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Option Hedging

Optimal Strategies in the Presence of Stochastic Volatility and Transaction Costs

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Abstract

In this thesis, a utility based model for hedging European options is derived when the volatility of the underlying is stochastic and when transaction costs are present.

Davis, Panas & Zariphopoulou (*SIAM J. Control and Optimization*. 31(2):470–493) and Hodges & Neuberger (*Rev. Futures and Markets*. 8:222–239) have earlier presented such a model for constant volatility of the underlying, which implies solving a *three-dimensional* free boundary problem for the option price. An optimal hedging strategy for the option writer is also given by the boundaries of the problem. Since solving this free boundary problem is computationally demanding, Whalley & Wilmott (*Mathematical Finance*. 3:307–324) suggested using asymptotic analysis in the case of small transaction costs to reach simple expressions for the hedging strategy. Although Whalley & Wilmott based their work on Davis *et al*, the authors proposed different hedging strategies.

Following the approach of Davis *et al* in the presence of a stochastic volatility gives a similar *four*-dimensional free boundary problem for the option price, and hedging strategies are derived using asymptotic analysis. The dimensionality of the problem is further reduced by using a technique presented in Fouque, Papanicolaou and Sircar (*Int. Journal of Theoretical and Appl. Finance*. 3:101–142), and the solutions are given as correction terms to the hedging strategies proposed by Whalley & Wilmott and Davis *et al*. The new stochastic volatility parameters used in these correction terms can be estimated from observed implied volatility surfaces and historical volatilities on the market.

Optionshedgning

Optimala strategier med stokastisk volatilitet och transaktionskostnader

Sammanfattning

I detta exjobb härleds nyttobaserad modell för hedgning av Europeiska köpoptioner, då volatiliteten för underligganden är stokastisk och då transaktionskostnader tas i beaktande.

Davis, Panas & Zariphopoulou (*SIAM J. Control and Optimization*. 31(2):470–493) och Hodges & Neuberger (*Rev. Futures and Markets*. 8:222–239) har tidigare presenterat en sådan modell för konstant volatilitet, där optionspriset ges av ett tredimensionellt fritt-randvärdesproblem. Ränderna till detta problem ger även en optimal hedgningstrategi för optionen. Eftersom detta fria-randvärdesproblem är beräkningsmässigt svårt att bemästra föreslog Whalley & Wilmott (*Mathematical Finance*. 3:307–324) en lösning av problemet via asymptotisk analys giltig för små transaktionskostnader. Denna asymptotiska analys ger lätthanterliga uttryck för ränderna och därmed den optimala hedgningstrategin. Trots att Whalley & Wilmott baserar sin analys på Davis o. a. föreslår dock författarna olika hedgningsstrategier.

Genom att utveckla modellen i Davis o. a. till att innefatta stokastisk volatilitet för underligganden får vi istället ett fyrdimensionellt fritt-randvärdesproblem där optimala hedgingstrateger härleds genom asymptotisk analys. Dimensionaliteten kan dock ytterligare reduceras och via en teknik föreslagen av Fouque, Papanicolaou and Sircar (*Int. Journal of Theoretical and Appl. Finance*. 3:101–142) erhålls slutligen uttryck bestående av korrektionstermer till strategierna i Whalley & Wilmott och Davis o. a. Dessa termer korrigerar för den stokastiska volatiliteten hos underligganden och kan enkelt skattas från implicita volatilitetsytor och historisk volatilitet observerad på marknaden.

Acknowledgements

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Jesper Carlsson

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Chapter 1

Introduction

Writing or buying an option always implies some exposure to financial risk. In the case of a European call option, the buyer of a contract pays for the right to buy a given amount of the underlying at a strike price K at the date of expiry T. If the market price for the commodity exceeds the strike price, the buyer will of course buy at the strike price and then sell the commodity on the market, and make a profit. Since the contract gives the buyer the right, but not the obligation to buy, he thus has the chance make an infinite profit and only faces the risk of losing what he initially paid for the contract.

The writer, on the other hand, has an obligation to deliver the commodity to the buyer at the contracted price and thus faces the risk of having to buy the commodity at market price and sell cheaply to the buyer. This can in theory lead to infinite losses, and the simplest method for a writer to reduce his/her risk is to own some amount of the underlying him/herself.

Option theory thus focuses on how to determine a fair price of an option and how the writer should hedge his risk via trading in the underlying. To determine what a buyer is willing to pay, and how much a writer should charge, for an option, it seems that something about the buyer's and the writer's attitude towards risk has to be known. Thus, no unique price for an option can exist. However, the publication of Black-Scholes formula in 1973 [3] showed that, under certain assumptions, a unique option price and a hedging strategy that completely hedges away the writer's risk, does exist. Such a hedging strategy is referred to as being "perfect" and it turns out to be equal to holding the number $\Delta = \frac{\partial f}{\partial S}$, i.e. the derivative of the option price with respect to the stock price, of underlying assets.

Although the theory, upon which Black-Scholes formula is based, has an appealing simplicity, it has a main drawback: It is based upon continuous trading in the underlying without any transaction costs. In real life transaction costs are always present, and continuous trading will thus imply unbounded costs for the writer, which no longer makes perfect hedging possible. Option valuation then has to be preference-dependent.

Hodges & Neuberger [8] and Davis *et al* [4] proposed a model, using measures of risk preference, which gives an option price and hedging strategy based on maximizing utility. Their approach thus finds an option price that can be hedged in an optimal sense to maximize the writer's utility. The proposed hedging strategy involves a band, centred on an ideal value, where the number of the underlying asset is allowed to vary. This type of hedging techniques is often referred to as a move-based-strategies.

The model presented in [8] and [4] involves solving a computationally demanding three dimensional free-boundary problem, and to give a simple expression for the hedging strategy, Whalley & Wilmott [14, 15] made an asymptotic expansion of the problem valid for the case of small transaction costs. This revealed explicit expressions for the boundaries of the problem.

