Pontryagin Approximations for Optimal Design

Jesper Carlsson

CSC, KTH

BIT Conference, 2006

Optimal shape problem

Find domain $D \subset \Omega \subset \mathbb{R}^d$ such that

$$\inf_{D \in \mathcal{D}_{ad}} \left\{ \int_D F(\varphi)\, \mathrm{d}x \,\middle|\, G(\varphi) = 0 \text{ in } D \right\},$$

Parameter design problem

Find characteristic function $\chi : \Omega \to \{0, 1\}$ such that

$$\inf_{\chi \in \mathcal{X}_{ad}} \left\{ \int_\Omega \chi F(\varphi)\, \mathrm{d}x \,\middle|\, G\chi(\varphi) = 0 \text{ in } \Omega \right\},$$

Issues

- sensitive to perturbations in data
- infimum may not be attained

Depending on $\mathcal{D}_{ad}$, a minimizing sequence $(\bar{D}_m, \varphi_m) \to (\bar{D}, \varphi)$ where $G(\varphi_m) = 0$ does not necessarily imply $G(\varphi) = 0$ or $\chi_{D_m} \to \chi_D$.

Optimal shape problem

Find domain $D \subset \Omega \subset \mathbb{R}^d$ such that

$$\inf_{D \in \mathcal{D}_{ad}} \left\{ \int_D F(\varphi)\, \mathrm{d}x \;\middle|\; G(\varphi) = 0 \text{ in } D \right\},$$

Parameter design problem

Find characteristic function $\chi : \Omega \to \{0, 1\}$ such that

$$\inf_{\chi \in \mathcal{X}_{ad}} \left\{ \int_\Omega \chi F(\varphi)\, \mathrm{d}x \;\middle|\; G\chi(\varphi) = 0 \text{ in } \Omega \right\},$$

Issues

- sensitive to perturbations in data
- infimum may not be attained

Depending on $\mathcal{D}_{ad}$, a minimizing sequence $(\bar{D}_m, \varphi_m) \to (\bar{D}, \varphi)$ where $G(\varphi_m) = 0$ does not necessarily imply $G(\varphi) = 0$ or $\chi_{D_m} \to \chi_D$.

Optimal shape problem

Find domain $D \subset \Omega \subset \mathbb{R}^d$ such that

$$\inf_{D \in \mathcal{D}_{ad}} \left\{ \int_D F(\varphi)\,\mathrm{d}x \;\middle|\; G(\varphi) = 0 \text{ in } D \right\},$$

Parameter design problem

Find characteristic function $\chi : \Omega \to \{0, 1\}$ such that

$$\inf_{\chi \in \mathcal{X}_{ad}} \left\{ \int_\Omega \chi F(\varphi)\,\mathrm{d}x \;\middle|\; G\chi(\varphi) = 0 \text{ in } \Omega \right\},$$

Issues

- sensitive to perturbations in data
- infimum may not be attained

Depending on $\mathcal{D}_{ad}$, a minimizing sequence $(\bar{D}_m, \varphi_m) \to (\bar{D}, \varphi)$ where $G(\varphi_m) = 0$ does not necessarily imply $G(\varphi) = 0$ or $\chi_{D_m} \to \chi_D$.

Value function

$$u(\phi, t) \equiv \inf_{\alpha} \left\{ g(\varphi_T) + \int_t^T h(\varphi_s, \alpha_s)\, \mathrm{ds} \;\middle|\; \partial_s \varphi = f(\varphi_s, \alpha_s),\ \varphi_t = \phi \right\},$$

$\varphi : [0, T] \times \Omega \rightarrow V$, $\alpha : \Omega \times [0, T] \rightarrow B$

Hamilton-Jacobi-Bellman equation

$$\partial_t u(\phi, t) + H\left(\partial_\phi u(\phi, t), \phi\right) = 0, \quad u(\cdot, T) = g(\varphi_T).$$

$$H(\lambda, \phi) \equiv \min_{a:\Omega \longrightarrow B} \{\langle \lambda, f(\phi, a)\rangle + h(\phi, a)\},$$

Pontryagin principle

$$-\partial_t \lambda_t = \langle \lambda_t, \partial_\varphi f(\varphi_t, \alpha_t)\rangle + \partial_\varphi h(\varphi_t, \alpha_t), \quad \lambda_T = \partial_{\varphi_T} g(\varphi_T)$$

$$\alpha_t \in \operatorname*{argmin}_{a:\Omega \longrightarrow B}\{\langle \lambda_t, f(\varphi_t, a)\rangle + h(\varphi_t, a)\}$$

