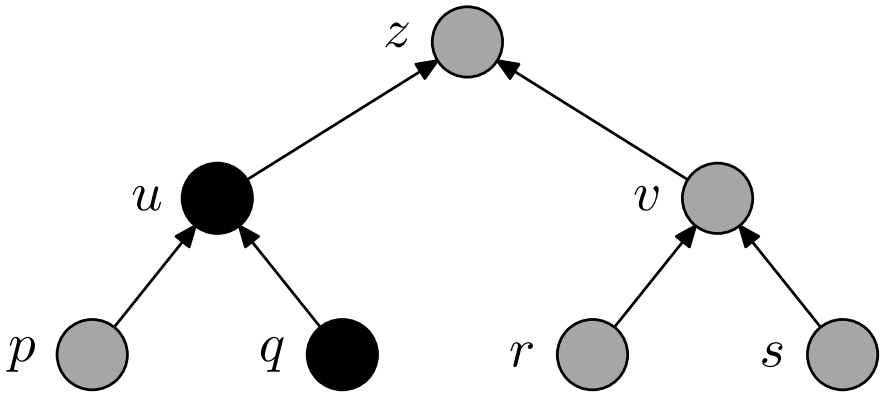

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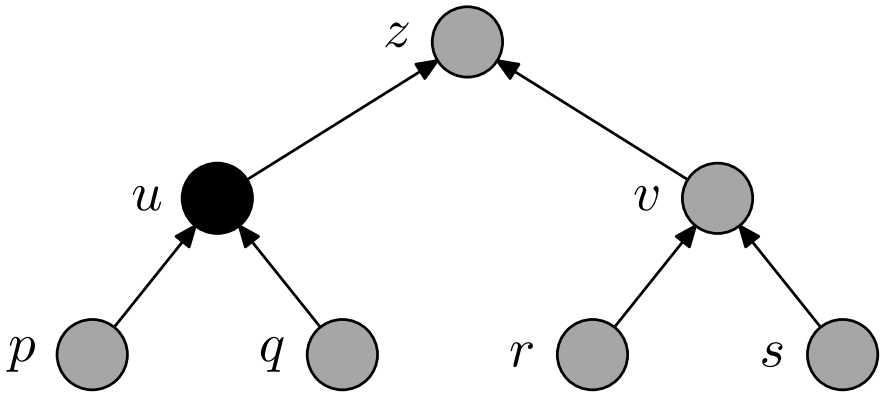
Example Pebbling and Pebbling Price



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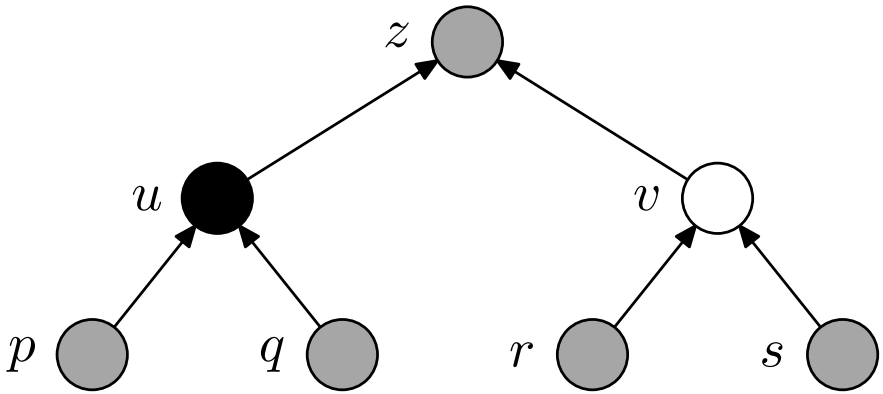