In [2] the performance of the move-based-strategy presented by [14, 15] was compared with the strategies of Black-Scholes [3] and Leland [11]. The strategy proposed by Leland is built upon Black-Scholes model using a modified volatility. Both [2] and [13], who also tested a dynamic programming approach by Lee [10], concluded that Whalley & Wilmott's strategy gave better results in a mean-variance framework.

In the discussion above we primarily focus on traded commodities, but we can also have a non-traded underlying such as a stock index or weather outcomes. The writer will then simply pay the buyer an amount, depending on some index, at time of maturity.

The theory presented here will solely treat European call options. This approach is simple and can in many cases be extended to options with other payoff functions as well as path-dependent options.

1.1 Constant vs. Stochastic Volatility

1.1.1 Black-Scholes Model and Delta Hedging

The most popular model for pricing and hedging derivatives was developed in the early 1970s by Fisher Black, Merton Scholes and Robert Merton [3]. For their contribution, Merton and Scholes received, in 1997, The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel.

The Black-Merton-Scholes model is based on the following assumptions:

1. The underlying, for example a stock, has a price process described by an Itô-process
$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t) \tag{1.1}$$

where the constants μ and σ are the drift and volatility of the stock. $Z(t)$ is a Brownian motion. Equation 1.1 states that prices are log-normally distributed i.e.

$$\log(\frac{S(t)}{S(0)}) \sim \phi((\mu - \sigma^2/2)t, \sigma^2 t) \tag{1.2}$$

where $\phi(m, n)$ denotes the normal distribution with mean m and variance n.

2. The price of an option is given by $f(S, t)$ and at the date of maturity, $t = T$, we have a payoff function $f(S, T) = g(S)$. For a European call option with strike price K the payoff function is given as

$$f(S, T) = \max(S - K, 0). \tag{1.3}$$

   That is, if $S(T) > K$, we make a profit of $S - K$.

3. The market has no arbitrage opportunities. That is, given an interest rate r, there can be no *risk-free* investments with a different rate of return. The amount in risk-free assets $B(t)$ follows the deterministic process

$$dB(t) = rB(t)dt. \tag{1.4}$$

From these assumptions one can construct a weighted risk-free portfolio of stocks and options. The model states that for each written option one has to keep, at all times, $t \in [0, T]$,

$$\Delta(t) = \frac{\partial f}{\partial S}$$

stocks in the portfolio to make it risk-free. This technique is called *delta* hedging and assumes continuous rebalancing of the portfolio.

The derivation, which can be found in Appendix A, also gives the so called Black-Scholes partial differential equation for the option price $f(S, t)$

$$\begin{cases} \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf & , 0 \le t < T \\ f(S, T) = g(S). \end{cases} \tag{1.5}$$

For a European call option the simple solution to this equation is

$$f(S, t) = SN[d_1(S, t)] - e^{-r(T-t)}KN[d_2(S, t)] \tag{1.6}$$

where N is the cumulative standard normal $\phi(0, 1)$ distribution and

$$\begin{aligned} d_1(S, t) &= \frac{1}{\sigma\sqrt{T-t}}\left\{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T - t)\right\} \\ d_2(S, t) &= d_1(S, t) - \sigma\sqrt{T - t}. \end{aligned} \tag{1.7}$$

The main drawback of this model is that it assumes continuous trading and constant, or at least deterministic volatility. In real life all trades are associated with transaction costs and continuous trading would imply unbounded hedging costs. Several studies, for example [6, 7, 12], have also showed that the market tends to have some features which can be better explained assuming *stochastic* instead of constant volatility in the price process from Equation 1.1.

1.1.2 Implied Volatility

One of the main reasons for discarding the assumption of constant volatility is the observation of an implied volatility surface. Estimating implied volatility involves looking at *observed* option prices f_{obs} on the market and solving the equation

$$f(S, t, \sigma, T, K) = f_{obs} \tag{1.8}$$

for σ. Here, f denotes the Black-Scholes price for the option.

Looking at different maturities and strike prices will in reality reveal different volatilities (see Dumas *et al* [5]), which clearly contradicts the assumption of constant volatility. A solution surface to 1.8 would normally look as in Figure 1.1. If the assumptions of Black-Scholes model were true, plotting $\sigma(S, T)$ would yield a constant surface. It is thus necessary to make some adjustments to the model.

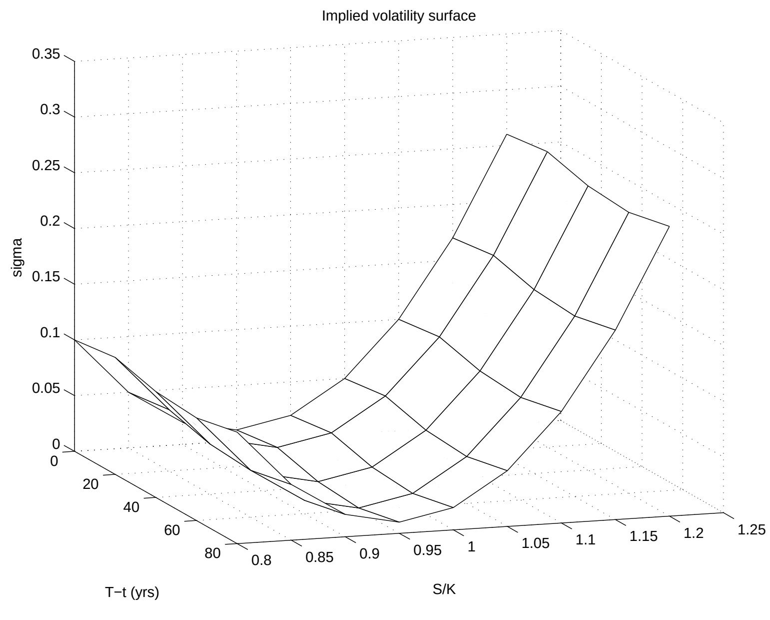

Figure 1.1. Example of an implied volatility surface.

One approach is “implied deterministic volatility” where volatility is modelled as a deterministic function of the price process $S(t)$

$$dS(t) = \mu S(t)dt + \sigma(S,t)S(t)dZ(t). \tag{1.9}$$

This gives the generalized Black-Scholes equation

$$\begin{cases} \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\partial^2 f}{\partial S^2}\sigma(S,t)^2 S^2 = rf \\ f(S,T) = g(S) \end{cases} \tag{1.10}$$

which has the advantage of both taking implied volatility into account and making it possible to hedge away the risk *completely* by trading only in the underlying (see [5]).