Hamiltonian system

$$\partial_t \varphi_t = \partial_\lambda H(\lambda_t, \varphi_t), \quad -\partial_t \lambda_t = \partial_\varphi H(\lambda_t, \varphi_t)$$

Value function

$$u(\phi,t) \equiv \inf_{\alpha} \left\{ g(\varphi_T) + \int_t^T h(\varphi_s,\alpha_s)\, \mathrm{ds} \;\middle|\; \partial_s\varphi = f(\varphi_s,\alpha_s),\ \varphi_t = \phi \right\},$$

$\varphi : [0,T]\times\Omega \to V$, $\alpha : \Omega\times[0,T] \to B$

Hamilton-Jacobi-Bellman equation

$$\partial_t u(\phi,t) + H\big(\partial_\phi u(\phi,t),\phi\big) = 0, \quad u(\cdot,T) = g(\varphi_T).$$

$$H(\lambda,\phi) \equiv \min_{a:\Omega\to B}\{\langle\lambda, f(\phi,a)\rangle + h(\phi,a)\},$$

Pontryagin principle

$$-\partial_t\lambda_t = \langle\lambda_t, \partial_\varphi f(\varphi_t,\alpha_t)\rangle + \partial_\varphi h(\varphi_t,\alpha_t), \quad \lambda_T = \partial_{\varphi_T} g(\varphi_T)$$

$$\alpha_t \in \operatorname*{argmin}_{a:\Omega\to B}\{\langle\lambda_t, f(\varphi_t,a)\rangle + h(\varphi_t,a)\}$$

Hamiltonian system

$$\partial_t\varphi_t = \partial_\lambda H(\lambda_t,\varphi_t), \quad -\partial_t\lambda_t = \partial_\varphi H(\lambda_t,\varphi_t)$$

Value function

$$u(\phi,t) \equiv \inf_{\alpha}\left\{ g(\varphi_T) + \int_t^T h(\varphi_s,\alpha_s)\,\mathrm{ds} \;\middle|\; \partial_s\varphi = f(\varphi_s,\alpha_s),\ \varphi_t = \phi \right\},$$

$\varphi : [0,T]\times\Omega \to V$, $\alpha : \Omega\times[0,T] \to B$

Hamilton-Jacobi-Bellman equation

$$\partial_t u(\phi,t) + H\big(\partial_\phi u(\phi,t),\phi\big) = 0, \quad u(\cdot,T) = g(\varphi_T).$$

$$H(\lambda,\phi) \equiv \min_{a:\Omega\to B}\{\langle\lambda, f(\phi,a)\rangle + h(\phi,a)\},$$

Pontryagin principle

$$-\partial_t\lambda_t = \langle\lambda_t, \partial_\varphi f(\varphi_t,\alpha_t)\rangle + \partial_\varphi h(\varphi_t,\alpha_t), \quad \lambda_T = \partial_{\varphi_T} g(\varphi_T)$$

$$\alpha_t \in \operatorname*{argmin}_{a:\Omega\to B}\{\langle\lambda_t, f(\varphi_t,a)\rangle + h(\varphi_t,a)\}$$

Hamiltonian system

$$\partial_t\varphi_t = \partial_\lambda H(\lambda_t,\varphi_t), \quad -\partial_t\lambda_t = \partial_\varphi H(\lambda_t,\varphi_t)$$

Value function

$$u(\phi,t) \equiv \inf_{\alpha} \left\{ g(\varphi_T) + \int_t^T h(\varphi_s,\alpha_s)\, \mathrm{d}s \,\middle|\, \partial_s\varphi = f(\varphi_s,\alpha_s),\ \varphi_t = \phi \right\},$$

$\varphi : [0,T]\times\Omega \to V$, $\alpha : \Omega\times[0,T] \to B$

Hamilton-Jacobi-Bellman equation

$$\partial_t u(\phi,t) + H\big(\partial_\phi u(\phi,t),\phi\big) = 0, \quad u(\cdot,T) = g(\varphi_T).$$

$$H(\lambda,\phi) \equiv \min_{a:\Omega\to B} \{\langle\lambda, f(\phi,a)\rangle + h(\phi,a)\},$$

