Pseudo-Boolean Solving and Optimization

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"Satisfiability: Theory, Practice, and Beyond" Boot Camp
Simons Institute for the Theory of Computing
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Organization of This Tutorial

Part I: Pseudo-Boolean Preliminaries

Part II: Pseudo-Boolean Solving

Part III: Pseudo-Boolean Optimization

Part IV: Mixed Integer Linear Programming

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Pseudo-Boolean?

Pseudo-Boolean function: $f : \{0,1\}^n \rightarrow \mathbb{R}$

Studied since 1960s in operations research and 0-1 integer linear programming [BH02]

Restricted version: f represented as linear form [focus of this tutorial]

Many problems expressible as optimizing value of linear pseudo-Boolean function under linear pseudo-Boolean constraints

Pseudo-Boolean vs. SAT

- Pseudo-Boolean format richer than conjunctive normal form (CNF)

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
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- Yet close enough to SAT to benefit from SAT solving advances
- Also possible synergies with 0-1 integer linear programming (ILP)

Pseudo-Boolean Constraints and Normalized Form

In this talk, pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \bowtie A$$

- $\bowtie \in \{\geq, \leq, =, >, <\}$
- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
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Convenient to use normalized form [Bar95] (without loss of generality)

$$\sum_i a_i \ell_i \geq A$$

- constraint always greater-than-or-equal
- $a_i, A \in \mathbb{N}$
- $A = \mathit{deg}(\sum_i a_i \ell_i \geq A)$ referred to as degree (of falsity)

Some Types of Pseudo-Boolean Constraints

1. Clauses are pseudo-Boolean constraints

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3. General constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Formulas, Decision Problems, and Optimization Problems

Pseudo-Boolean (PB) formula
Conjunction of pseudo-Boolean constraints
$F \doteq C_1 \land C_2 \land \cdots \land C_m$

Pseudo-Boolean Solving (PBS)
Decide whether F is satisfiable/feasible

Pseudo-Boolean Optimization (PBO)
Find satisfying assignment to F that minimizes objective function $\sum_i w_i \ell_i$
(Maximization: minimize $-\sum_i w_i \ell_i$)

Approaches for Pseudo-Boolean Problems

What we will discuss in this talk:

1. Pseudo-Boolean (PB) solving and optimization
2. MaxSAT solving
3. Integer linear programming (ILP) — or, more generally, mixed integer linear programming (MIP)

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Conversion to disjunctive clauses

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  - MINISAT+ [ES06]
  - OPEN-WBO [MML14]
  - NAPS [SN15]

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Native reasoning with pseudo-Boolean constraints

- PRS [DG02]
- Galena [CK05]
- Pueblo [SS06]
- Sat4j [LP10]
- RoundingSat [EN18]

Re-encoding to CNF

- CNF encoding can be exponentially larger than PB encoding
- Use extension variables for more compact encoding
- High-level idea: new variables = gates in circuit evaluating PB constraint
- *See Part II of pre-recorded tutorial for some concrete examples:*
  - sequential counter encoding [Sin05]
  - totalizer encoding [BB03] and generalized totalizer encoding [JMM15]
  - adder network encoding [ES06]

CNF Encoding Desiderata

Generalized arc consistency (GAC)

For F_C encoding PB constraint C and ρ partial assignment, want:

- If C propagates under ρ, then F_C should yield same propagations
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Quick summary

- Possible to achieve both GAC and polynomial-size encoding [BBR09]
- But complicated; and in practice does not seem better than generalized totalizer [JMM15]?
- Rich literature on encodings — see SAT handbook for more references

Performance of CDCL-Based Pseudo-Boolean Solving

- CDCL-based pseudo-Boolean can be very competitive (sometimes beating native pseudo-Boolean solvers hands down)
- Extension variables potentially gives solver lots of power
  - Allows branching over complex statements
  - Can learn clauses corresponding to polytopes in original problem
- But performance gain from extension variables seems quite sensitive to input order [EGNV18]
- And sometimes extension variables cannot make up for CDCL being exponentially weaker than pseudo-Boolean reasoning [EGNV18]

Some Research Questions

1. How to find best possible CNF encodings of PB constraints for given problem?
   - Trade-offs between propagation strength and encoding size?
   - Rigorous mathematical insights?
2. Understand complementary strengths of CDCL-based and "native" cutting-planes-based PB solving?
   - Theoretical results on computational complexity?
   - Harness complementary strengths in applied solvers?
3. How to make sure re-encoding into CNF is guaranteed to be correct?

"Native" Pseudo-Boolean Conflict-Driven Search

Want to do "same thing" as in conflict-driven clause learning (CDCL) SAT solving [MS96, BS97, MMZ+01] but with pseudo-Boolean constraints without re-encoding

- Variable assignments
  1. Always propagate forced assignment if possible
  2. Otherwise make assignment using decision heuristic

- At conflict
  1. Do conflict analysis to derive new constraint
  2. Add new constraint to instance
  3. Backjump by rolling back decisions so that learned constraint propagates asserting literal (flipping it to opposite value)

Propagation, Conflict, and Slack

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Represent as $\rho = \{$(ordered) set of literals assigned true$\}$

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$$slack(\textstyle\sum_i a_i \ell_i \geq A; \rho) = \displaystyle\sum_{\ell_i \text{ not falsified by } \rho} a_i - A$$

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Consider $C : x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

ρ	$slack(C;\rho)$	comment

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$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

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Note that constraint can be conflicting though not all variables assigned

Conflict Analysis Invariant

Consider example CDCL analysis *[more details in pre-recorded Part II]*

$(u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$

$\boxed{w \stackrel{\mathsf{d}}{=} 0}$

$\boxed{u \stackrel{\overline{u} \vee w}{=} 0}$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$\boxed{y \stackrel{u \vee x \vee y}{=} 1}$

$\boxed{z \stackrel{x \vee \overline{y} \vee z}{=} 1}$

$\boxed{\stackrel{\overline{y} \vee \overline{z}}{\bot}}$

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$\overline{y} \lor \overline{z}$ falsified by trail $\rho = \{\overline{w}, \overline{u}, \overline{x}, y, z\}$