The main disadvantage of this approach is the difficulty to produce stable implied volatility surfaces that can be used over time. It also *only* uses derivatives data and not price data to estimate the model parameters. Neither can it simulate the behaviour of clustering volatility which is a phenomena observed on most stock markets (see [6, 7, 12]). In real life volatility tends to have periods of high volatility followed by periods of low volatility, so called volatility clusters or changes of regimes.

1.1.3 Stochastic Volatility

Another way to simulate market behaviour more accurately is to adopt a mean reverting stochastic volatility model as in [6, 7, 12].

In short, such a model can be described by the processes

$$\begin{aligned} dS(t) &= \mu S(t)dt + f(Y)S(t)dZ_1(t) \\ dY(t) &= \alpha(m - Y(t))dt + \beta d\hat{Z}(t) \end{aligned} \tag{1.11}$$

where $\hat{Z}(t)$ is a Brownian motion correlated to Z_1, defined as

$$\hat{Z}(t) = \rho Z_1(t) + \sqrt{1-\rho^2} Z_2(t) \tag{1.12}$$

where ρ is the correlation, and $Z_1(t)$ and $Z_2(t)$ are independent Brownian motions.

The volatility process $f(Y)$ is here a positive increasing function of the mean reverting Ohrstein-Uhlenbeck process $Y(t)$.

This model has the following nice features that reflect observed market behaviour:

1. Volatility is positive
2. Volatility is fast mean-reverting but persistent, i.e. the volatility drift is pulled towards the mean value m and the volatility itself is clustering. An example of clustering volatility can be seen in Figure 1.2.

3. Volatility changes are correlated to asset price changes. When volatility goes up prices typically goes down. This is reflected by a negative ρ. This correlation causes a skew in the asset price distribution, which better reflects market data than the ordinary lognormal distribution from Equation 1.1. In Figure 1.3 distributions of returns for different volatility processes can be seen. It is evident that longer tails and thus a higher skew can be observed for the stochastic volatility process.

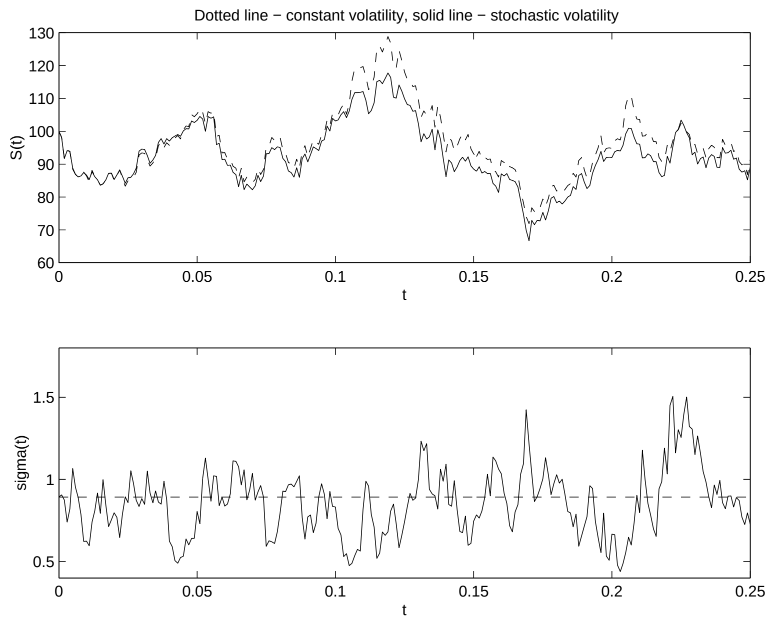

Figure 1.2. Example of stochastic volatility process: α=200, σ =0.95, ν=0.25, $m = log(\sigma) - \nu^2$. $T = 0.25$ years.

Assuming stochastic volatility, and following the derivation of the ordinary Black-Scholes formula in Appendix A, it is obvious that it is impossible to form a risk less portfolio by trading only in the underlying. This follows from the use of two independent Brownian motions.

In theory, one could hedge away the risk completely if volatility itself was a traded asset. In the case of a constant market price of volatility risk $\overline{\gamma}$, then it is possible to derive a PDE corresponding to (1.11) for the price $f(t, S, Y)$ (see [7]).

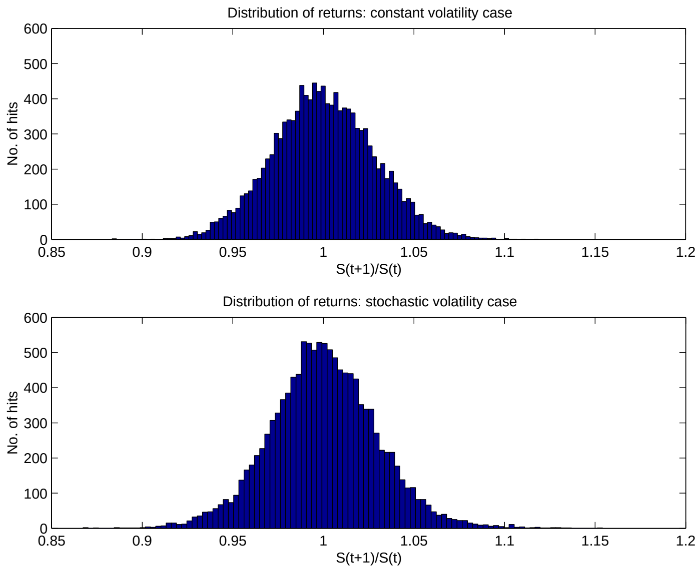

Figure 1.3. Distributions of returns

1.2 Hedging with Transaction Costs

As mentioned above, continuous hedging would involve enormous or even infinite trading costs. Hence, it is necessary to rebalance in discrete time steps, which violates the assumptions of the models above and results in a hedging error. The gain from lower transaction costs thus results in an increased risk.