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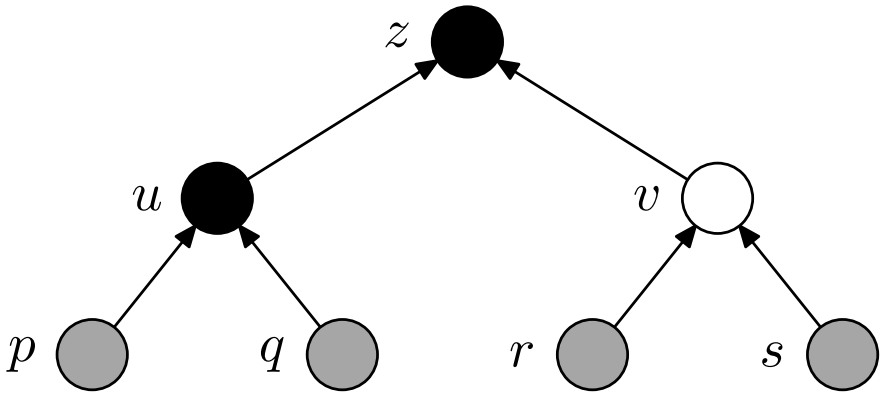
Example Pebbling and Pebbling Price



- Cost of pebbling:
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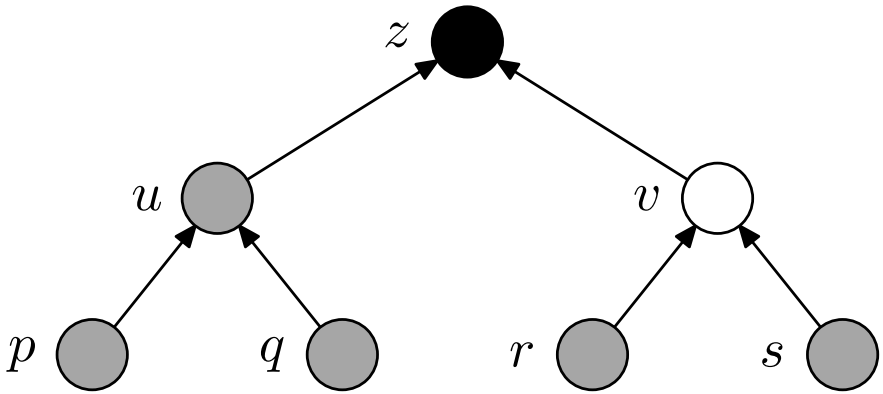
- Black-white pebbling price $\mathit{BW\text{-}Peb}(G)$ of DAG G:
  minimal cost of any pebbling

Example Pebbling and Pebbling Price



- Cost of pebbling:
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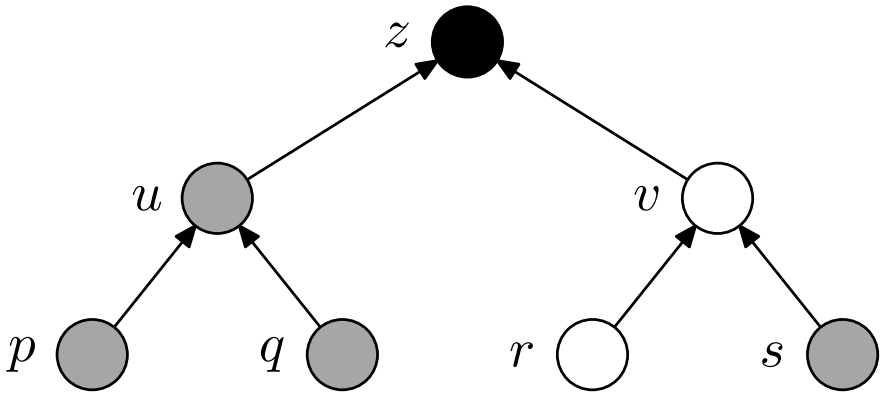
- Black-white pebbling price *BW-Peb*(G) of DAG G:
  minimal cost of any pebbling

