On the Semantics of Local Characterizations for Linear-Invariant Properties

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Joint work with Arnab Bhattacharyya, Elena Grigorescu, and Ning Xie

Property Testing

Given (huge) object

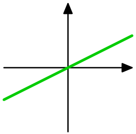
Want to know whether it has certain property or not

No time to read all of input, but can make random access queries

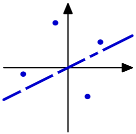
Distinguish:

- object has property
- object is far from having property

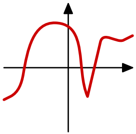
Example: Decide whether given function linear



YES



DON'T CARE



NO

A Little Bit More Formally

distance$(f, g) = \Pr_{\mathbf{x} \sim D}\big[f(\mathbf{x}) \neq g(\mathbf{x})\big]$
distance$(f, \mathcal{P}) = \min_{g \in \mathcal{P}}\{\text{distance}(f, g)\}$

Property tester $T(\delta, \epsilon_1, \epsilon_2, q)$, $\epsilon_1 < \epsilon_2$

- Makes q queries to input f
- If $f \in \mathcal{P}$, accepts with probability at least $1 - \epsilon_1$
- If f δ-far from property, rejects with probability at least ϵ_2

A tester has

- one-sided error if $\epsilon_1 = 0$
- constant query complexity if q independent of input size

This talk: "testable" = with one-sided error and constant query complexity

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A Short (Non-)History

- Initiated in [Babai-Fortnow-Lund '91] and [Blum-Luby-Rubinfeld '93]
- Formalized in [Rubinfeld-Sudan '96] and [Goldreich-Goldwasser-Ron '98]
- Rich literature on testing of
  - graphs (bipartiteness, k-colourability, ...),
  - algebraic functions (linearity, low-degree polynomials, ...),
  - distributions (statistical distance, entropy, ...),
  - other properties
- [Ron '08], [Ron '09], and [Sudan '10] nice surveys

What Makes a Property Testable?

- Many ingenious result, but somewhat ad hoc solutions
- Would like to find underlying explanation what makes a property testable
- Well understood for (dense) graphs [Alon-Fischer-Newman-Shapira '06]
- Less so for algebraic functions [Kaufman-Sudan '08]
- Starting point for this work

Invariances and Constraints

One-sided tester must see violation of local constraint

- bipartiteness: small non-bipartite subgraph
- linearity: $\mathbf{x}$ and $\mathbf{y}$ s.t. $f(\mathbf{x}) + f(\mathbf{y}) \neq f(\mathbf{x} + \mathbf{y})$

Properties have invariances

- graph properties the same under relabelling of vertices
- linear functions remain linear if composed with linear transformation of domain

Many algebraic properties are linear-invariant — interesting class to study

Linear-Invariant Properties

Linear invariance

Property $\mathcal{P}$ is linear-invariant if for all linear maps $L : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ it holds that $f \in \mathcal{P} \Rightarrow f \circ L \in \mathcal{P}$

Two questions:

1. Which linear-invariant properties are testable?
2. What are these properties?

Described syntactically by local constraints, but syntactically distinct properties can collapse into semantically identical property!

Recent testability results essentially ignore this issue

This work: initiate systematic study of the semantics of linear-invariant properties

Our Results in (Very) Brief

- Develop techniques for determining whether two syntactically distinct specifications encode semantically distinct properties
- Show for fairly broad class of properties that techniques provide necessary and sufficient conditions
- Corollary: recent testability results indeed provide infinite number of new, testable properties

Outline

Some Notation

- Study functions $f : \mathsf{D} \to \mathsf{R}$ from domain D to range R
- Domain vector space for linear invariance to make sense
- In this talk usually $\mathsf{D} = \mathbb{F}_2^n$ (but other base fields possible)
- Focus on range $\mathsf{R} = \{0, 1\}$ (but again other choices possible)
- L always linear transformation
- $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ unit vectors in ambient space

Testing Linear-Invariant Properties

Consider tester T for linear-invariant property $\mathcal{P}$ that randomly queries f at $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ to decide whether $f \in \mathcal{P}$ or not

- Constant query complexity $\Rightarrow$ w.l.o.g. non-adaptive
- One-sided error $\Rightarrow$ must see local violation
- $\mathcal{P}$ linear-invariant and "nontrivial" $\Rightarrow$ vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ linearly dependent
- Should make same decision for f and $f \circ L$ for any $L \Rightarrow$ linear dependencies only thing that matters
- So can get query points by encoding linear dependencies as fixed vectors in $\mathbb{F}^r$, $r \leq k$, and applying random $L : \mathbb{F}^r \to \mathbb{F}^n$