The framework for hedging in the presence of transaction costs can be divided into two main categories, time-based and move-based strategies. The general objective is to minimize the variance of the hedging error given some accepted level of transaction costs. The hedging error is defined as the difference between the replicating portfolio and the payoff of the option.

Time-based strategies involve rebalancement of the portfolio at equally distant dates, while the rebalancing in move-based strategies is controlled by moves in the underlying asset.

We will here focus on a move-based method proposed by Davis, Panas and Zariphopoulou [4] and further developed by Whalley & Wilmott [14, 15]. We will briefly present the method in the following sections, and the method will be thoroughly derived for the stochastic volatility case, in the next chapter.

1.2.1 Utility Theory

The model presented by [4] is based on a maximized utility approach, that is, the writer wants to find a hedging strategy that maximizes the expected utility of his final wealth W. In order to make things clearer we will here discuss some important properties in utility theory.

A utility function U is used as a measure to make a rational choice between the different strategies available to the writer, and ranks the alternatives according to the motto "more is better". This motto places one general restriction on the utility function: it has to be an increasing continuous function.

Now, consider that we face two alternatives of wealth: One with a deterministic outcome W_d and one with a stochastic outcome W_s. Also assume that $W_d = E(W_s)$. Which alternative should we choose?

In utility theory, this answer can only be determined by the concept of risk aversion. Risk aversion tells us if someone is "risk-seeking", "risk-neutral" or "risk-averse".

The definitions are:

1. A risk-seeker has a utility function such that $U(E(W)) < E(U(W))$.
   The risk-seeker prefers the stochastic alternative.
2. A risk-neutral person has a utility function such that $U(E(W)) = E(U(W))$.
   The risk-neutral is indifferent between the two alternatives.
3. A risk-averse person has a utility function such that $U(E(W)) > E(U(W))$.
   The risk-averse prefers the deterministic alternative.

To determine the degree of risk aversion, we define the absolute risk aversion function, which measures the risk aversion for a given level of wealth,

$$ARA(W) = \frac{U''(W)}{U'(W)}. \tag{1.13}$$

In the future we will only use the constant risk aversive exponential utility function

$$U(W) = 1 - e^{-\gamma W}. \tag{1.14}$$

The degree of risk aversion here equals γ and is independent of the level of wealth. A large γ thus indicates a higher degree of risk aversion.

1.2.2 The Utility Maximization Model

Consider an optimal control problem, where the writer of an option has to allocate money between risk-less investments $B(t)$ and the underlying security, here a stock, $S(t)$ over the time interval $[0, T]$ to maximize his expected utility of final wealth. The number of shares invested in the underlying is described by the control function $y(S, t)$, which is restricted to a set of all possible trading strategies $\mathcal{T}$. The problem can then be described as

$$\max_{y \in \mathcal{T}} E[U(W(T))]. \tag{1.15}$$

To derive an option price we define two portfolios. One containing stocks and risk free assets, and one containing stocks, risk free assets and a short position in a call option. We denote the respective portfolios with index w and wo, indicating “with” or “without” an option position. For each portfolio we then have a value function

$$J_i(t, S, B_i, y_i) = \max_{y \in \mathcal{T}} E[U(W_i(T))] \qquad , i = w, wo. \tag{1.16}$$

This is actually a conditional expectation given $S(t) = S, B(t) = B, y(t) = y$, but to keep the notation as simple as possible we will just write as in (1.16) in the future.

The final wealth of the portfolios, in the case of writing a European call option with strike price K, are

$$W_i(T) = \begin{cases} y_i S + B_i - k(S, y_i) & , i = wo \\ \\ y_i S + B_i - k(S, y_i) - \max\{S - K, 0\} & , i = w \end{cases} \tag{1.17}$$

where the variable $k(S, y_i)$ denotes the transaction cost when trading y_i shares at the price S.

We now define the minimum initial wealth that delivers a non-negative maximum expected utility of final wealth as

$$\hat{B}_i = \inf\{B : J_i(t = 0, S, B, y = 0) \geq 0\}. \tag{1.18}$$

If the writer starts with a portfolio only containing money invested in the risk-free asset, and possibly a short position in an option, $\hat{B}_i$ is the invested amount for which the writer is *indifferent* between doing nothing and following the optimal strategy. Davis *et al* [4] proposed that the option value then could be expressed as (see Section 2.6 for proof)

$$P = \hat{B}_w - \hat{B}_{wo} \tag{1.19}$$

and the hedging strategy proposed by [4] to achieve this price is then

$$y(S, t) = y_w(S, t) \tag{1.20}$$

where y_w is the optimal control strategy for the portfolio with an option obligation. In [14, 15] the control strategy proposed is

$$y(S, t) = y_w(S, t) - y_{wo}(S, t). \tag{1.21}$$

The reason for taking the difference is unclear, and we will later compare these two strategies against each other.

Using the exponential utility function (1.14) it can be shown that Equation 1.16 give rise to a Hamilton-Jacobi-Bellman equation. The original derivation in [4] is almost identical to the one found in next chapter. From the HJB equation three optimal strategies can then be determined. This is summarized in a variational equation

$$\begin{aligned} min\{\ & V_y - (S + \tfrac{\partial k}{\partial y}), -[V_y - (S + \tfrac{\partial k}{\partial y})], \\ & V_t - rV + \mu S V_S + \tfrac{\sigma^2 S^2}{2}(V_{SS} - \tfrac{\gamma}{\delta(t,T)} V_S^2)\} = 0 \end{aligned} \tag{1.22}$$

where V is the maximized utility of wealth held in stocks only, and $\delta(t,T) = e^{-r(T-t)}$ is a discount factor.