Pontryagin principle

$$-\partial_t\lambda_t = \langle\lambda_t, \partial_\varphi f(\varphi_t,\alpha_t)\rangle + \partial_\varphi h(\varphi_t,\alpha_t), \quad \lambda_T = \partial_{\varphi_T} g(\varphi_T)$$

$$\alpha_t \in \operatorname*{argmin}_{a:\Omega\to B}\{\langle\lambda_t, f(\varphi_t,a)\rangle + h(\varphi_t,a)\}$$

Hamiltonian system

$$\partial_t\varphi_t = \partial_\lambda H(\lambda_t,\varphi_t), \quad -\partial_t\lambda_t = \partial_\varphi H(\lambda_t,\varphi_t)$$

Compliance optimization

Minimization problem

$$\inf_{\alpha:\Omega\rightarrow\{\alpha_-,1\}} \left\{ l(\varphi) \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V,\ \int_\Omega \alpha \,\mathrm{d}x = C \right\}$$

Compliance

$$l(\varphi) = \int_\Omega f_b \cdot \varphi dx + \int_{\Gamma_N} f_s \cdot \varphi ds,$$

Energy functional

$$a_\alpha(\varphi, v) = \int_\Omega \alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(v) dx.$$

Alternative formulation

$$\inf_{\alpha:\Omega\rightarrow\{\alpha_-,1\}} \left\{ l(\varphi) + \eta \int_\Omega \alpha \,\mathrm{d}x \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V \right\}.$$

Compliance optimization

Minimization problem

$$\inf_{\alpha:\Omega\rightarrow\{\alpha_{\_},1\}} \left\{ l(\varphi) \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V,\ \int_\Omega \alpha \, \mathrm{d}x = C \right\}$$

Compliance

$$l(\varphi) \equiv \int_\Omega f_b \cdot \varphi dx + \int_{\Gamma_N} f_s \cdot \varphi ds,$$

Energy functional

$$a_\alpha(\varphi, v) \equiv \int_\Omega \alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(v) dx.$$

Alternative formulation

$$\inf_{\alpha:\Omega\rightarrow\{\alpha_{\_},1\}} \left\{ l(\varphi) + \eta \int_\Omega \alpha \, \mathrm{d}x \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V \right\}.$$

Compliance optimization

Minimization problem

$$\inf_{\alpha:\Omega\rightarrow\{\alpha_{\_},1\}} \left\{ l(\varphi) \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V,\ \int_\Omega \alpha \,\mathrm{d}x = C \right\}$$

Compliance

$$l(\varphi) \equiv \int_\Omega f_b \cdot \varphi dx + \int_{\Gamma_N} f_s \cdot \varphi ds,$$

Energy functional

$$a_\alpha(\varphi, v) \equiv \int_\Omega \alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(v) dx.$$

Alternative formulation

$$\inf_{\alpha:\Omega\rightarrow\{\alpha_{\_},1\}} \left\{ l(\varphi) + \eta \int_\Omega \alpha \,\mathrm{d}x \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V \right\}.$$

Compliance optimization

Minimization problem

$$\inf_{\alpha:\Omega\to\{\alpha\_,1\}} \left\{ l(\varphi) \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V,\ \int_\Omega \alpha \, \mathrm{d}x = C \right\}$$

Compliance

$$l(\varphi) \equiv \int_\Omega f_b \cdot \varphi dx + \int_{\Gamma_N} f_s \cdot \varphi ds,$$

Energy functional

$$a_\alpha(\varphi, v) \equiv \int_\Omega \alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(v) dx.$$

Alternative formulation

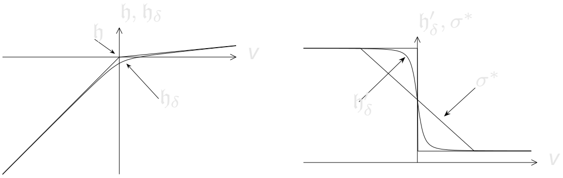
$$\inf_{\alpha:\Omega\to\{\alpha\_,1\}} \left\{ l(\varphi) + \eta \int_\Omega \alpha \, \mathrm{d}x \,\middle|\, a_\alpha(\varphi, v) = l(v),\ \forall v \in V \right\}.$$

Lagrangian

$$\mathcal{L}(\varphi,\lambda,\alpha) = I(\varphi) + I(\lambda) + \int_\Omega \alpha \Big(\underbrace{\eta - \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(\lambda)}_{\mathfrak{v}} \Big) \, \mathrm{d}x$$