What a Tester Must Do (Intuitively)

Summing up, it seems that what a tester has to do is:

1. Fix linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{F}^r$, $r \leq k$,
2. Apply random $L : \mathbb{F}^r \rightarrow \mathbb{F}^n$ to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$
3. Reject f if pattern $\langle f(L(\mathbf{v}_1)), f(L(\mathbf{v}_2)), \ldots, f(L(\mathbf{v}_k)) \rangle$ in set of "forbidden patterns" $S \subseteq \mathrm{R}^k$; accept otherwise

A Syntactic Specification of Linear-Invariant Properties

Hence, natural to describe linear-invariant properties in terms of matroid freeness

(Linear) matroid M: bunch of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{F}^r$ for $r \leq k$

Matroid freeness property

A function $f : \mathbb{F}^n \rightarrow$ R is (M, S)-free if for all $L : \mathbb{F}^r \rightarrow \mathbb{F}^n$ pattern $\langle f(L(\mathbf{v}_1)), \ldots, f(L(\mathbf{v}_k)) \rangle$ is not in $S \subseteq$ R^k

Any linear-invariant property testable with one-sided error* can be expressed as intersection of matroid freeness properties [Bhattacharyya-Grigorescu-Shapira '10]

(*) Modulo technical assumption that tester doesn't depend in any essential way on dimension n

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Examples of Matroid Freeness Properties

1. Linearity
   $M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$
   $S = \{001, 111\}$

2. Subspace
   $M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$
   $S = \{110\}$

3. Triangle freeness
   $M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$
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4. Degree-d polynomial (with zero constant term)
   $M = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d+1]\}$
   $S = \left\{\sigma \in \{0,1\}^{2^{d+1}-1} \,\middle|\, \text{parity of } \sigma \text{ odd}\right\}$

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Full Linear Matroid

Full linear matroid of dimension d

$F_d = \{\sum_{i\in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d]\}$

Any matroid freeness property intersection of F_d-freeness properties (forbid all labels $r \in \mathsf{R}$ for vectors we don't care about)

Also any (F_d, S)-freeness property intersection of properties forbidding each $\sigma \in S$

So understanding (F_d, σ)-freeness properties for a single pattern σ would be great!

Partial Linear Matroid

Seems a bit too hard for the moment...

So consider instead

Partial matroid of weight w

$F_d^{\leq w} = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d], |I| \leq w\}$

Understanding $(F_d^{\leq w}, \sigma)$-freeness properties also hard, but here we can do something

And already $w = 2$ interesting!

A Canonical Matroid Freeness Tester

As we discussed, the tester for (M, σ)-freeness seems obvious:

1. Consider the matroid vectors $M = \{\mathbf{v_1}, \ldots, \mathbf{v}_k\} \subseteq \mathbb{F}^r$
2. Apply random $L : \mathbb{F}^r \to \mathbb{F}^n$ to get $\{L(\mathbf{v_1}), \ldots, L(\mathbf{v}_k)\} \subseteq \mathbb{F}^n$
3. Reject f if $\langle f(L(\mathbf{v_1})), \ldots, f(L(\mathbf{v}_k))\rangle = \sigma$; accept otherwise

Clearly this test never gives false negatives (by definition)

But will it detect with high probability that f is far from (M, σ)-free?

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Testability Results for Matroid Freeness Properties (1/2)

- [Green '05]:
  $(F_2, 111)$-freeness testable
- [Bhattacharyya-Chen-Sudan-Xie '09]:
  $(F_d^{\leq 2}, 1^*)$-freeness testable
- [Kráľ-Serra-Vena '09], [Shapira '09]:
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Also true if $\sigma = 0^*$ (by symmetry)

All these properties are monotone / anti-monotone

Not too hard to show that they cannot all be the same [Bhattacharyya-Chen-Sudan-Xie '09]

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Testability Results for Matroid Freeness Properties (2/2)

If $\sigma \notin \{0^*, 1^*\}$, then property is (potentially) non-monotone

- [Bhattacharyya-Chen-Sudan-Xie '09]:
  $(\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \sum_{i=1}^{k} \mathbf{e}_i\}, \sigma)$-freeness testable
- [Bhattacharyya-Grigorescu-Shapira '10]
  $(F_d^{\leq 2}, \sigma)$-freeness testable

But what are these properties?