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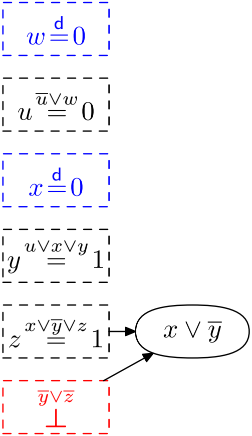

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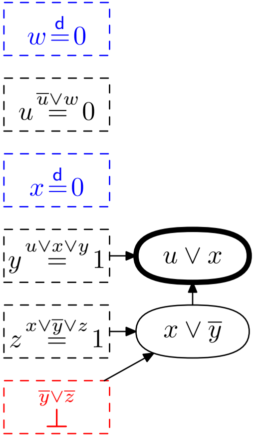

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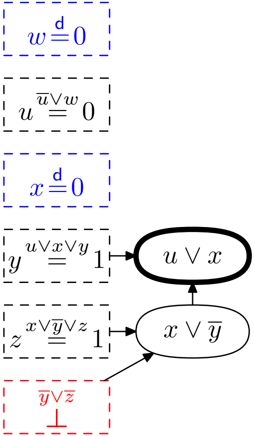

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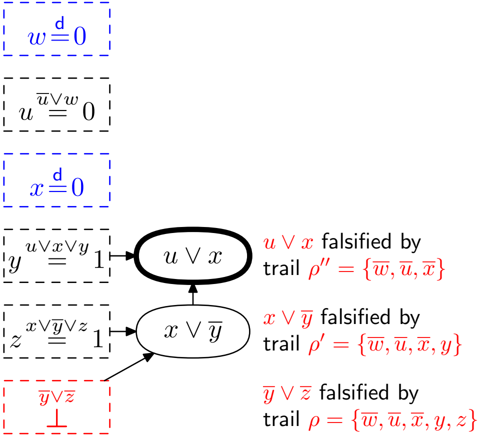

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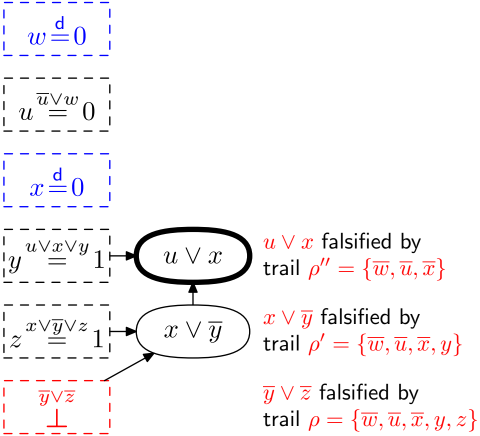

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Learn asserting constraint: after backjump, some variable guaranteed to flip

Generalized Resolution

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$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

by adding clauses as pseudo-Boolean constraints

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

(Recall $z + \overline{z} = 1$)

Generalized Resolution

Can mimic resolution step

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

by adding clauses as pseudo-Boolean constraints

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

(Recall $z + \overline{z} = 1$)

Generalized resolution rule (from [Hoo88, Hoo92])
Positive linear combination so that some variable cancels

$$\frac{a_1 x_1 + \sum_{i \geq 2} a_i \ell_i \geq A \qquad b_1 \overline{x}_1 + \sum_{i \geq 2} b_i \ell_i \geq B}{\sum_{i \geq 2} \left(\frac{c}{a_1} a_i + \frac{c}{b_1} b_i\right) \ell_i \geq \frac{c}{a_1} A + \frac{c}{b_1} B - c} \; [c = \operatorname{lcm}(a_1, b_1)]$$

Saturation

Actually, don't get quite the right constraint in mimicking of resolution

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$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min\{a_i, A\} \cdot \ell_i \geq A}$$

Sound over integers, not over rationals (need such rules for SAT solving)

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[Generalized resolution as defined in [Hoo88, Hoo92] includes fix above, but convenient here to make the two separate steps explicit]

Analyze Conflict with Generalized Resolution + Saturation!

$$
\begin{aligned}
C_1 &\doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4 \\
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\end{aligned}
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(Note: same constraint can propagate several times!)

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- Resolve reason$(x_3, \rho) \doteq C_1$ with C_2 over x_3 to get resolve(C_1, C_2, x_3)

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- Applying saturate$(x_4 \geq 1)$ does nothing
- Non-negative slack w.r.t. $\rho' = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1\}$ — not conflicting!

What Went Wrong? And What to Do About It?

Accident report

- Generalized resolution sound over the reals
- Given $\rho' = \{x_1 = 0, x_2 = 1\}$, over the reals have
  - $C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$ propagates $x_3 \geq \frac{1}{2}$
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- So after resolving away x_3, "can't see any conflict"

Remedial action

- Strengthen propagation to $x_3 \geq 1$ also over the reals
- I.e., want reason C with $slack(C; \rho') = 0$
- **Fix** (non-obvious): Apply weakening

$$\text{weaken}(\textstyle\sum_i a_i\ell_i \geq A, \ell_j) = \sum_{i \neq j} a_i\ell_i \geq A - a_j$$

  to reason constraint and then saturate
- Approach in [CK05] (seems to go back to observations in [Wil76])

Try to Reduce the Reason Constraint

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
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$$\text{resolve } x_3 \dfrac{\text{saturate } \dfrac{\text{weaken } x_2 \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}}{2x_1 + 2x_3 + x_4 \geq 2} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}$$

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Bummer! Still non-negative slack — not conflicting

Try Again to Reduce the Reason Constraint...

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Negative slack — conflicting!

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Negative slack — conflicting!