The boundary values to (1.22) are given as

$$V_i(T) = \begin{cases} y_i(S,T)S(T) - k(S, y_i(S,T)) & , i = wo \\ y_i(S,T)S(T) - k(S, y_i(S,T)) - \max\{S - K, 0\} & , i = w. \end{cases} \tag{1.23}$$

This variational equation is a free boundary problem, consisting of three different solution regions in (S, y)-space at time t, as seen in Figure 1.4. One where it is optimal to sell stocks, one where it is optimal to buy stocks and one where it is optimal to do nothing. The proposed option value is now

$$P = V_{wo} - V_w. \tag{1.24}$$

Solving a free boundary problem in three dimensions is computationally expensive, and [14, 15] hence proposed making an asymptotic expansion of V to reduce the complexity of the problem. Assuming proportional transaction costs

$$k(S, t, y) = \kappa S y(S, t) \tag{1.25}$$

of order λ, the proposed expansion is

$$V(S, t, y) = yS + V_0(S, t, y) + \lambda^{1/4} V_1(S, t, y) + \dots \tag{1.26}$$

which reveals the option price

$$P(S, t) = V^{BS}(S, t) + \lambda^{1/2}(V_2^{wo} - V_2^{w}) + \dots \tag{1.27}$$

where V^{BS} is the solution to the ordinary Black-Scholes equation with boundary value

$$V^{BS}(S, T) = \max\{S - K, 0\} \tag{1.28}$$

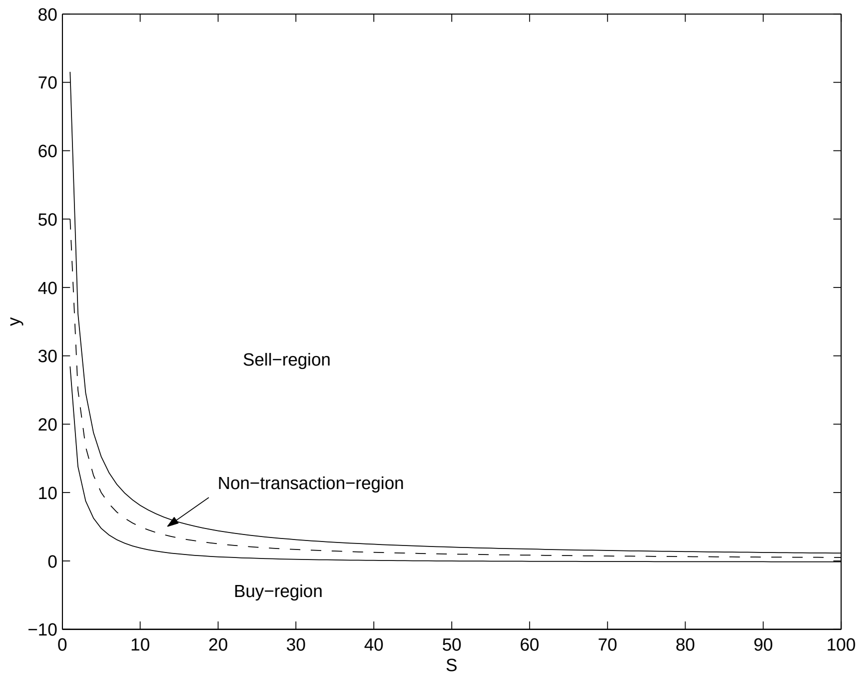

Figure 1.4. Solution regions: If (S, y) is located outside the non-transaction region the number of shares, y, will change until a boundary is hit. The dashed line in the middle indicates the ideal number of shares when no transaction costs are present.

and V_2^i is the solution to a perturbed Black-Scholes equation.

The main result from the asymptotic expansion is that it yields expressions for the boundaries between the solution regions. The optimal hedging strategy, found by [14, 15], for each portfolio is to do nothing if

$$y_i^* + Q_i^- \leq y_i(S,t) \leq y_i^* + Q_i^+ \tag{1.29}$$

where Q_i^+ and Q_i^- denotes the upper and lower boundaries and y_i^* denotes the ideal strategy in case of no transaction costs. The ideal strategy is

$$y_i^*(S,t) = \begin{cases} V_S^{BS} + \frac{\delta(t,T)(\mu - r)}{\gamma S \sigma^2} & , i = w \\ \frac{\delta(t,T)(\mu - r)}{\gamma S \sigma^2} & , i = wo \end{cases} \tag{1.30}$$

and the boundaries are given by

$$Q_i^+ = -Q_i^- = \left(\frac{3\kappa S \delta(t,T)}{2\gamma} (\frac{\partial y_i^*}{\partial S})^2 \right)^{1/3}. \tag{1.31}$$

According to the strategy presented by Davis *et al* [4] in Equation 1.20, the center of the non-transaction region is

$$y^* = y_w^* = V_S^{BS} + \frac{\delta(t,T)(\mu - r)}{\gamma S \sigma^2} \tag{1.32}$$

and the boundaries by

$$Q = Q_w. \tag{1.33}$$

On the other hand, according to the Whalley-Wilmott strategy [14, 15] in Equation 1.21 the center is given by

$$y^* = y_w^* - y_{wo}^* = V_S^{BS} \tag{1.34}$$

and the boundaries are

$$Q = Q_w - Q_{wo}. \tag{1.35}$$

In Figure 1.5 an example of hedging according to the strategy proposed by [14, 15] is showed.

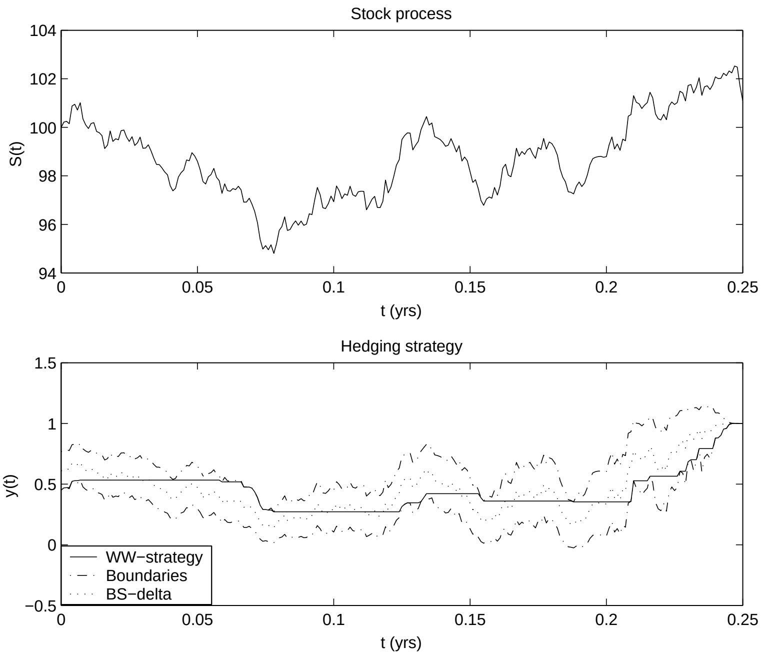

Figure 1.5. Example of hedging between boundaries. We see that for Whalley-Wilmott's strategy hedging strategy the number of shares, y, is held constant until a boundary is hit.