Example Pebbling and Pebbling Price



- Cost of pebbling:
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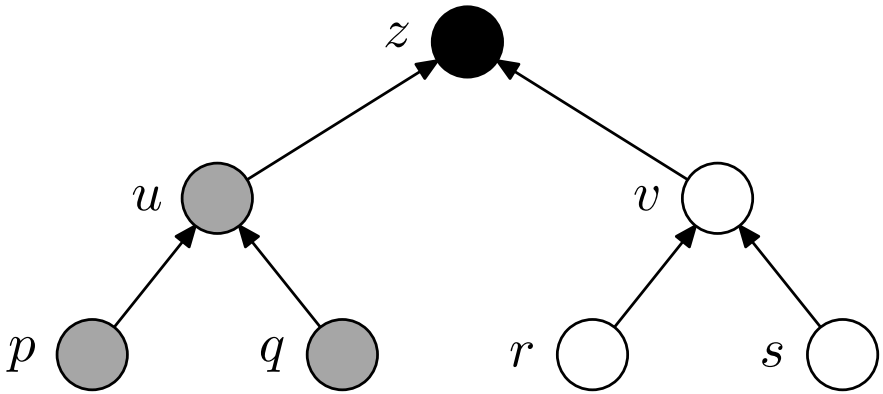

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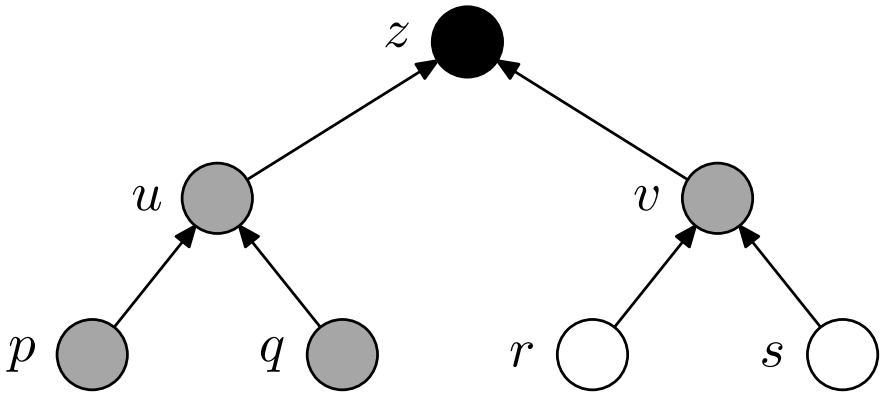

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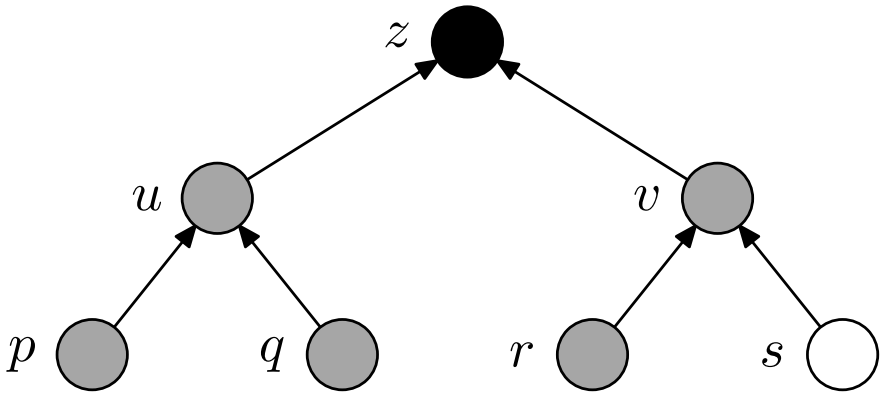

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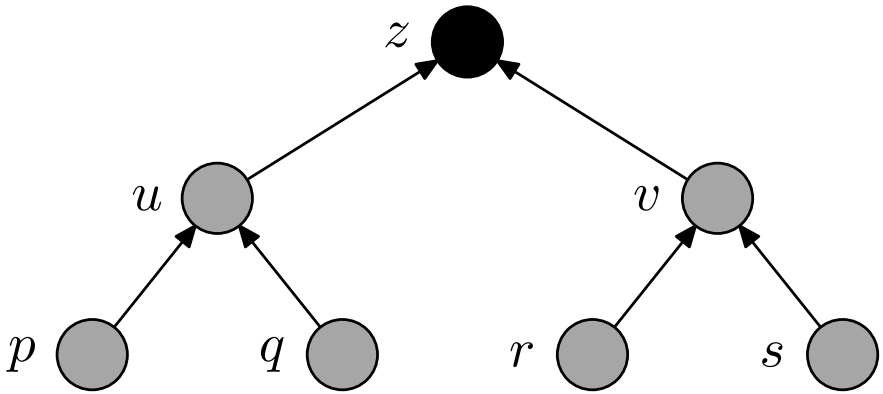