Backjump propagates to conflict without solver making any decisions

Done! Conflict without decisions $\Rightarrow$ formula unsatisfiable

Reason Reduction Using Saturation [CK05]

$\mathrm{reduceSat}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell, \rho)$

while $\mathit{slack}(\mathsf{resolve}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell); \rho) \geq 0$ **do**
  $\ell' \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ not falsified by ρ;
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Why does this work? *[Explained more carefully in pre-recorded Part II]*

- Slack is subadditive

$$\mathit{slack}(c \cdot C + d \cdot D; \rho) \leq c \cdot \mathit{slack}(C; \rho) + d \cdot \mathit{slack}(D; \rho)$$

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- Saturation decreases slack — reach 0 when max #literals weakened

Pseudo-Boolean Conflict Analys

analyzePBconflict(C_{confl}, ρ)

while C_{confl} *not asserting* **do**
- $\ell \leftarrow$ literal assigned last on trail ρ;
- **if** $\overline{\ell}$ *occurs in* C_{confl} **then**
  - $C_{\mathrm{reason}} \leftarrow$ reason(ℓ, ρ);
  - $\boldsymbol{C_{\mathrm{reason}} \leftarrow \mathrm{reduceSat}(C_{\mathrm{reason}}, C_{\mathrm{confl}}, \ell, \rho)}$;
  - $C_{\mathrm{confl}} \leftarrow$ resolve$(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell)$;
  - $C_{\mathrm{confl}} \leftarrow$ saturate(C_{confl});
- **end**
- $\rho \leftarrow \mathit{removeLast}(\rho)$;

end
return C_{confl};

Reduction of reason new compared to CDCL — everything else the same
Essentially conflict analysis used in SAT4J [LP10]

Some Problems Compared to CDCL

- Compared to clauses harder to detect propagation for constraints like

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- Generalized resolution for general pseudo-Boolean constraints
  $\Rightarrow$ lots of lcm computations
  $\Rightarrow$ coefficient sizes can explode (expensive arithmetic)

- For CNF inputs, degenerates to resolution!
  $\Rightarrow$ CDCL but with super-expensive data structures

The Cutting Planes Proof System

Cutting planes as defined in theory literature [CCT87] doesn't use saturation but instead division (a.k.a. Chvátal-Gomory cut)

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$$

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil}$$

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Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil}$$

- Cutting planes with division implicationally complete
- Cutting planes with saturation is **not** [VEG+18]
- Can division yield stronger conflict analysis?

Using Division to Reduce the Reason

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Terminate immediately!

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$\mathrm{reduceDiv}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell, \rho)$

$c \leftarrow \mathit{coeff}(C_{\mathrm{reason}}, \ell)$;
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    $\ell_j \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ such that $\overline{\ell}_j \notin \rho$ and $c \nmid \mathit{coeff}(C, \ell_j)$;
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So why does **this** work? *[Again, more details in pre-recorded Part II]*

- Sufficient to get reason with slack 0 since
  1. $\mathit{slack}(C_{\mathrm{confl}}; \rho) < 0$
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- Weakening doesn't change slack $\Rightarrow$ always $0 \leq \mathit{slack}(C_{\mathrm{reason}}; \rho) < c$

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$\mathrm{reduceDiv}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell, \rho)$

$c \leftarrow \mathit{coeff}(C_{\mathrm{reason}}, \ell)$;
while $\mathit{slack}(\mathsf{resolve}(C_{\mathrm{confl}}, \mathsf{divide}(C_{\mathrm{reason}}, c), \ell); \rho) \geq 0$ **do**
  $\ell_j \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ such that $\overline{\ell}_j \notin \rho$ and $c \nmid \mathit{coeff}(C, \ell_j)$;
  $C_{\mathrm{reason}} \leftarrow \mathsf{weaken}(C_{\mathrm{reason}}, \ell_j)$;
end
return $\mathsf{divide}(C_{\mathrm{reason}}, c)$;

So why does **this** work? *[Again, more details in pre-recorded Part II]*

- Sufficient to get reason with slack 0 since
  1. $\mathit{slack}(C_{\mathrm{confl}}; \rho) < 0$
  2. slack is subadditive
- Weakening doesn't change slack $\Rightarrow$ always $0 \leq \mathit{slack}(C_{\mathrm{reason}}; \rho) < c$
- After max #weakenings have $0 \leq \mathit{slack}(\mathsf{divide}(C_{\mathrm{reason}}, c); \rho) < 1$

Division vs. Saturation

- Higher conflict speed when PB reasoning doesn't help [EN18]
- Seems to perform better when PB reasoning crucial [EGNV18]
- Keeps coefficients small — can (often) do fixed-precision arithmetic
- But SAT4J still better for some circuit verification problems [LBD+20]
- And still equally hard to detect propagation
- Also, still degenerates to resolution for CNF inputs
- Sometimes very poor performance even on infeasible 0-1 LPs!

Some PB Solving Challenges I: Input Format

1. Preprocessing/presolving: Important in SAT solving and integer linear programming, but not done in PB solvers — why?
    - Follow up on preliminary work on PB preprocessing in [MLM09]?
    - Use presolver PaPILO [PaP] from MIP solver SCIP [SCI]?
2. CNF: How to go beyond conflict-driven clause learning CDCL for decision problems encoded in CNF?
3. Cardinality constraint detection: Proposed as preprocessing [BLLM14] or inprocessing [EN20] — not yet competitive in practice
4. Robustness: Make PB solvers less sensitive to presence of extra constraints (anecdotally, CDCL solvers seem more stable)

Some PB Solving Challenges II: Conflict Analysis

1. Choice of Boolean rule:
   - Division, saturation, or select adaptively?
   - Or some other cut rule from ILP?
   - Try to avoid irrelevant literals? [LMMW20]
2. Many more degrees of freedom than in CDCL:
   - Skip resolution steps when slack very negative?
   - How aggressively to weaken reason in reduction step? [LMW20]
   - Learn general PB constraints or more limited form?
   - How far to backjump when learned constraint asserting at many levels?
   - How large precision to use in integer arithmetic?
3. Do constraint minimization à la [SB09, HS09]?
4. How to assess quality of learned constraints?
5. Theoretical potential and limitations poorly understood [VEG$^+$18]
   - Separations of subsystems of cutting planes?
   - In particular, is division reasoning stronger than saturation? [GNY19]

Some PB Solving Challenges III: Solver Heuristics

Many heuristics more or less copied from CDCL — maybe tailor more carefully to PB setting?

1. Variable selection: VSIDS [MMZ^{+}01] or VMTF [Rya04] or something else?
2. Variable bumping: Consider different bumping score depending on
   - whether literal falsified,
   - whether literal cancels,
   - coefficient of literal and/or degree of constraint?
3. Phase saving: Standard as in [PD07], multiple phases [BF20], or something else?
4. Different "modes" for SAT-focused and UNSAT-focused search?