It is worth noting that as γ goes to infinity the price equals the Black-Scholes price, and the hedging ratio is equal to the option delta.

More about the expansion can be found in Chapter 3.

Chapter 2

Utility Theory

In this chapter we will derive the “utility maximized” approach introduced in section 1.2, but assuming that volatility is in fact stochastic and mean-reverting.

2.1 The Model

Consider the problem of how a writer of an option should hedge his position in the underlying to minimize his risk. If he does not hedge his position by holding some amount of the underlying he will face a great risk of losing money at the time of delivery. On the other hand, if he hedges continuously he will in practice get enormous trading costs.

A model to face this problem for European stock options was, as mentioned in Chapter 1, proposed by Davis, Panas and Zariphopoulou [4] in 1993 by using a maximal utility approach along with the concept of stochastic control. We will here derive a similar technique but also assume that the stock price process has a mean-reverting stochastic volatility contrary to the constant volatility assumption of [4]. The volatility is given as a function $f(Y)$ of a stochastic volatility process $Y(t)$.

First, define as in [4]:

1. One portfolio with value function W_{wo} containing $y_{wo}(t)$ stocks and the amount $B_{wo}(t)$ in riskfree assets (cash). The stock price is driven by a stochastic process $S(t)$.

2. One portfolio with value function W_w containing $y_w(t)$ stocks, the amount $B_w(t)$ in riskfree assets and a short position in an option with strike price K.

3. A concave increasing utility function U with $U(0) = 0$. The utility function defines the writer’s attitude towards risk.

Index w and wo thus indicates if the portfolios contain a short position in an option or not.

The final value at time of delivery for each of the two portfolios is given as

$$W(T, S, B, y, Y) = \begin{cases} B + c(y, S) & , i = wo \\ B + 1_{\{S \leq K\}} c(y, S) & \\ +1_{\{S > K\}} [c(y - 1, S) + K] & , i = w \end{cases} \tag{2.1}$$

where $c(y, S)$ denotes the cash value of liquidating our stock positions. The cash value is given as:

$$c(y, S) = \begin{cases} (1 + \kappa) y S & , y < 0 \\ (1 - \kappa) y S & , y > 0 \end{cases} \tag{2.2}$$

Note that this expression differs slightly from the boundary values given in (1.17), since we have restricted the transaction costs to be proportional.

The model by [4] tries to maximize the writer's utility of wealth at time of delivery by finding an optimizing trading strategy for $y(t)$ and $B(t)$ over the time interval [0,T]. To achieve this we have to know the state dynamics of the problem.

2.2 Market Dynamics

The state dynamics of our market model are given as

$$\begin{aligned} dB &= rBdt - (1 + \kappa) S dL_{buy} + (1 - \kappa) S dL_{sell} \\ dy &= dL_{buy} - dL_{sell} \\ dS &= \mu S dt + f(Y) S dZ_1 \\ dY &= \alpha(m - Y)dt + \beta(\rho dW + \sqrt{1 - \rho^2} dZ_2) \end{aligned} \tag{2.3}$$

where the variables are:
$B(t)$ - amount invested in bank (cash) with interest rate r,
$y(t)$ - number of stocks,
$S(t)$ - stock price process,
$f(Y)$ - volatility,
$Y(t)$ - volatility driving process,
$L_{buy}(t)$, $L_{sell}(t)$ - cumulative number of stocks bought or sold up to time t.
$Z_1(t)$ and $Z_2(t)$ are independent Brownian motions.

and the constants are:
κ - bid/ask spread i.e. the transaction cost per unit of traded stock,
μ - drift of stock price process,
α - rate of mean reversion of $Y(t)$,

β - volatility of the volatility driving process $Y(t)$ and
ρ - correlation between $Z_1(t)$ and $\hat{Z} = (\rho Z_1 + \sqrt{1-\rho^2} Z_2)$.

Note that the pair $(B(t),y(t))$ is the solution to the first two equations in (2.3) corresponding to the processes $(L_{buy}(t),L_{sell}(t))$ which can be seen as control processes.

$L_{buy}(t)$ and $L_{sell}(t)$ are both monotone increasing $\Re^+$-valued controls with $L_j(0^-) = 0$. The index, j, here belongs to the set $\{sell, buy\}$. A jump in the beginning is possible so $L_j(0)$ can be positive.

2.3 Admissible Controls

To ensure a solution to our control problem we must put two restrictions on the control processes $L_j(t)$.

1. $L_j(t)$ must be non-decreasing right continuous processes adapted to $\mathcal{F}$, the σ-algebra generated by the price process $S(t)$.

2. Due to the existence of transaction costs, the number of shares $y(t)$ and consequently the controls $L_j(t)$ must have finite variation.

2.4 Value Function

Our objective is now to find a trading strategy, i.e. to find $B(t)$ and $y(t)$ (or equivalently $L_j(t)$) over the time interval $[0, T]$, that maximizes the terminal utility of each of the value functions.

This is carried out by defining the following value functions for the portfolios

$$J_i(t, S, B_i, y_i, Y) = \sup_{L_{buy}, L_{sell}} E[U(W_i(T)] \quad i = w, wo. \tag{2.4}$$

So far nothing else other than the inclusion of an extra variable Y differs from the original problem derived in [4].

