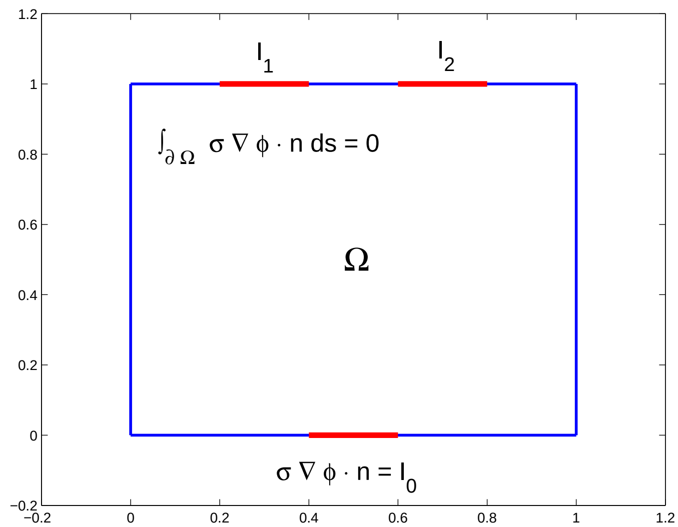
A Multigrid Method for Topology Optimization

Jesper Carlsson
Mattias Sandberg
Anders Szepessy

The Problem



Minimize power-loss in conductive medium i.e.

$$\min_{\sigma}\left(\int_{\partial\Omega} I\varphi ds + \eta \int_{\Omega} \sigma dx\right)$$

s.t.

$$\mathrm{div}(\sigma\nabla\varphi(x)) = 0 \quad x \in \Omega, \qquad \sigma\frac{\partial\varphi}{\partial n}|_{\partial\Omega} = I$$

$$\sigma : \Omega \rightarrow [\sigma_-, \sigma_+].$$

Approaches

- Interior-Point
- Topological Gradient
- Homogenization

Pontryagin Principle

Lagrangian:

$$\mathcal{L} = \int_{\partial\Omega} I\varphi ds + \eta \int_{\Omega} \sigma dx + \int_{\Omega} \mathrm{div}(\sigma \nabla \varphi(x))\lambda dx$$
$$= \int_{\partial\Omega} I(\varphi + \lambda) ds + \int_{\Omega} \sigma \underbrace{(-\nabla\varphi \cdot \nabla\lambda + \eta)}_{v} dx.$$

Hamiltonian:

$$H = \min_{\sigma} \int_{\Omega} \sigma v dx + \ldots = \int_{\Omega} \underbrace{\min_{\sigma} \sigma v}_{s(v)} dx + \ldots.$$

Control:

$$\sigma(v) = s'(v) = \sigma_{+} 1_{\{v<0\}} + \sigma_{-} 1_{\{v>0\}}.$$

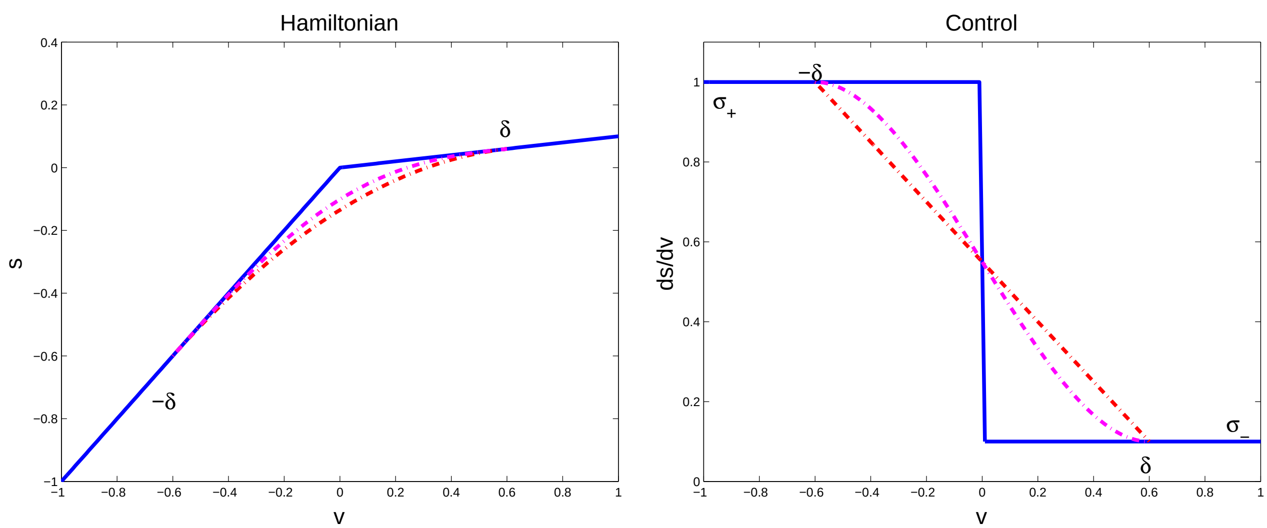
Goal:

Consistency with infinite dimensional H-J eq.