Hamiltonian

$$H(\varphi,\lambda) = I(\varphi) + I(\lambda) + \int_\Omega \underbrace{\min_{\alpha \in \{\alpha_-, 1\}} \{\alpha \mathfrak{v}\}}_{\mathfrak{h}(\mathfrak{v})} \, \mathrm{d}x$$

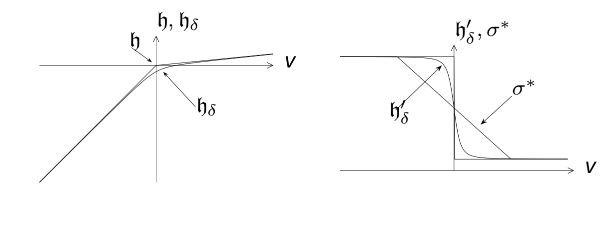


Lagrangian

$$\mathcal{L}(\varphi,\lambda,\alpha) = I(\varphi) + I(\lambda) + \int_\Omega \alpha\Big(\underbrace{\eta - \varepsilon_{ij}(\varphi)E_{ijkl}\varepsilon_{kl}(\lambda)}_{\mathfrak{v}}\Big)\,\mathrm{d}x$$

Hamiltonian

$$H(\varphi,\lambda) = I(\varphi) + I(\lambda) + \int_\Omega \underbrace{\min_{\alpha\in\{\alpha_-,1\}}\{\alpha\mathfrak{v}\}}_{\mathfrak{h}(\mathfrak{v})}\,\mathrm{d}x$$



Regularized Hamiltonian system

$$\partial_\lambda H_\delta(\lambda, \varphi) = \partial_\varphi H_\delta(\lambda, \varphi) = 0$$

By symmetry $\varphi = \lambda$ we get

$$\int_\Omega \mathfrak{h}'_\delta\left(\eta - \varepsilon_{mn}(\varphi) E_{mnop} \varepsilon_{op}(\varphi)\right) \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(v)\, \mathrm{d}x = l(v), \quad \forall v \in V$$

Pontryagin principle

$$\alpha = \mathfrak{h}'_\delta\left(\eta - \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}(\varphi)\right)$$

Regularized Hamiltonian system

$$\partial_\lambda H_\delta(\lambda,\varphi) = \partial_\varphi H_\delta(\lambda,\varphi) = 0$$

By symmetry $\varphi = \lambda$ we get

$$\int_\Omega \mathfrak{h}'_\delta\Big(\eta - \varepsilon_{mn}(\varphi)E_{mnop}\varepsilon_{op}(\varphi)\Big)\varepsilon_{ij}(\varphi)E_{ijkl}\varepsilon_{kl}(v)\,\mathrm{d}x = l(v), \quad \forall v \in V$$

Pontryagin principle

$$\alpha = \mathfrak{h}'_\delta\Big(\eta - \varepsilon_{ij}(\varphi)E_{ijkl}\varepsilon_{kl}(\varphi)\Big)$$

Regularized Hamiltonian system

$$\partial_\lambda H_\delta(\lambda, \varphi) = \partial_\varphi H_\delta(\lambda, \varphi) = 0$$

By symmetry $\varphi = \lambda$ we get

$$\int_\Omega \mathfrak{h}'_\delta\Big(\eta - \varepsilon_{mn}(\varphi)E_{mnop}\varepsilon_{op}(\varphi)\Big)\varepsilon_{ij}(\varphi)E_{ijkl}\varepsilon_{kl}(v)\,\mathrm{d}x = l(v), \quad \forall v \in V$$

Pontryagin principle

$$\alpha = \mathfrak{h}'_\delta\Big(\eta - \varepsilon_{ij}(\varphi)E_{ijkl}\varepsilon_{kl}(\varphi)\Big)$$