2.5 Hamilton-Jacobi-Bellman Equation

From Equation 2.4 we see that the dynamic programming principle gives rise to a Hamilton-Jacobi-Bellman equation. To derive the HJB-equations, [4] defines a smaller class of trading strategies such that $L_j(t)$ are absolutely continuous processes given by

$$L_j(t) = \int_0^t dl_j(\tau), \qquad 0 \le l_j < k \tag{2.5}$$

where l_j are uniformly bounded by $k < \infty$. When $k \to \infty$ this set of admissible controls will approach the set of admissible controls defined in Section 2.3.

From (2.4) we see that the portfolios have the same corresponding Hamilton-Jacobi-Bellman PDE but with different boundary values.

The PDE is given by

$$\begin{aligned} \max_{0 \le l_{buy}, l_{sell} \le k} \quad & \left\{ [J_y - (1+\kappa)SJ_B] l_{buy} - [J_y - (1-\kappa)SJ_B] l_{sell} \right\} \\ & + J_t + rBJ_B + \mu SJ_S + \tfrac{f(Y)^2 S^2}{2} J_{SS} \\ & + \alpha(m - Y)J_Y + f(Y)\rho\beta SJ_{SY} + \tfrac{\beta^2}{2} J_{YY} = 0 \end{aligned} \tag{2.6}$$

and the boundary values by

$$J(T, S, B, y, Y) = W_i(T, S, B, y, Y) \tag{2.7}$$

for each portfolio.

We can easily derive three optimal trading strategies:

1. $[J_y - (1-\kappa)SJ_B] < 0$ and $[J_y - (1+\kappa)SJ_B] \le 0$

   $\Rightarrow l_{buy} = 0, l_{sell} = k$
2. $[J_y - (1-\kappa)SJ_B] \ge 0$ and $[J_y - (1+\kappa)SJ_B] > 0$

   $\Rightarrow l_{buy} = k, l_{sell} = 0$
3. $[J_y - (1-\kappa)SJ_B] \le 0$ and $[J_y - (1+\kappa)SJ_B] \ge 0$

   $\Rightarrow l_{buy} = l_{sell} = 0$

The first case implies selling stocks at the maximal rate k, the second buying stocks at the maximal rate k, and finally doing nothing. All other strategies are non-optimal since they imply buying and selling at the same time.

From Equation 2.6 we see that as $k \to \infty$ the three trading strategies imply an equation valid for each strategy:

1. Selling is optimal $\Leftrightarrow J_y - (1+\kappa)SJ_B = 0$

2. Buying is optimal $\Leftrightarrow J_y - (1-\kappa)SJ_B = 0$

3. Doing nothing is optimal $\Leftrightarrow$

$$J_t + rBJ_B + \mu SJ_S + \frac{f(Y)^2 S^2}{2} J_{SS} + \alpha(m-Y)J_Y + f(Y)\rho\beta SJ_{SY} + \frac{\beta^2}{2} J_{YY} = 0$$

Since none of these equations can hold at the same time they will form a free boundary problem dividing the (S, y)-space into three regions, the sell region, the buy region and the non-transaction region. This can be summarized in the form of a variational equation

$$\max\Big\{ J_y - (1+\kappa)SJ_B, -[J_y - (1-\kappa)SJ_B], \quad J_t + rBJ_B + \mu SJ_S + \tfrac{f(Y)^2 S^2}{2} J_{SS} \quad + \alpha(m-Y)J_Y + f(Y)\rho\beta SJ_{SY} + \tfrac{\beta^2}{2} J_{YY} \Big\} = 0 \tag{2.8}$$

with boundary value

$$J(T, S, B, y, Y) = W_i(T, S, B, y, Y) \tag{2.9}$$

for each of the two portfolios.

2.6 Option Pricing

To determine the option price we define as in [4]

$$\hat{B}_i = \inf\{B_i | J_i(0, S, B_i, y_i = 0, Y) \geq 0\}. \tag{2.10}$$

which is the minimum initial wealth which delivers a non-negative maximum expected utility of final wealth for each of the portfolios (following an admissible trading strategy starting with initial capital B_i and $y_i = 0$ shares).

$\hat{B}_i$ has the following interpretation:

1. For the portfolio containing a short position of an option the writer is indifferent between “doing nothing” and “accepting $\hat{B}_w$ and writing the option”.

2. While for the portfolio containing only stocks and riskfree assets the writer is indifferent between “investing the amount $-\hat{B}_{wo}$ and following the utility maximizing trading strategy” and “doing nothing”. Note that $\hat{B}_{wo}$ is negative since $J_i(t, S, B_i, y_i, Y) \geq 0$.

The option writing price proposed by [4] is then given by

$$P = \hat{B}_w - \hat{B}_{wo}. \tag{2.11}$$

The primary justification for this price is that, for constant volatility, it coincides with the Black-Scholes price in the case of no transaction costs. [4] also showed that if a self-financing replicating portfolio exists, i.e. a strategy starting with $(B(0), y(0)) = (B, 0)$ and ending with $(B(T), y(T)) = 1_{\{S>K\}}(-K, 1)$, the price is given by $P = B$. The process $Y(t)$ does not affect this result.

To see this we give a heuristic proof:

1. Let π be an element of the form $(B(t), y(t))$ belonging to the space of admissible strategies $\mathcal{T}(B)$ starting with $(B(0), y(0)) = (B, 0)$. Assume that $\mathcal{T}(B)$ is a linear space and $J_i(B)$ are continuous and strictly increasing functions.

2. Since $\mathcal{T}(B)$ is a linear space a strategy $\pi \in \mathcal{T}(B)$ can be divided into two arbitrary separate strategies $\pi = \pi_1 + \pi_2$ where $\pi_1 \in \mathcal{T}(B_1)$ and $\pi_2 \in \mathcal{T}(B_2)$ with $B = B_1 + B_2$.

   Let π_1 be a self-financing replicating strategy, i.e. a strategy starting with $(B_1, 0)$ and ending with $1_{\{S>K\}}(-K, 1)$, and let $\hat{B}_w = B$.