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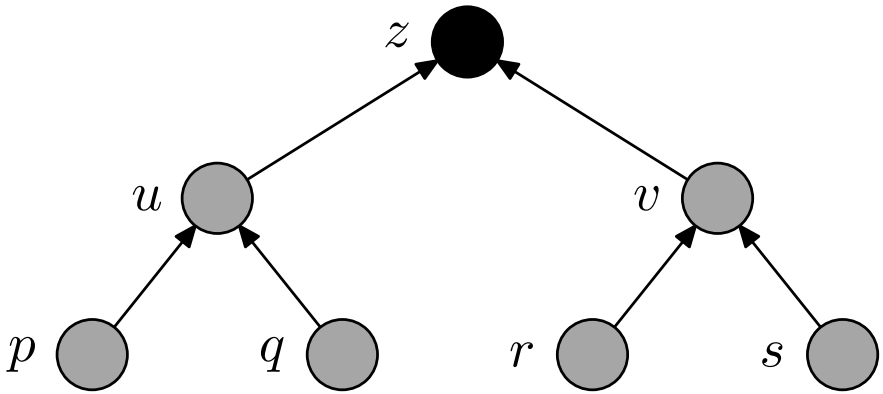
Example Pebbling and Pebbling Price



- Cost of pebbling:
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Pebbling Contradiction

CNF formula encoding pebble game on DAG G with unique target z and all non-source vertices having indegree 2

Associate d variables $v_1, \ldots, v_d$ with every vertex $v \in V(G)$

The dth degree pebbling contradiction Peb_G^d over G says that:

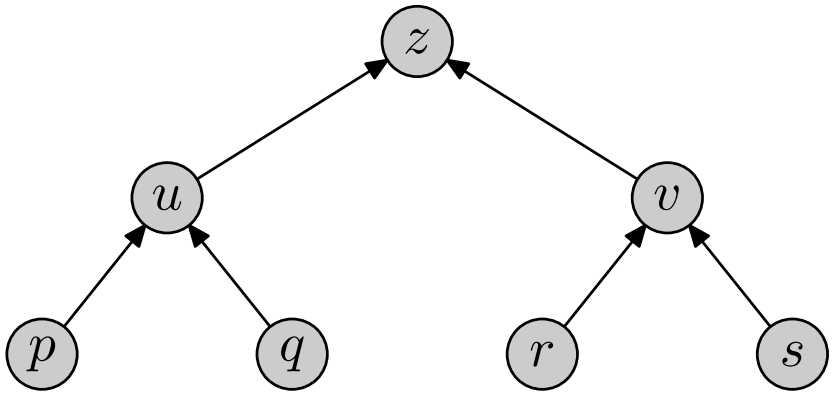
- All source vertices have at least one true variable
- For the target z all variables are false
- Truth and falsity propagate according to pebble game rules

Studied by Bonet et al. (1998), Raz & McKenzie (1999), Ben-Sasson & Wigderson (1999) and others

1st Degree Pebbling Contradiction $Peb^1_{T_2}$

Our example formula:
1st degree pebbling contradiction over binary tree of height 2