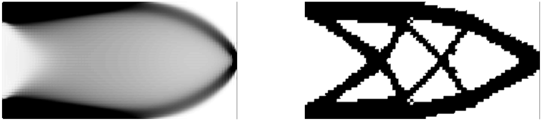


Figure: 80×40 mesh

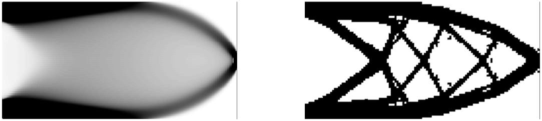


Figure: 240×120 mesh

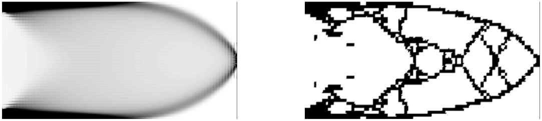


Figure: 80 $\times$ 40 mesh

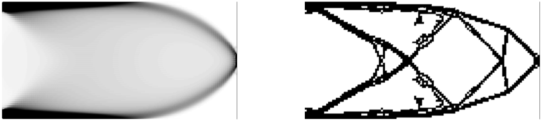


Figure: 240 $\times$ 120 mesh

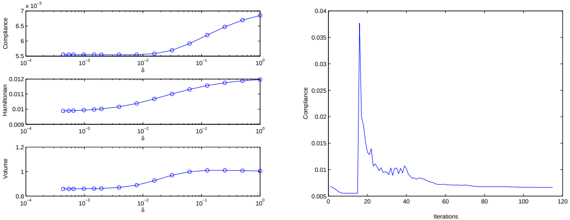


Figure: 240 × 120 mesh

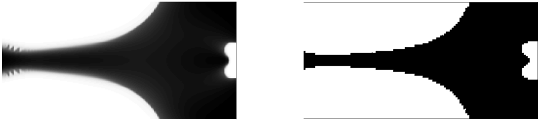


Figure: 240 × 120 mesh

Reconstruction

Minimization problem

$$\inf_{\alpha:\Omega\to\{\alpha_-,1\}} \left\{ \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, \mathrm{d}s \;\middle|\; a_\alpha(\varphi, v) = l(v), \ \forall v \in V \right\}$$

Hamiltonian

$$H(\varphi,\lambda) = \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, \mathrm{d}s + l(\lambda) + \int_\Omega \underbrace{\min_{\alpha\in\{\alpha_-,1\}} \{-\alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}\}}_{\mathfrak{h}(\mathfrak{v})} \, \mathrm{d}x$$

Regularized Hamiltonian system

$$a_{\mathfrak{h}'_\delta}(\varphi, v) \, \mathrm{d}x = l(v), \qquad \forall v \in V$$

$$a_{\mathfrak{h}'_\delta}(\lambda, w) \, \mathrm{d}x = 2\int_{\Gamma_N} (\varphi - \varphi_{meas}) \cdot w \, \mathrm{d}s \qquad \forall w \in V$$

Reconstruction

Minimization problem

$$\inf_{\alpha:\Omega\to\{\alpha_-,1\}} \left\{ \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, ds \;\middle|\; a_\alpha(\varphi, v) = l(v), \; \forall v \in V \right\}$$

Hamiltonian

$$H(\varphi,\lambda) = \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, ds + l(\lambda) + \int_\Omega \underbrace{\min_{\alpha\in\{\alpha_-,1\}} \{-\alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}\}}_{\mathfrak{h}(\mathfrak{v})} \, dx$$

Regularized Hamiltonian system

$$a_{\mathfrak{h}'_\delta}(\varphi, v)\, dx = l(v), \qquad \forall v \in V$$

$$a_{\mathfrak{h}'_\delta}(\lambda, w)\, dx = 2\int_{\Gamma_N} (\varphi - \varphi_{meas}) \cdot w \, ds \qquad \forall w \in V$$

Reconstruction

Minimization problem

$$\inf_{\alpha:\Omega\to\{\alpha_-,1\}} \left\{ \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, ds \;\middle|\; a_\alpha(\varphi, v) = l(v), \ \forall v \in V \right\}$$