Understanding Matroid Freeness Properties

Lemma (Bhattacharyya-Chen-Sudan-Xie '09)

If $k > 2$ and σ has even number of 0's and 1's, then $(\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \sum_{i=1}^{k} \mathbf{e}_i\}, \sigma)$-free functions = constant functions.

Proof: Suppose $\exists\ \mathbf{x}, \mathbf{y}$ s.t. $f(\mathbf{x}) = 0$, $f(\mathbf{y}) = 1$. Let L send 0-labelled vectors to $\mathbf{x}$ and 1-labelled vectors to $\mathbf{y}$. Then when evaluating f on $L(M)$ we see σ.

Lemma (Bhattacharyya-Chen-Sudan-Xie '09)

If $k \geq 2$ and σ has one 0 and even number of 1's, then $(\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \sum_{i=1}^{k} \mathbf{e}_i\}, \sigma)$-free functions = subspace functions.

Proof: For any $\mathbf{x}, \mathbf{y}$ s.t. $f(\mathbf{x}) = f(\mathbf{y}) = 1$, let L send all 1-labelled vectors except one to $\mathbf{x}$, the last 1-labelled vector to $\mathbf{y}$, and the only 0-labelled vector to $\mathbf{x} + \mathbf{y}$. If we never see σ when evaluating, f is subspace indicator function.

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A Property Collapse

In fact, [Bhattacharyya-Chen-Sudan-Xie '09] show:

All $(\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \sum_{i=1}^{k} \mathbf{e}_i\}, \sigma)$-freeness properties collapse into one of 9 properties, all previously known testable!

What about properties in [Bhattacharyya-Grigorescu-Shapira '10]?

Unclear...

Need to understand what matroid freeness properties mean!

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Some More Notation and Terminology

Matroid $M = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{F}^r$ for $r \leq k$
Forbidden pattern $\sigma = \langle \sigma_1, \ldots, \sigma_k \rangle \in \mathsf{R}^k$

Say $f : \mathbb{F}^n \to \mathsf{R}$ contains (M, σ) at L if
$\big\langle f(L(\mathbf{v}_1)), f(L(\mathbf{v}_2)), \ldots, f(L(\mathbf{v}_k)) \big\rangle = \sigma$

Other matroid $N = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\} \subseteq \mathbb{F}^s$ for $s \leq \ell$
Forbidden pattern $\tau = \langle \tau_1, \ldots, \tau_\ell \rangle \in \mathsf{R}^\ell$

Refer to (M, σ) and (N, τ) as labelled matroids with

- vector $\mathbf{v}_i$ labelled by σ_i
- vector $\mathbf{w}_j$ labelled by τ_j

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How to Relate the Structure of Two Matroids?

Matroid homomorphism $\phi : M \to N$

- linear map from $\mathbb{F}^r$ to $\mathbb{F}^s$
- sends every $\mathbf{v}_i \in M$ to some $\mathbf{w}_j \in N$

Labelled matroid homomorphism from (M, σ) to (N, τ)

- homomorphism
- label-preserving, i.e., if $\mathbf{w}_j = \phi(\mathbf{v}_i)$ then $\tau_j = \sigma_i$

Say (M, σ) embeds into (N, τ); denoted $(M, \sigma) \hookrightarrow (N, \tau)$

An Easy Observation

Homomorphisms imply property containment

Observation

If $(M, \sigma) \hookrightarrow (N, \tau)$*, then* (M, σ)*-freeness* $\subseteq (N, \tau)$*-freeness.*

Proof: If $\phi : M \to N$ is a homomorphism and f contains (N, τ) at a linear transformation L, then f contains (M, σ) at $L \circ \phi$.

What about the other direction?

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What about the other direction?

Dichotomy Theorem for Monotone Properties

Labelled homomorphisms completely determine relations between monotone matroid freeness properties

Theorem

Let M and N be any matroids. Then one of two cases holds:

1. *If $(M, 1^*) \hookrightarrow (N, 1^*)$, then $(M, 1^*)$-freeness is contained in $(N, 1^*)$-freeness.*
2. *Otherwise, $(M, 1^*)$-freeness is* **far** *from being contained in $(N, 1^*)$-freeness.*

(2nd case means there are $(M, 1^*)$-free functions f for which a constant fraction of values needs changing to get $(N, 1^*)$-freeness)

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(2nd case means there are $(M, 1^*)$-free functions f for which a constant fraction of values needs changing to get $(N, 1^*)$-freeness)