See [Wal20] for a first in-depth investigation of some of these questions

Some PB Solving Challenges IV: Efficiency and Correctness

1. Efficient unit propagation for PB constraints is a major challenge — latest news in [Dev20], but still much left to do
2. Efficient detection of assertiveness during conflict analysis
3. Efficient and concise proof logging for pseudo-Boolean solving (shameless self-plug: ongoing work on PB proof checker VeriPB [Ver19, GMN20b] in [EGMN20, GMN20a, GMM+20, GN21])

Organization of This Tutorial

Part I: Pseudo-Boolean Preliminaries

Part II: Pseudo-Boolean Solving

Part III: Pseudo-Boolean Optimization

Part IV: Mixed Integer Linear Programming

MaxSAT Problem

Pseudo-Boolean optimization and MaxSAT solving intimately connected, so let's do a detour and define MaxSAT

Weighted partial MaxSAT problem

Input: Soft clauses $C_1, \ldots, C_m$ with weights $w_i \in \mathbb{R}^+$, $i \in [m]$
Hard clauses $C_{m+1}, \ldots, C_M$

Goal: Find assignment ρ such that

- for all hard clauses $C_{m+1}, \ldots, C_M$ it holds that $\rho(C_j) = 1$
- ρ maximizes $\sum_{\rho(C_i)=1, i \in [m]} w_i$

- All hard clauses must be satisfied
- Maximize weight of satisfied soft clauses = Minimize penalty of falsified soft clauses
- Write $(C)_w$ for clause C with weight w ($w = \infty$ for hard clause)

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance

$$(\overline{x})_5$$
$$(y \lor \overline{z})_4$$
$$(\overline{y} \lor z)_3$$
$$(x \lor y \lor z)_\infty$$
$$(x \lor \overline{y} \lor \overline{z})_\infty$$

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance

$$(\overline{x})_5$$
$$(y \vee \overline{z})_4$$
$$(\overline{y} \vee z)_3$$
$$(x \vee y \vee z)_\infty$$
$$(x \vee \overline{y} \vee \overline{z})_\infty$$

PBO instance

$$\min \; 5b_1 + 4b_2 + 3b_3$$
$$b_1 + \overline{x} \geq 1$$
$$b_2 + y + \overline{z} \geq 1$$
$$b_3 + \overline{y} + z \geq 1$$
$$x + y + z \geq 1$$
$$x + \overline{y} + \overline{z} \geq 1$$

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance

$$(\overline{x})_5$$
$$(y \vee \overline{z})_4$$
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$$b_2 + y + \overline{z} \geq 1$$
$$b_3 + \overline{y} + z \geq 1$$
$$x + y + z \geq 1$$
$$x + \overline{y} + \overline{z} \geq 1$$

So-called blocking variable transformation
Variables b_i are blocking or relaxation variables

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance	**PBO instance**
	$\min \; 5b_1 + 4b_2 + 3b_3$
$(\overline{x})_5$	$b_1 + \overline{x} \geq 1$
$(y \vee \overline{z})_4$	$b_2 + y + \overline{z} \geq 1$
$(\overline{y} \vee z)_3$	$b_3 + \overline{y} + z \geq 1$
$(x \vee y \vee z)_\infty$	$x + y + z \geq 1$
$(x \vee \overline{y} \vee \overline{z})_\infty$	$x + \overline{y} + \overline{z} \geq 1$

So-called blocking variable transformation
Variables b_i are blocking or relaxation variables

Optimal solution $\rho = \{x = 0, y = 1, z = 0\}$ with penalty 3

From Pseudo-Boolean Optimization to MaxSAT/WBO

"MaxSAT instance" but with PB constraints:
Weighted Boolean Optimization [MMP09]

From Pseudo-Boolean Optimization to MaxSAT/WBO

"MaxSAT instance" but with PB constraints:
Weighted Boolean Optimization [MMP09]

PBO instance

$$\begin{aligned} \min\ & \textstyle\sum_{i=1}^{n} w_i \ell_i \\ & C_1 \\ & C_2 \\ & \vdots \\ & C_M \end{aligned}$$

From Pseudo-Boolean Optimization to MaxSAT/WBO

"MaxSAT instance" but with PB constraints:
Weighted Boolean Optimization [MMP09]

PBO instance

$$\min \sum_{i=1}^{n} w_i \ell_i$$
$$C_1$$
$$C_2$$
$$\vdots$$
$$C_M$$

MaxSAT/WBO instance

$$(\bar{\ell}_1)_{w_1}$$
$$\vdots$$
$$(\bar{\ell}_n)_{w_n}$$
$$(C_1)_\infty$$
$$\vdots$$
$$(C_M)_\infty$$

Main Approaches for MaxSAT Solving (and PBO)

1. Linear search SAT-UNSAT (LSU) (or model-improving search)
2. Core-guided search
3. Implicit hitting set (IHS) algorithm

Main Approaches for MaxSAT Solving (and PBO)

1. Linear search SAT-UNSAT (LSU) (or model-improving search)
2. Core-guided search
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Will describe all of these algorithms as trying to

- minimize $\sum_{i=1}^{n} w_i \ell_i$
- subject to collection of PB constraints $F = C_1 \land \cdots \land C_m$ (possibly clausal)

Linear Search SAT-UNSAT (LSU) Algorithm

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Set $\rho_{\text{best}} = \emptyset$ and repeat the following:

1. Run SAT/PB solver
2. If solver returns `UNSATISFIABLE`, output ρ_{best} and terminate
3. Otherwise, let $\rho_{\text{best}} :=$ returned solution ρ
4. Add constraint $\sum_{i=1}^{n} w_i \ell_i \leq -1 + \sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$
5. Start over from the top

Linear Search Toy Example

1. Given PB formula F and objective function
$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$

Linear Search Toy Example

1. Given PB formula F and objective function $\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$
2. Solver run on F returns $\rho_1 = \{x_1 = x_2 = x_3 = x_6 = 0; x_4 = x_5 = 1\}$