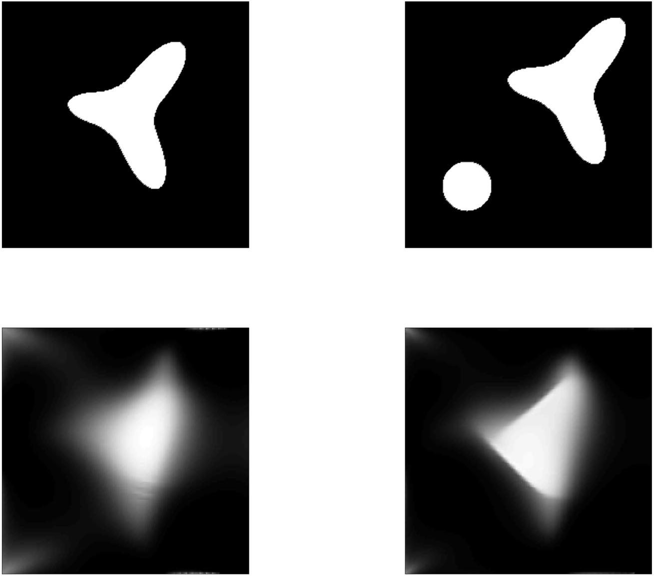
Hamiltonian

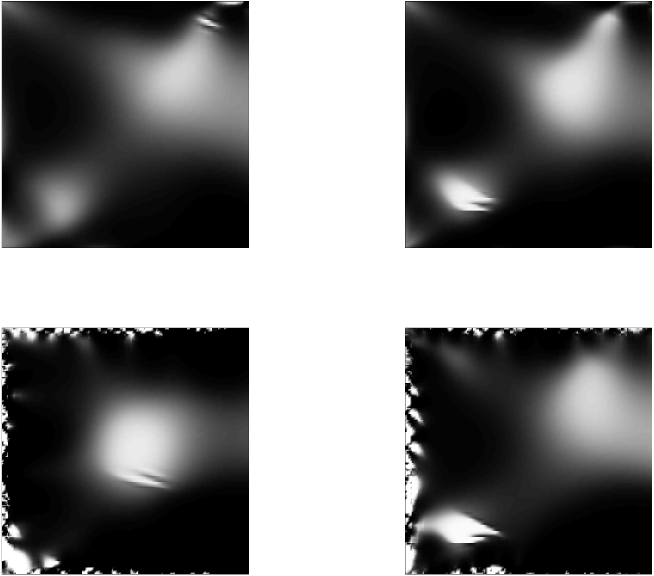
$$H(\varphi, \lambda) = \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, ds + l(\lambda) + \int_\Omega \underbrace{\min_{\alpha\in\{\alpha_-,1\}} \{-\alpha \varepsilon_{ij}(\varphi) E_{ijkl} \varepsilon_{kl}\}}_{\mathfrak{h}(\mathfrak{v})} \, dx$$

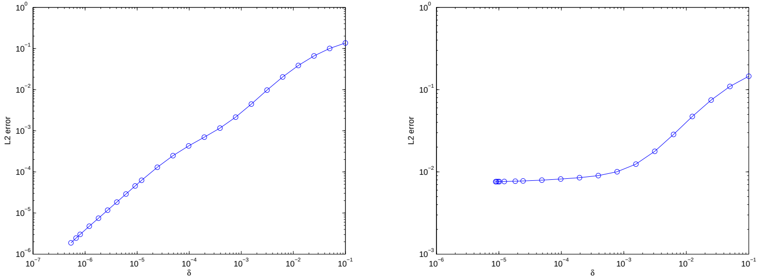
Regularized Hamiltonian system

$$a_{\mathfrak{h}'_\delta}(\varphi, v)\, dx = l(v), \qquad \forall v \in V$$

$$a_{\mathfrak{h}'_\delta}(\lambda, w)\, dx = 2\int_{\Gamma_N} (\varphi - \varphi_{meas}) \cdot w \, ds \qquad \forall w \in V$$






L2 error
δ
L2 error
δ

Future work

- Reconstruction from acoustic and elastic wave propagation
- Reconstruction using optimal input data

Reconstruction from optimal input data

$$\sup_{\beta} \inf_{\alpha} \left\{ \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, ds \right\}$$

$$a_\alpha(\varphi, v) = l_\beta(v), \ \forall v \in V$$

$$a_\alpha(\varphi_{meas}, w) = l_\beta(w), \ \forall w \in V$$

$$\|\beta\|_{L^2(\Gamma_N)} \leq 1$$

Future work

- Reconstruction from acoustic and elastic wave propagation
- Reconstruction using optimal input data

Reconstruction from optimal input data

$$\sup_{\beta} \inf_{\alpha} \left\{ \int_{\Gamma_N} |\varphi - \varphi_{meas}|^2 \, ds \right\}$$

$$a_\alpha(\varphi, v) = l_\beta(v), \; \forall v \in V$$

$$a_\alpha(\varphi_{meas}, w) = l_\beta(w), \; \forall w \in V$$

$$\|\beta\|_{L^2(\Gamma_N)} \leq 1$$