Dichotomy Theorem for Non-monotone Properties

For non-monotone properties things get (much) messier, but we have the following result (to be stated in more detail later)

Theorem (Informal)

For a fairly broad class of $F_d^{\leq 2}$-freeness properties we have:

1. *If $(M, \sigma) \hookrightarrow (N, \tau)$, then (M, σ)-freeness $\subseteq$ (N, τ)-freeness.*
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Canonical Function

Encode (N, τ) into canonical function $f_{(N,\tau)} : \mathbb{F}^n \to \mathsf{R}$

Split $\mathbf{x} \in \mathbb{F}^n$ into $\mathbf{y}|\mathbf{z}$ for $\mathbf{y} \in \mathbb{F}^s$, $\mathbf{z} \in \mathbb{F}^{n-s}$

$$f_{(N,\tau)}(\mathbf{x}) = f_{(N,\tau)}(\mathbf{y}|\mathbf{z}) = \begin{cases} \tau_j & \text{if } \mathbf{y} = \mathbf{w}_j \\ b & \text{otherwise} \end{cases}$$

where b is some "padding value"

Example: consider (N, τ) for

- $N = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$
- $\tau = \langle 10011 \rangle$

in $\mathbb{F}_2^4$ as shown on the right

	$\mathbf{w}_1/1$	$\mathbf{w}_2/0$	
$\mathbf{w}_3/0$			
$\mathbf{w}_4/1$			
			$\mathbf{w}_5/1$

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	1	0	
0			
1			
			1

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b	1	0	b
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1	b	b	b
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Cheating Proof of Dichotomy Theorems

- Consider functions from $\mathbb{F}^n$ to $\mathsf{R} \cup \{b\}$ for $b \notin \mathsf{R}$
- Easy to show $f_{(N,\tau)}$ far from (N,τ)-free
- Suppose $f_{(N,\tau)}$ contains (M,σ) at $L : M \to \mathbb{F}^n$
- Let $\pi : \mathbb{F}^n \to \mathbb{F}^s$ be projection $\pi(\mathbf{x}) = \pi(\mathbf{y}|\mathbf{z}) = \mathbf{y}$
- Then $\pi \circ L$ is homomorphism from (M,σ) to (N,τ)
  1. clearly linear map from M to $\mathbb{F}^s$
  2. in fact, must map M to N since $\mathbb{F}^s \setminus N$ labelled by $b \notin \mathsf{R}$
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The Problem

We don't have $b \notin \mathrm{R}$! And padding with $b \in \mathrm{R}$ destroys structure

b	1	0	b
0	b	b	b
1	b	b	b
b	b	b	1

b	1	0	b
0	b	b	b
1	b	b	b
b	b	b	1

Monotone case: All labels are 1 so $b = 0$ is a "free" padding value — cheating proof works

Non-monotone case: Much harder — have to argue that some choice of $b \in \mathrm{R}$ "saves enough structural info" about matroid

- Don't know how to do this in general
- In fact, not even true in general

The Problem

We don't have $b \notin$ R! And padding with $b \in$ R destroys structure

0	1	0	0
0	0	0	0
1	0	0	0
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1	1	0	1
0	1	1	1
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Recipe for Separating Properties

1. Find labelled matroids (M, σ) and (N, τ) that don't embed into one another
2. Apply dichotomy theorems

Already discussed 2nd component — let's turn to 1st component

Revisiting Partial Matroids of Weight 2

Recall: Intersections of (F_d, σ)-freeness properties capture all matroid freeness properties

Full linear matroid of dimension d

$F_d = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d]\}$

Analogously, intersections of $(F_d^{\leq 2}, \sigma)$-freeness properties capture (almost) all matroid freeness properties currently known testable

Partial matroid of weight 2

$F_d^{\leq 2} = \{\mathbf{e}_i, \, \mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i \neq j \leq d\}$

More About Matroid Homomorphisms

- If (M, σ) submatroid of (N, τ), then clearly $(M, \sigma) \hookrightarrow (N, \tau)$
- E.g. $(F_2^{\leq 2}, 1^*) \hookrightarrow (F_3^{\leq 2}, 1^*)$
- But homomorphisms can be trickier than that — also $(F_3^{\leq 2}, 1^*) \hookrightarrow (F_2^{\leq 2}, 1^*)$!
- That is, $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_2 + \mathrm{e}_3\}$ can be mapped linearly into $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2\}$
- So $(F_2^{\leq 2}, 1^*)$-freeness $=$ $(F_3^{\leq 2}, 1^*)$-freeness — yet another "property collapse"

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Non-Homomorphism Results

For dimension $d \geq 3$ no such homomorphism surprises

Lemma

If $d > c \geq 3$, then $(F_d^{\leq 2}, \sigma) \not\to (F_c^{\leq 2}, \tau)$ for any σ, τ.