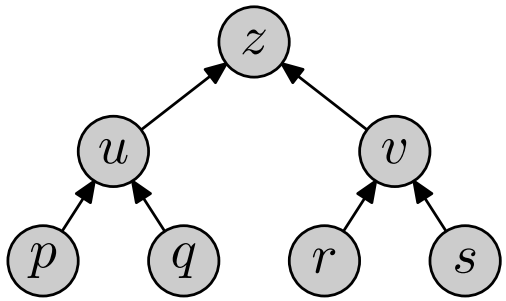
$$
\begin{aligned}
& p \\
\wedge\ & q \\
\wedge\ & r \\
\wedge\ & s \\
\wedge\ & (\overline{p} \vee \overline{q} \vee u) \\
\wedge\ & (\overline{r} \vee \overline{s} \vee v) \\
\wedge\ & (\overline{u} \vee \overline{v} \vee z) \\
\wedge\ & \overline{z}
\end{aligned}
$$



2nd Degree Pebbling Contradiction $Peb^2_{T_2}$

The "expanded formula" is a 2nd degree pebbling contradiction:

$$
\begin{aligned}
&(p_1 \vee p_2) \\
\wedge\ &(q_1 \vee q_2) \\
\wedge\ &(r_1 \vee r_2) \\
\wedge\ &(s_1 \vee s_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_2 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_2 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{r}_1 \vee \overline{s}_1 \vee v_1 \vee v_2)
\end{aligned}
$$



$$
\begin{aligned}
\wedge\ &(\overline{r}_1 \vee \overline{s}_2 \vee v_1 \vee v_2) \\
\wedge\ &(\overline{r}_2 \vee \overline{s}_1 \vee v_1 \vee v_2) \\
\wedge\ &(\overline{r}_2 \vee \overline{s}_2 \vee v_1 \vee v_2) \\
\wedge\ &(\overline{u}_1 \vee \overline{v}_1 \vee z_1 \vee z_2) \\
\wedge\ &(\overline{u}_1 \vee \overline{v}_2 \vee z_1 \vee z_2) \\
\wedge\ &(\overline{u}_2 \vee \overline{v}_1 \vee z_1 \vee z_2) \\
\wedge\ &(\overline{u}_2 \vee \overline{v}_2 \vee z_1 \vee z_2) \\
\wedge\ &\overline{z}_1 \\
\wedge\ &\overline{z}_2
\end{aligned}
$$

Making the Connection

Look at resolution refutation of pebbling contradiction $Peb^d_{T_h}$ over binary tree T_h of height h:

- Interpret clauses "on blackboard" in terms of pebbles (our second resolution refutation example of this)
- Show refutation of $Peb^d_{T_h}$ yields black-white pebbling of T_h (but get very ill-behaved pebbling which requires new rules)
- Show many pebbles in graph $\Rightarrow$ many clauses on blackboard (if degree $d \geq 2$)
- Generalizing known bound $BW\text{-}Peb(T_h) = \Theta(h)$ to new pebble game gives bound on refutation space $Sp(Peb^d_{T_h} \vdash 0) = \Theta(h) = \Theta(\log(\text{formula size}))$

Conclusion

First lower bound on space in resolution which

- is not the consequence of a lower bound on width
- but instead separates the two measures

Open Problems

Extend to arbitrary DAGs

Our proof works only for binary trees (pebbling price collapses for general DAGs because of new rules)

Conjecture 1

For an arbitrary DAG G and $d \geq 2$ it holds for the pebbling contradiction Peb_G^d of degree d over G that $Sp(Peb_G^d \vdash 0) = \Omega(BW\text{-}Peb(G))$.

Would yield almost optimal separation $\Omega(n/\log n)$ between space and width

Best conceivable is $\Omega(n)$

Open Problems (cont.)

Generalize to k-DNF resolution and prove space hierarchy

k-DNF resolution: lines in proof not disjunctive clauses but disjunctions of conjunctions of size $\leq k$

Conjecture 2

For k-DNF resolution refutations of pebbling contradictions defined over complete binary trees T_h of height h, fixing k it holds that $Sp_{\mathfrak{Res}(k+1)}\left(Peb_{T_h}^{k+1} \vdash 0\right) = \mathcal{O}(1)$ but $Sp_{\mathfrak{Res}(k)}\left(Peb_{T_h}^{k+1} \vdash 0\right) = \Omega(h)$.

Would show that k-DNF resolution proof systems for increasing k form strict space hierarchy

References

Full-length version of this paper published as
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Thank you for your attention!