Linear Search Toy Example

1. Given PB formula F and objective function $\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$
2. Solver run on F returns $\rho_1 = \{x_1 = x_2 = x_3 = x_6 = 0; x_4 = x_5 = 1\}$
3. Yields objective value $0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 0 = 9$, so add

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$

Linear Search Toy Example

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$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$

4. Solver run on F plus this new constraint returns $\rho_2 = \{x_1 = x_3 = x_5 = x_6 = 0; x_2 = x_4 = 1\}$

Linear Search Toy Example

1. Given PB formula F and objective function $\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$
2. Solver run on F returns $\rho_1 = \{x_1 = x_2 = x_3 = x_6 = 0; x_4 = x_5 = 1\}$
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$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$

4. Solver run on F plus this new constraint returns $\rho_2 = \{x_1 = x_3 = x_5 = x_6 = 0; x_2 = x_4 = 1\}$
5. Yields objective value 6, so add

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 5$$

Linear Search Toy Example

1. Given PB formula F and objective function $\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$
2. Solver run on F returns $\rho_1 = \{x_1 = x_2 = x_3 = x_6 = 0; x_4 = x_5 = 1\}$
3. Yields objective value $0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 0 = 9$, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$
4. Solver run on F plus this new constraint returns $\rho_2 = \{x_1 = x_3 = x_5 = x_6 = 0; x_2 = x_4 = 1\}$
5. Yields objective value 6, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 5$$
6. Now solver returns `UNSATISFIABLE`

Linear Search Toy Example

1. Given PB formula F and objective function $\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$
2. Solver run on F returns $\rho_1 = \{x_1 = x_2 = x_3 = x_6 = 0; x_4 = x_5 = 1\}$
3. Yields objective value $0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 0 = 9$, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$
4. Solver run on F plus this new constraint returns $\rho_2 = \{x_1 = x_3 = x_5 = x_6 = 0; x_2 = x_4 = 1\}$
5. Yields objective value 6, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 5$$
6. Now solver returns `UNSATISFIABLE`
7. Hence, minimum value of objective function subject to F is 6

Linear vs. Binary Search?

What if we run binary search instead of linear search?

Conventional wisdom appears to be that linear search is better

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Two possible explanations:

1. In theory, objective value could decrease by just 1 every time — in practice, tend to get much larger jumps
2. Potentially very different cost for
    - SAT calls (feasible instances where solver will find solution)
    - UNSAT calls (where solver determines no solution exists)

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2. Potentially very different cost for
   - SAT calls (feasible instances where solver will find solution)
   - UNSAT calls (where solver determines no solution exists)

Properties of linear search SAT-UNSAT:

- Can get **some decent** solution quickly, even if not optimal one
- Important for anytime solving (when time is limited and something is better than nothing)
- But get no estimate of how good the solution is

Core-Guided Pseudo-Boolean Search

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Core-guided PB search: assume optimistically that objective can reach best imaginable value; derive contradiction if not possible

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Set $val_{\text{best}} = 0$ and repeat the following:

1. Run pseudo-Boolean solver with assumptions (pre-made decisions) $\ell_i = 0$ for all ℓ_i in objective function

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2. If solver returns `SATISFIABLE`, output val_{best} and terminate
3. Otherwise learn constraint $\sum_{i=1}^{k} \ell_i \geq A$ over assumption variables
4. Introduce new variables $z_j \Leftrightarrow \sum_{i=1}^{k} \ell_i \geq j$
5. Update objective function and val_{best} using $\sum_{i=1}^{k} \ell_i = A + \sum_{j=A+1}^{k} z_j$

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- Minimize $\sum_{i=1}^{n} w_i \ell_i$
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6. Start over from top with updated objective function

Core-Guided Pseudo-Boolean Search

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
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4. Introduce new variables $z_j \Leftrightarrow \sum_{i=1}^{k} \ell_i \geq j$
5. Update objective function and val_{best} using $\sum_{i=1}^{k} \ell_i = A + \sum_{j=A+1}^{k} z_j$
6. Start over from top with updated objective function
7. Let's explain by example *[done more slowly in pre-recorded Part III]*

Core-Guided Search Toy Example (1/3)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

Core-Guided Search Toy Example (1/3)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

2. Run solver on F with assumptions $x_i = 0$, $i \in [6]$

Core-Guided Search Toy Example (1/3)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

2. Run solver on F with assumptions $x_i = 0,\ i \in [6]$
3. Suppose solver returns PB core constraint

$$3x_2 + 2x_3 + x_4 + x_5 \geq 4 \tag{2}$$

Core-Guided Search Toy Example (1/3)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

2. Run solver on F with assumptions $x_i = 0,\ i \in [6]$
3. Suppose solver returns PB core constraint

$$3x_2 + 2x_3 + x_4 + x_5 \geq 4 \tag{2}$$

4. Round to nicer-to-work-with cardinality core constraint

$$x_2 + x_3 + x_4 + x_5 \geq 2 \tag{3}$$

Core-Guided Search Toy Example (1/3)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

2. Run solver on F with assumptions $x_i = 0$, $i \in [6]$
3. Suppose solver returns PB core constraint

$$3x_2 + 2x_3 + x_4 + x_5 \geq 4 \tag{2}$$

4. Round to nicer-to-work-with cardinality core constraint

$$x_2 + x_3 + x_4 + x_5 \geq 2 \tag{3}$$

5. Introduce new, fresh variables y_3 and y_4 and constraints

$$x_2 + x_3 + x_4 + x_5 = 2 + y_3 + y_4 \tag{4a}$$

$$y_3 \geq y_4 \tag{4b}$$

to enforce that y_j means "$x_2 + x_3 + x_4 + x_5 \geq j$"

Core-Guided Search Toy Example (2/3)

6. Multiply (4a) by 2 and add to (1) to cancel x_2 and get updated, equivalent objective function

$$x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \tag{5}$$

and update $val_{\text{best}} = 4$

Core-Guided Search Toy Example (2/3)

6. Multiply (4a) by 2 and add to (1) to cancel x_2 and get updated, equivalent objective function

$$x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \tag{5}$$

   and update $val_{\text{best}} = 4$

7. Run solver on F assuming all literals in (5) being 0

Core-Guided Search Toy Example (2/3)