Lemma

If $d \geq 3$ and σ and τ have distinct number of labels of each type, then $(F_d^{\leq 2}, \sigma) \not\to (F_d^{\leq 2}, \tau)$.

Partial Matroids and Dichotomy Theorems

To be able to apply dichotomy theorems, focus on partial matroids with

- All non-basis vectors labelled by 1
- Basis vectors labelled 0 or 1
- So w.l.o.g. because of symmetry study labelled matroids $(F_d^{\leq 2}, 0^c 1^*)$ for $c \leq d$

Denote $(F_d^{\leq 2}, 0^c 1^*)$-freeness by $\mathcal{F}_d^{\leq 2}[\neg 0^c 1^*]$ for brevity

(Notation $f \in \mathcal{F}_d^{\leq 2}[\neg 0^c 1^*]$ means that evaluating f on any set of vectors $\{\mathbf{x}_i, \mathbf{x}_i + \mathbf{x}_j \mid 1 \leq i \neq j \leq d\} \subseteq \mathbb{F}^n$ we do **not** see pattern $\langle 0^c 1^* \rangle$)

$\mathbf{e}_1$
$\mathbf{e}_2$
$\mathbf{e}_3$
$\mathbf{e}_4$
$\mathbf{e}_1 + \mathbf{e}_2$
$\mathbf{e}_1 + \mathbf{e}_3$
$\mathbf{e}_1 + \mathbf{e}_4$
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$\mathbf{e}_1$
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$\mathbf{e}_3$
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$\mathbf{e}_1 + \mathbf{e}_2 \,/\, 1$
$\mathbf{e}_1 + \mathbf{e}_3 \,/\, 1$
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$\mathbf{e}_1 + \mathbf{e}_2 \,/\, 1$
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Denote $(F_d^{\leq 2}, 0^c1^*)$-freeness by $\mathcal{F}_d^{\leq 2}[\neg 0^c1^*]$ for brevity

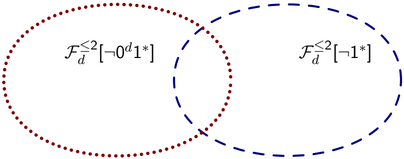
(Notation $f \in \mathcal{F}_d^{\leq 2}[\neg 0^c1^*]$ means that evaluating f on any set of vectors $\{\mathbf{x}_i, \mathbf{x}_i + \mathbf{x}_j \mid 1 \leq i \neq j \leq d\} \subseteq \mathbb{F}^n$ we do **not** see pattern $\langle 0^c1^* \rangle$)

Two Nested Hierarchies Venn Diagram-Style

Compare

(a) $\mathcal{F}_{d}^{\leq 2}[\neg 0^{d}1^{*}]$: basis 0, rest 1

(b) $\mathcal{F}_{d}^{\leq 2}[\neg 1^{*}]$: all labels 1



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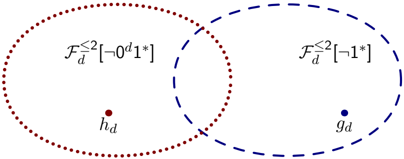

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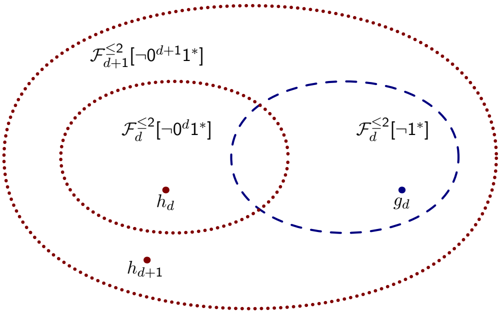

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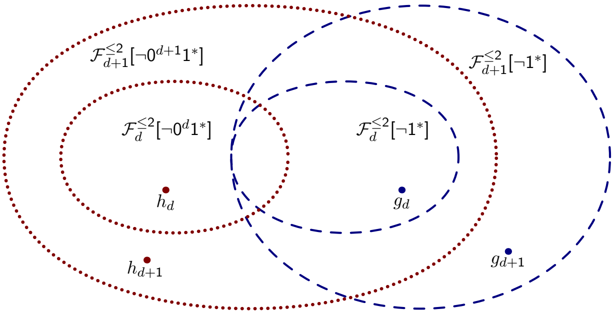