Regularization

Following Sandberg

$$H^{\delta}(\lambda, \varphi) = \int_{\Omega} s_{\delta}(v)dx + \int_{\partial\Omega} I(\varphi + \lambda)ds$$



The Pontryagin principle yields

$$0 = \dot{\varphi} = H^{\delta}_{\lambda}$$
$$0 = \dot{\lambda} = -H^{\delta}_{\varphi}.$$

Symmetry implies $\varphi = \lambda$ and

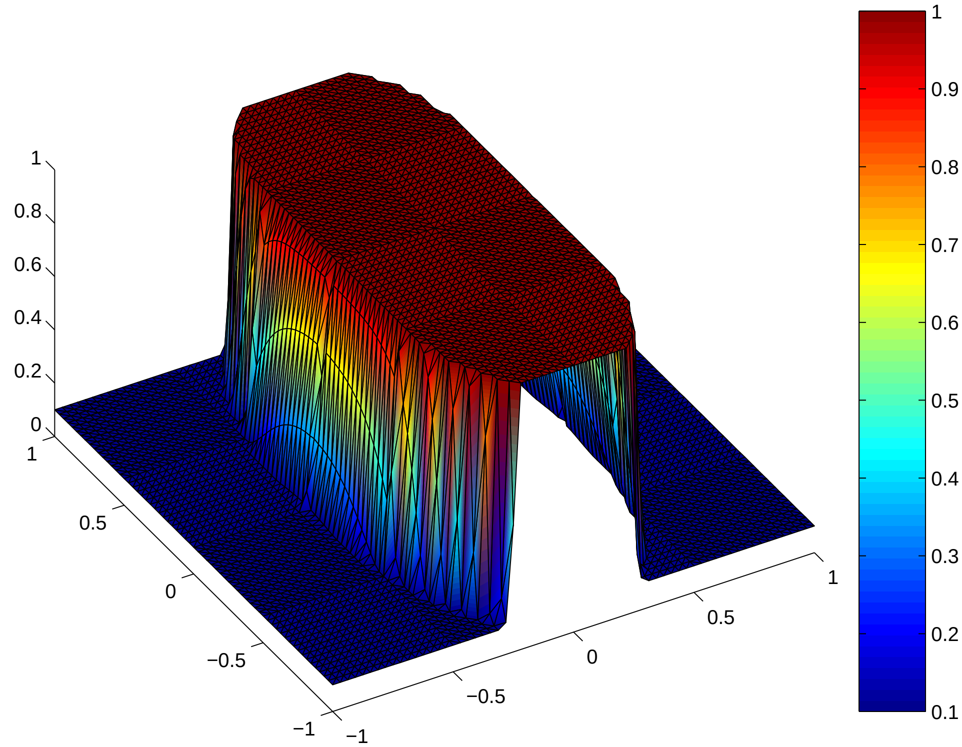
$$\text{div}(s'_{\delta}(v)\nabla\varphi(x)) = 0 \quad x \in \Omega, \qquad s'_{\delta}\frac{\partial\varphi}{\partial n}|_{\partial\Omega} = I.$$