6. Multiply (4a) by 2 and add to (1) to cancel x_2 and get updated, equivalent objective function

$$x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \tag{5}$$

   and update $val_{\text{best}} = 4$

7. Run solver on F assuming all literals in (5) being 0
8. Suppose solver returns the clausal core constraint

$$x_4 + x_5 + x_6 + y_3 \geq 1 \tag{6}$$

Core-Guided Search Toy Example (2/3)

6. Multiply (4a) by 2 and add to (1) to cancel x_2 and get updated, equivalent objective function

$$x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \tag{5}$$

   and update $val_{\text{best}} = 4$

7. Run solver on F assuming all literals in (5) being 0
8. Suppose solver returns the clausal core constraint

$$x_4 + x_5 + x_6 + y_3 \geq 1 \tag{6}$$

9. Introduce new variables z_2, z_3, z_4 and the constraints

$$x_4 + x_5 + x_6 + y_3 = 1 + z_2 + z_3 + z_4 \tag{7a}$$

$$z_2 \geq z_3 \tag{7b}$$

$$z_3 \geq z_4 \tag{7c}$$

   to enforce that z_j means "$x_4 + x_5 + x_6 + y_3 \geq j$"

Core-Guided Search Toy Example (3/3)

10. Multiply (7a) by 2 and add to (5) to get 3rd equivalent objective

$$x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

and update $val_{\text{best}} = 6$

Core-Guided Search Toy Example (3/3)

10. Multiply (7a) by 2 and add to (5) to get 3rd equivalent objective

$$x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

and update $val_{\text{best}} = 6$

11. For 3rd time run solver on F, assuming all literals in (8) being 0

Core-Guided Search Toy Example (3/3)

10. Multiply (7a) by 2 and add to (5) to get 3rd equivalent objective

$$x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

and update $val_{\text{best}} = 6$

11. For 3rd time run solver on F, assuming all literals in (8) being 0
12. Suppose solver reports it is possible to achieve

$$\rho = \{x_1 = x_3 = x_5 = x_6 = y_4 = z_2 = z_3 = z_4 = 0\} \tag{9}$$

Core-Guided Search Toy Example (3/3)

10. Multiply (7a) by 2 and add to (5) to get 3rd equivalent objective

$$x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

and update $val_{\text{best}} = 6$

11. For 3rd time run solver on F, assuming all literals in (8) being 0
12. Suppose solver reports it is possible to achieve

$$\rho = \{x_1 = x_3 = x_5 = x_6 = y_4 = z_2 = z_3 = z_4 = 0\} \tag{9}$$

13. Hence, have recovered optimal solution 6
(as before for linear search)

Properties of (Pure) Core-Guided Search

- Can get decent lower bounds on solution quickly
- Helps to cut off parts of search space that are "too good to be true"
- But find no actual solution until the final, optimal one
- Also, no estimate of how good the lower bound is
- Linear search much better at finding solutions — how to get the best of both worlds?

Improvements of Core-Guided Search (1/2)

Weight stratification [ABGL12]

Set only literals with largest weight in objective to $0 \Rightarrow$

1. More compact core; or
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Hybrid/interleaving search [ADMR15]

Switch back and forth repeatedly between core-guided and linear search — cumbersome in CNF-based solver, but fairly cheap (and efficient) in native pseudo-Boolean solver [DGD$^+$21]

Improvements of Core-Guided Search (2/2)

Core minimization
In CDCL-based solver, try to get smaller core clauses. For PB solver, not so clear how to do this (constraint minimization also interesting problem in general for PB conflict analysis)

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Lazy variables [MJML14, DGD+21]
For real-world instances, rewriting of objective function can introduce huge numbers of new variables, slowing down the solver — so don't introduce all variables in one go but only lazily as needed

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Inference strength of core-guided search?

- Extension variables very strong in theory, but hard to use in practice
- Core-guided search provides principled way of introducing them
- Can we characterize the power of this method?

Evaluation of Core-Guided PB Solver in [DGD$^+$21]

RoundingSat variants with core-guided (CG) and linear search (LSU)

#instances solved to optimality; highlighting **1st**, **2nd**, and **3rd** best

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	PB16opt (1600)	MIPopt (291)	KNAP (783)	CRAFT (985)
Hybrid (interleave CG & LSU)	**968**	**78**	306	**639**
HybridCl (w/ clausal cores)	937	75	298	**618**
HybridNL (w/ non-lazy variables)	936	70	186	607
HybridClNL (w/ both)	917	67	203	612
RoundingSat (only LSU)	853	75	341	309
CoreGuided (only CG)	911	61	43	595
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Sat4j	773	61	**373**	105
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Significant improvement over PB state of the art, but MIP still better

Implicit Hitting Set (IHS) Algorithm (1/2)

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$ (consider clausal constraints)

As in core-guided search, use solving with assumptions, but maintain collection $\mathcal{K}$ of learned core clauses

$$\begin{aligned}
C_1 &\doteq \ell_{1,1} \vee \ell_{1,2} \vee \cdots \vee \ell_{1,k_s} \\
C_2 &\doteq \ell_{2,1} \vee \ell_{2,2} \vee \cdots \vee \ell_{2,k_s} \\
&\vdots \\
C_s &\doteq \ell_{s,1} \vee \ell_{s,2} \vee \cdots \vee \ell_{s,k_s}
\end{aligned}$$

lower-bounding objective function

Implicit Hitting Set (IHS) Algorithm (2/2)

Set $\mathcal{K} = \emptyset$ and repeat the following:

1. Compute minimum hitting set for $\mathcal{K}$, i.e., $H = \{\ell_i\}$ s.t.
   - $H \cap C \neq \emptyset$ for all $C \in \mathcal{K}$ (H is hitting set)
   - $\sum_{\ell_i \in H} w_i$ minimal among H with this property.
2. Run the solver with assumptions $\{\ell_i = 1 \mid \ell_i \in H\} \cup \{\ell_j = 0 \mid \ell_j \notin H\}$
3. If solver found solution, it must be optimal (since hitting set is optimal), so return solution with value $\sum_{\ell_i \in H} w_i$
4. Otherwise, solver returns new core C_{s+1} — add it to $\mathcal{K}$ and start over from top

Implicit Hitting Set vs. Core-Guided

- IHS and core-guided approaches for MaxSAT seem orthogonal [Bac21]
- For MaxSAT problems with many interchangeable soft clauses core-guided seems better (i.e., when it is not important exactly which of these clauses end up in core)
- For MaxSAT problems with many distinct weights, IHS seems better

Relation between IHS and core-guided search?