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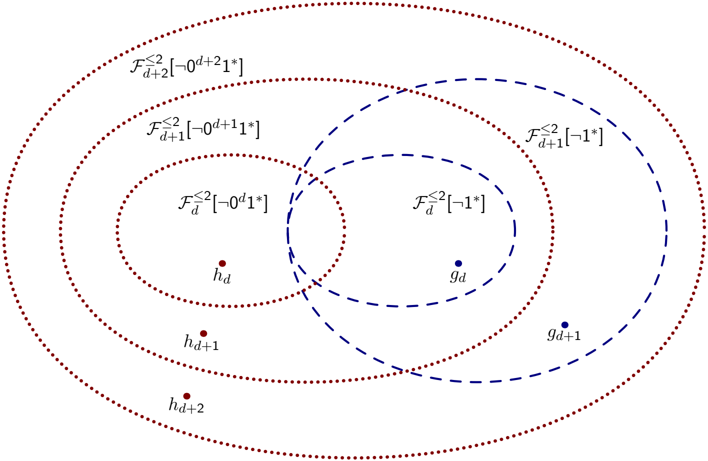

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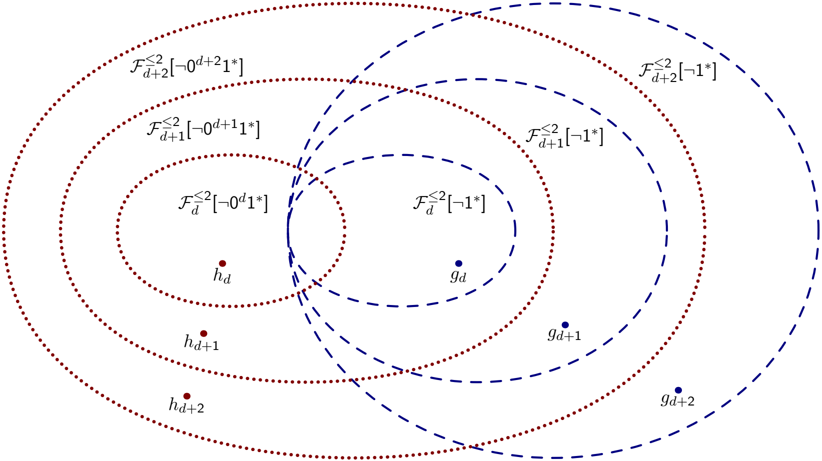

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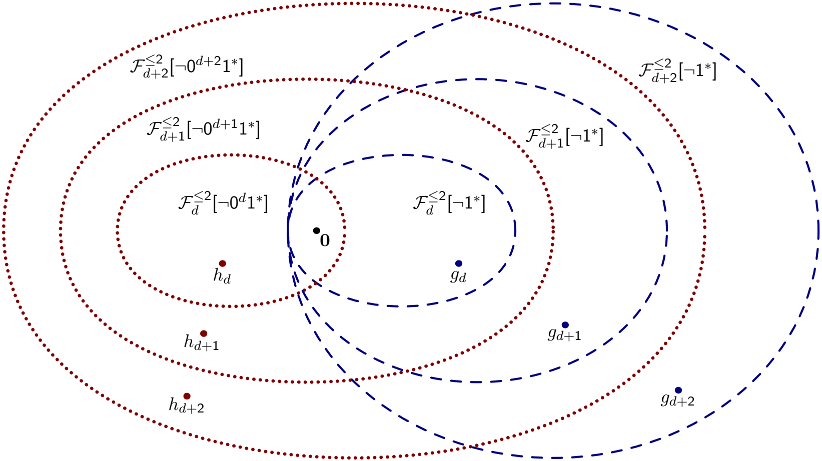

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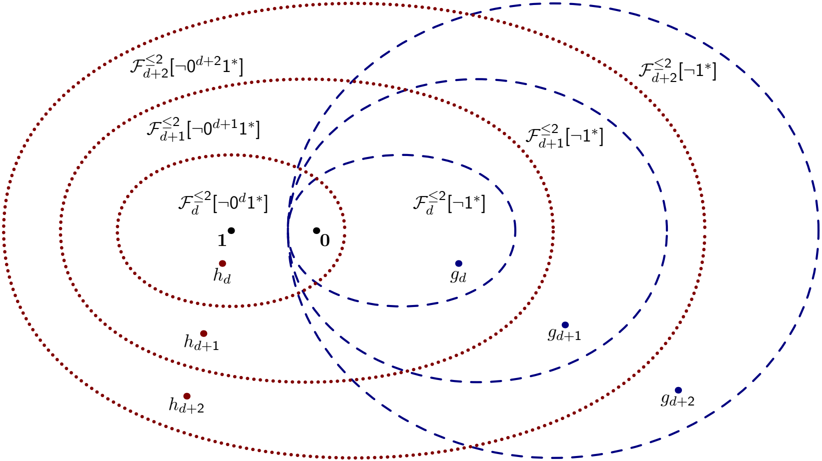