Multigrid

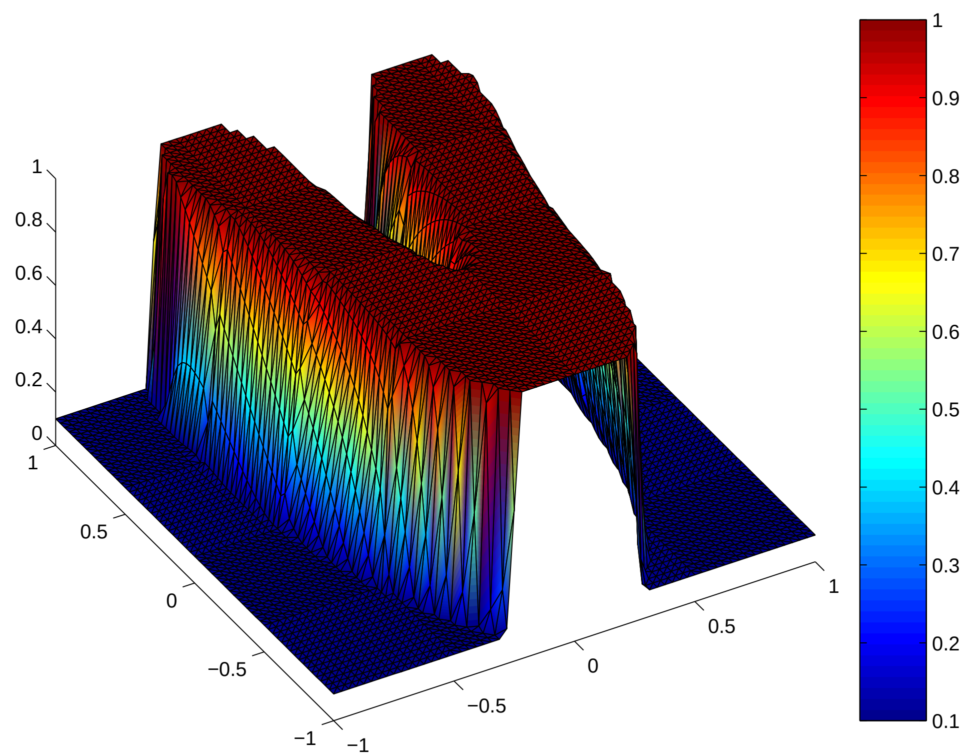
Solved by nonlinear multigrid and two tricks:

- adaptive damping in smoothing steps; help for $s''_\delta \simeq 0$,

- increasing number of smoothing steps on coarser meshes; to improve the corrections.



1
0.8
0.6
0.4
0.2
0
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1



1
0.8
0.6
0.4
0.2
0
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1

Consistency with Infinite Dimensional H-J eq.

$$u(\varphi, t) = \int_0^T \left(\int_{\partial\Omega} I\varphi ds + \eta \int_\Omega \sigma dx \right) dt$$

$$\int_\Omega (\partial_t \varphi w + s'(v) \nabla \varphi \cdot \nabla w) dx = -\int_{\partial\Omega} I w ds$$
$$\int_\Omega (-\partial_t \lambda w + s'(v) \nabla \lambda \cdot \nabla w) dx = -\int_{\partial\Omega} I w ds$$

$$\lim_{T \to \infty} \frac{u(\varphi, t) - \bar{u}(\varphi, t)}{T}$$

Using the Gateaux derivative $\partial_\varphi u = \lambda$ and

$$\partial_t u + H(\partial_\varphi u, \varphi) = 0$$

Optimal Control

$$\frac{dX}{dt} = f(X_t, \alpha_t),$$
$$\inf_{\alpha \in \mathcal{A}} \Big(g(X_T) + \int_0^T h(X_s, \alpha_s) ds \Big),$$

with given data f, X_0, g
and controls $\mathcal{A} = \{\alpha : [0, T] \to B\}$.

The Lagrange Principle (h=0)

$$\mathcal{L} = \int_0^T \lambda_t \cdot \Big(f(X_t, \alpha_t) - X_t' \Big) \, dt + g(X_T)$$

Leads to the Hamiltonian system

$$\delta\lambda : \quad X_t' = f(X_t, \alpha_t), \quad X_0 \text{ given},$$

$$\delta X : \quad -\lambda'(t)_i = \partial_{x_i} f(X_t, \alpha_t) \cdot \lambda_t, \quad \lambda_T = g'(X_T),$$

$$\delta\alpha : \quad \alpha_t^* = \text{argmin}_{\alpha \in B} \, \lambda_t^* \cdot f(X_t^*, \alpha).$$

based on the Pontryagin principle.

Dynamic Programming (h=0)

Gives

$$u(x,t) \equiv \inf_{\alpha} g(X_T;\ X_t = x)$$

as a solution of the non-linear HJB partial differential equation

$$\partial_t u(x,t) + \underbrace{\min_{\alpha \in B} \left(f(x,\alpha) \cdot \nabla u(x,t)\right)}_{H(\nabla u(x,t),x)} = 0, \quad t < T,$$

$$u(\cdot, T) = g.$$

Approximation

Hamilton-Jacobi by FEM or FD:

(+) Global minimum is found, (-) Not $d \gg 1$.

The Lagrange Principle:

(-) Local minimum hard to find, (+) $d \gg 1$!