Provide a more precise theoretical comparison of IHS and core-guided search with simulations and/or separations

(Some theoretical work on related problems in, e.g., [FMSV20, MIB$^+$19])

Some More Open Questions

Combine IHS and core-guided search in MaxSAT solving?

Recent work on this in [BBP20]

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Combine IHS and core-guided search in MaxSAT solving?

Recent work on this in [BBP20]

Combine IHS with pseudo-Boolean optimization?

- In PB setting, cores will not be subsets of clauses but PB constraints $C_1, \ldots, C_s$ over objective function literals
- Hitting set H is partial assignment guaranteed to satisfy all constraints $C_1, \ldots, C_s$
- Want to find minimum-cost set H of literals (w.r.t. objective function) with this property
- Not implemented in native PB solvers (to best of my knowledge)

Organization of This Tutorial

Part I: Pseudo-Boolean Preliminaries

Part II: Pseudo-Boolean Solving

Part III: Pseudo-Boolean Optimization

Part IV: Mixed Integer Linear Programming

Mixed Integer Linear Programming

Mixed integer linear program

- Minimize $\sum_j a_j x_j$
- Subject to $\sum_j a_{i,j} x_j \leq A_i$, $i = 1, \ldots, m$
- $x_j \in \mathbb{N}$ for $j = 1, \ldots, n$
- $x_j \in \mathbb{R}_{\geq 0}$ for $j = n+1, \ldots, N$

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- Real-valued variables
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- Linear constraints
- Integer-valued variables
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- No real-valued variables: integer linear program (ILP)
- $0 \leq x_j \leq 1$ for all j: 0-1 ILP
- Vacuous objective $\sum_j 0 \cdot x_j$: decision problem
- But MIP best for optimization

Two Differences Compared to SAT/PB

Academia vs. industry

- Best solvers are commercial and closed-source
- E.g., CPLEX [CPL], GUROBI [Gur], and XPRESS [Xpr]
- Academic solvers like SCIP [SCI] are excellent but not as good

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Search vs. backtracking

- SAT/PB: Fast decisions; careful, slow(er) conflict analysis
- MIP: Lots of time & effort on decisions; backtracking not so advanced

MIP Solving at a High Level

1. Preprocessing (called presolving)
2. Linear programming + branch-and-bound
3. Add cutting planes ruling out infeasible LP-solutions (branch-and-cut method going back to [Gom58])
4. Heuristics for quickly finding good feasible solutions

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Let's discuss items 2 and 3 very briefly
[Please refer to pre-recorded Part IV of tutorial for more details]

Linear Programming Relaxation

Linear Programming Relaxation (LPR)

- Minimize $\sum_j a_j x_j$
- Subject to $\sum_j a_{i,j} x_j \leq A_i$, $i = 1, \ldots, m$
- ~~$x_j \in \mathbb{N}$ for $j = 1, \ldots, n$~~ $x_j \in \mathbb{R}_{\geq 0}$ for $j = 1, \ldots, n$
- $x_j \in \mathbb{R}_{\geq 0}$ for $j = n+1, \ldots, N$

- Fast to solve (just linear programming)
- LP solution x^* yields lower bound
- Or, if x^* "accidentally" feasible, have optimal solution
- Use simplex algorithm — will have many LP calls for same problem with different variable bounds; need efficient hot restarts

LP-Based Branch-and-Bound

Branch-and-bound

Choose integer-valued x_j and $B \in \mathbb{N}$

- Solve MIP plus constraint $x_j \geq B$
- Solve MIP plus constraint $x_j \leq B - 1$

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Creates (growing) branch-and-bound tree of subproblems
Prune subproblem/node when

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Branch on

- Variables
- General linear constraints (powerful but difficult)
  Corresponds to stabbing planes proof system [BFI+18]

Branch-and-Cut

General cutting plane method

1. Solve LP relaxation
2. If solution x^* feasible for MIP $\Rightarrow$ found optimum
3. Otherwise generate and add constraint $\sum_j b_j x_j \leq B$ that is
   - valid for MIP
   - violated by LP solution x^*
4. Repeat from the top

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PB solving rules division and saturation are examples of cut rules

Branch-and-cut

- Run branch-and-bound
- But in each subproblem, use cutting plane method to repeatedly
    - solve LP relaxation
    - add cut

Example Cut: Mixed Integer Rounding (MIR)

Mixed integer rounding (MIR) cut [MW01] applied to (normalized) pseudo-Boolean constraint

$$\sum_i a_i \ell_i \geq A$$

with divisor $d \in \mathbb{N}^+$ produces constraint

$$\sum_i \Big(\min(a_i \bmod d, A \bmod d) + \left\lfloor \frac{a_i}{d} \right\rfloor (A \bmod d) \Big) \ell_i \geq \left\lceil \frac{A}{d} \right\rceil (A \bmod d)$$

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Concretely, MIR cut with divisor 3 applied on

$$x + 2y + 3z + 4w + 5u \geq 5$$

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For comparison, standard division by 3 and multiplication by 2 produces

$$2x + 2y + 2z + 4w + 4u \geq 4$$

Numerics and Correctness

Numerics

- Use floating point for efficiency reasons
- Can lead to rounding errors
- Exact MIP solvers like [CKSW13]
  - are significantly slower
  - don't support the full range of state-of-the-art techniques

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Proof logging / certification

- Currently not available for state-of-the-art solvers
- Though known that even best commercial solvers sometimes give wrong results
- Some work on proof logging in [CGS17] — challenges:
  - How to capture wide diversity of techniques?
  - What is a convenient format?
  - How to generate proofs efficiently on-the-fly?

Some Interesting MIP Questions

1. Develop better heuristics to branch on general linear constraints (cf. stabbing planes [BFI+18])
2. Design stronger conflict analysis operating directly on linear constraints (borrow ideas from native pseudo-Boolean solvers?)
3. Provide rigorous understanding of MIP solver performance
4. Develop families of theory benchmarks and computational complexity results for them (cf. SAT solving and proof complexity [BN21])
5. Steal best MIP ideas and use for pseudo-Boolean solving?!