A Family of Hierarchies for Partial Matroid of Weight 2

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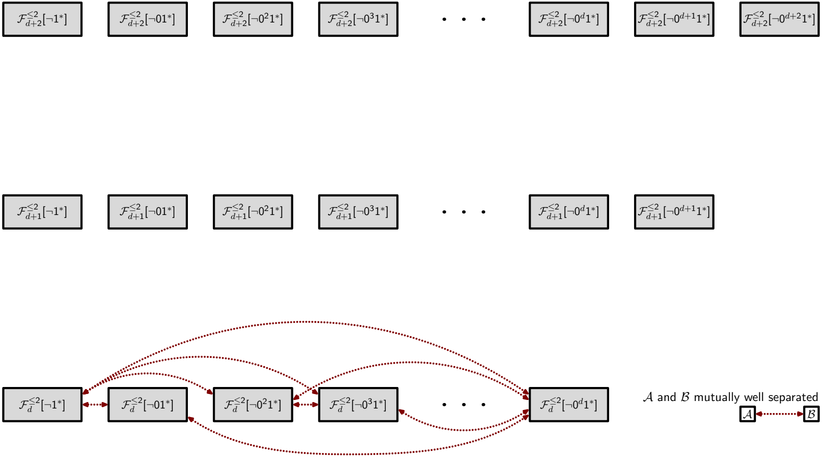

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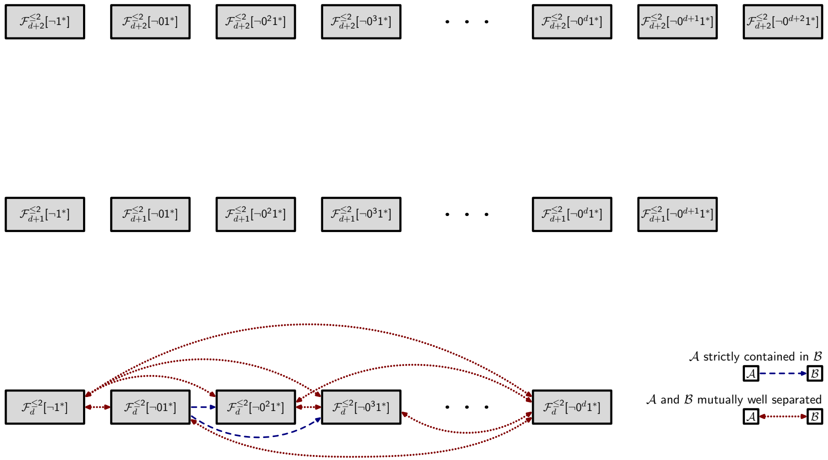

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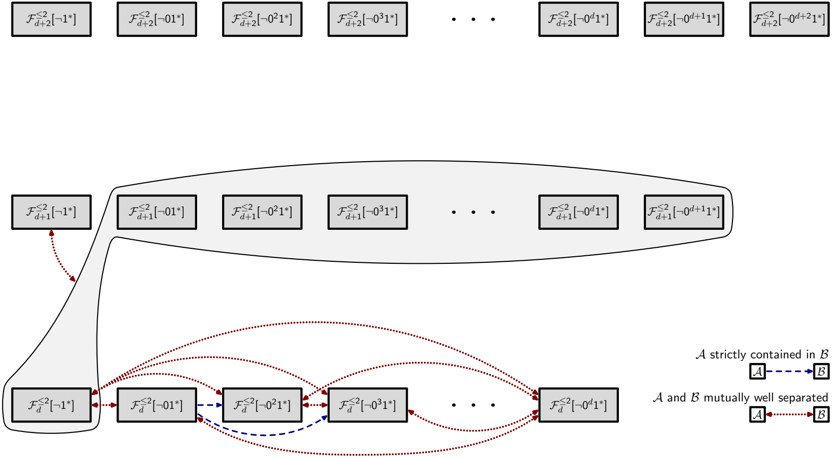

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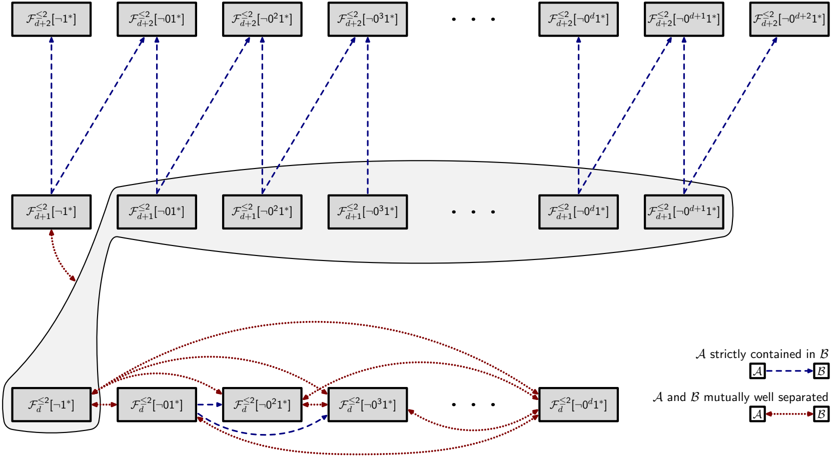

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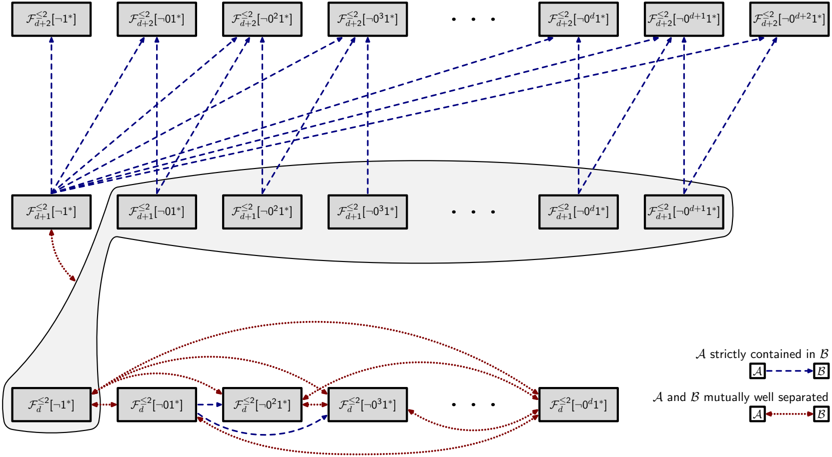

Some Comments for the Record

- Results are slightly more general than stated in this talk
  (E.g. apply also for properties currently not known testable)

- A fair bit of other technicalities swept under the rug

- Canonical functions are great but don't always work
  There are cases where they can't separate distinct properties

- Homomorphisms are also great but don't always work either
  We saw examples where (M, σ)-freeness $\subseteq$ (N, τ)-freeness although $(M, \sigma) \not\to (N, \tau)$

  (But for our examples (M, σ) "almost" embeds into (N, τ) if we are also allowed to map to $\mathbf{0}$-vector...)

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How Far Does This Approach Extend?

Open Problem 1

Can these techniques be generalized to deal with

1. any $(F_d^{\leq 2}, \sigma)$-freeness property?
2. any $(F_d^{\leq w}, \sigma)$-freeness property for $w > 2$?

Open Problem 2

Is it true for any labelled matroids (M, σ) and (N, τ) that (M, σ)-freeness $\subseteq$ (N, τ)-freeness if and only if (M, σ) embeds into $(N, \tau) \cup \{\mathbf{0}\}$?

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When Does the Dichotomy Hold?

Open Problem 3

Does a dichotomy always hold for any two linear-invariant properties $\mathcal{P}$ and $\mathcal{Q}$ in the sense that

- either $\mathcal{P}$ is contained in $\mathcal{Q}$
- or $\mathcal{P}$ is **far** from being contained in $\mathcal{Q}$?

Summing up

- Active line of research in property testing to characterize testable properties in terms of their invariances
- If we want to understand linear-invariant properties, then matroid freeness is a fundamental concept
- However, syntactic specifications of matroid freeness properties don't say much about semantic meaning — on the contrary can be downright misleading
- This work initiates systematic study of the semantics of (local characterizations of) linear-invariant properties
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Thank you for your attention!