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   [next and final topic]

Combining PB Solving and Mixed Integer Programming

Pseudo-Boolean solvers

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Why not merge the two to get the best of both worlds of SAT-style conflict-driven search and MIP-style branch-and-cut?

Balance Time Allocation for PB and LP Solving?

High-level idea: Give pseudo-Boolean solver access to LP solver

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Need to carefully balance time allocation for PB solver and LP solver

Backtracking from LP Infeasibility?

What to do if LP call shows LP relaxation infeasible under current trail?

- Obviously, PB solver should backtrack
- But can only do conflict analysis on violated PB constraint
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More subtle issue:

- Efficient LP solvers use inexact floating-point arithmetic
- How to incorporate into Boolean solver that must maintain perfectly sound reasoning?

Sharing of Cut Constraints?

Cut constraints from LP solver

- When LP relaxation feasible, MIP solver generates cut constraint to remove the found LP solution
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Cut constraints from PB solver

- PB solvers learns new constraints at high rate from conflict analysis
- These learned constraints can also be viewed as cuts
- Should such constraints be passed from PB solver to LP solver?

Report on Attempted PB-LP Integration [DGN21]

1. Interleave incremental LP solving within conflict-driven PB search
   - Limit LP solver time by enforcing total #LP pivots $\leq$ #PB conflicts
   - Only run LP solver when this condition holds
   - Abort if $> P$ pivots in single LP call; but if so also double limit P to avoid wasted LP calls in future

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2. When LP solver detects that LP relaxation infeasible
   - Farkas' lemma $\Rightarrow$ linear combination of constraints violated by trail
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4. Also explore letting PB solver pass learned constraints to LP solver

Relation to MIP Solvers with Conflict Analysis?

MIP solvers also combine constraint propagation and SAT-style clause learning with LP solving

- Implemented in SCIP [ABKW08]
- And also in closed-source solvers (see [AW13])

Important to understand similarities and differences — let's give high-level description of PB search and conflict analysis phrased in MIP language

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Pseudo-Boolean search

1. Make decision to assign free variable to 0 or 1
2. Propagate all assignments implied by some linear constraint until saturation
3. If no contradiction, go to step 1
4. Otherwise some constraint C violated $\Rightarrow$ trigger conflict analysis

PB Conflict Analysis “in MIP Language”

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6. Backjump by undoing further assignments in reverse chronological order until D is no longer violated
7. Switch back to search phase

Comparison to MIP Propagation and Conflict Analysis

Propagation in SCIP

- Fast, simple propagation in PB solvers
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Arithmetic

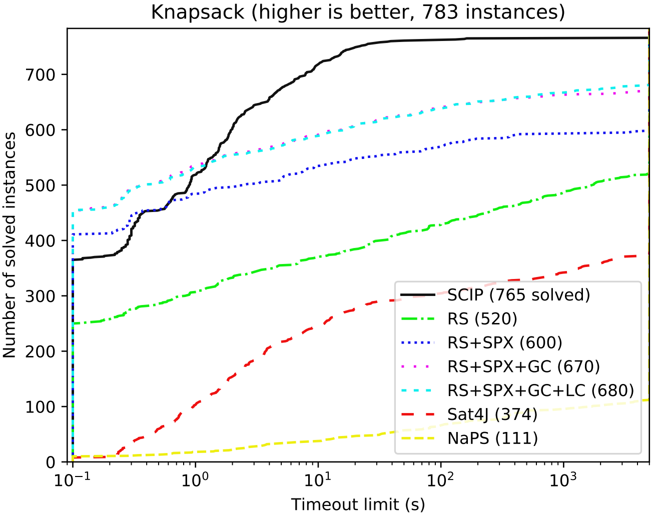
- SCIP uses floating point
- Reasoning steps in PB solver computed with exact integer arithmetic
- No issues with possible rounding errors

Experimental Results for Knapsack Benchmarks [Pis05]

ROUNDINGSAT (RS) enhanced with

- LP solver SOPLEX (SPX) (from SCIP)
- Gomory cuts (GC)
- shared learned PB cuts (LC)

compared to other solvers



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	SCIP	RS	+SPX	+GC	+LC	Sat4j	NaPS
PB16dec (1783)	1123	**1472**	**1453**	**1452**	1451	1432	1400
PB16opt (1600)	**1057**	862	**988**	986	**993**	776	896
MIPdec (556)	**264**	203	**263**	**261**	259	169	170
MIPopt (291)	**125**	78	101	**102**	**102**	62	65

Performance of Integrated PB-LP Solver

1. Best of both worlds?
   - At least well-rounded performance
   - Hybrid PB-LP solver always competitive with best solver
   - Pretty dramatic improvements for optimization problems compared to pseudo-Boolean state of the art
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3. Sharing Gomory cuts and learned cuts not so helpful
   - Except for knapsack benchmarks, where they help a lot
   - And maybe we could/should fine-tune how sharing is done?

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Estimate usefulness of different types of constraints

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Constraints learned after Farkas-based conflicts

- Less useful than regular learned constraints
- But big spread in usage measurements

Future Research Directions for PB-LP Integration (1/2)

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5. Use MIR cuts and/or other MIP cut rules to improve pseudo-Boolean conflict analysis

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8. Export pseudo-Boolean conflict analysis to MIP
9. Use hybrid PB-LP solver to solve 0-1 MIP problems
   - PB solver decides on Boolean variables and propagates
   - LP solver takes care of real-valued variables

Summing up

- Pseudo-Boolean optimization powerful and expressive framework
- Can be attacked with methods from
  - SAT solving and MaxSAT solving
  - "Native" cutting-planes-based pseudo-Boolean reasoning
  - Mixed integer linear programming
- Approaches with complementary strengths — room for synergies?
- Some highly nontrivial challenges regarding
  - Algorithm design
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  - Theoretical understanding
- But maybe also quite a bit of low-hanging fruit?
- And in any case lots of fun questions to work on! ☺

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Thank you for your attention!

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