   This gives that

$$\begin{aligned}
0 = J_w(B) &= \sup_{\pi\in\mathcal{T}(B)} E[U(B(T) + 1_{\{S\le K\}}c(y_w(T), S(T)) \\
&\quad +1_{\{S>K\}}(K + c(y_w(T) - 1, S(T))))] \\
&= \sup_{\pi_2\in\mathcal{T}(B-B_1)} E[U(B_1(T) + B_2(T) + 1_{\{S\le K\}}c(y_1(T) + y_2(T), S(T)) \\
&\quad +1_{\{S>K\}}(K + c(y_1(T) + y_2(T) - 1, S(T))))] \\
&= \sup_{\pi_2\in\mathcal{T}(B-B_1)} E[U(B_2(T) - K1_{\{S>K\}} + 1_{\{S\le K\}}c(y_2(T), S(T)) \\
&\quad +1_{\{S>K\}}(K + c(y_2(T), S(T))))] \\
&= \sup_{\pi_2\in\mathcal{T}(B-B_1)} E[U(B_2(T) + c(y_2(T), S(T))] \\
&= J_{wo}(B - B_1)
\end{aligned}$$

   Since $J_w(B) = J_{wo}(B - B_1) = 0$ and $B = \hat{B}_w$, we have $B - B_1 = \hat{B}_{wo}$ and $P = \hat{B}_w - \hat{B}_{wo} = B_1$. The price of the contract is thus the same as the amount invested in the replicating portfolio. We also see that the strategy π_1 which leads to the price B_1 is given by $\pi_1 = \pi - \pi_2 = \pi_w - \pi_{wo}$.

Another explanation for the price proposed by [4] is that the cash holdings when writing an option involves one component to hedge the claim and one to play the market. The second component must thus be subtracted to achieve the correct price.

2.7 Reducing Dimension

Summarizing the last sections, we now have a free boundary problem in five dimensions. To reduce the problem one dimension we let the utility function to be $U(x) = 1 - e^{-\gamma x}$ and rewrite J_i according to [14] in the form

$$J_i(t, S, B_i, y_i, Y) = 1 - e^{-\frac{\gamma}{\delta(t,T)}(B_i + V_i(t,S,y_i,Y))} \tag{2.12}$$

where V_i is the expected value of the utility maximized wealth of the risky asset discounted to time t, and $\delta(t, T) = e^{-r(T-t)}$ is a discount factor. B_i is the amount held in the risk-free asset at time t.

This transforms the problem to

$$\begin{aligned} \min\Big\{ & V_y - (1+\kappa)S, -[V_y - (1-\kappa)S], \\ & V_t - rV + \mu S V_S + \tfrac{f(Y)^2 S^2}{2}(V_{SS} - \tfrac{\gamma}{\delta(t,T)}V_S^2) + \alpha(m - Y)V_Y \\ & + f(Y)\rho\beta S(V_{SY} - \tfrac{\gamma}{\delta(t,T)}V_S V_Y) + \tfrac{\beta^2}{2}(V_{YY} - \tfrac{\gamma}{\delta(t,T)}V_Y^2)\Big\} = 0 \end{aligned} \tag{2.13}$$

with boundary values

$$V_i(T, S, y, Y) = \begin{cases} c(y, S) & , i = wo \\ 1_{\{S \le K\}} c(y, S) & \\ +1_{\{S > K\}}[c(y-1, S) + K] & , i = w \end{cases} \tag{2.14}$$

From (2.12) it is evident that $\inf\{B_i | J_i(0, S, B_i, y_i = 0, Y) \ge 0\}$ is achieved only by letting $B_i = -V_i$. Thus we have $\hat{B}_i = -V_i$ and the following option price

$$P = \hat{B}_w - \hat{B}_{wo} = V_{wo} - V_w. \tag{2.15}$$

Note that if $y(T)$ is restricted to be positive we get

$$V_i(T, S, y, Y) = \begin{cases} (1-\kappa)yS & , i = wo \\ (1-\kappa)yS - \max\{S - K, 0\} + 1_{\{S > K\}}\kappa S & , i = w \end{cases} \tag{2.16}$$

and

$$V_{wo}(T, S, y, Y) - V_w(T, S, y, Y) = \max\{S - K, 0\} - 1_{\{S > K\}}\kappa S. \tag{2.17}$$

Chapter 3

Asymptotic Expansion

In the asymptotic expansion we focus on the non-transaction region for each portfolio i.e.

$$\begin{aligned} &V_t - rV + \mu S V_S + \tfrac{f(Y)^2 S^2}{2}(V_{SS} - \tfrac{\gamma}{\delta(t,T)} V_S^2) + \alpha(m-Y)V_Y \\ &+ f(Y)\rho\beta S(V_{SY} - \tfrac{\gamma}{\delta(t,T)} V_S V_Y) + \tfrac{\beta^2}{2}(V_{YY} - \tfrac{\gamma}{\delta(t,T)} V_Y^2) = 0. \end{aligned} \tag{3.1}$$

In Section 3.1 we will make an asymptotic expansion in the transaction costs, to find a hedging interval which determines whether it is necessary to rebalance or not for all times $[0, T]$. In Section 3.2 an expansion in the rate of mean-reversion is made to eliminate the non-observable quantity Y. For the interested reader, the complete derivation of the results presented in this chapter can be found in Appendix B.

3.1 Asymptotic Expansion in λ

To reduce the dimension of the problem we will try an asymptotic expansion in λ. The λ term is introduced to measure the size of transaction costs. When rebalancing there is an associated cost $k(S, y) = \kappa S y$ of size $O(\lambda)$.

As in [14, 15] we propose the following expansion in λ:

$$V(S, t, y, Y) = yS + V_0(S, t, y, Y) + \lambda^{1/4} V_1(S, t, y, Y) + \dots \tag{3.2}$$

From (2.16) we also note, by comparing orders of λ, that

$$V_0^i(T, S, y_i, Y) = \begin{cases} 0 & , i = wo \\ -\max\{S - K, 0\} & , i = w \end{cases} \tag{3.3}$$

so the boundary value for $-V_0^w$ is the same as in the Black-Scholes valuation of a European call option.

The reason for expanding V in orders of $\lambda^{1/4}$ may seem somewhat unclear, but it has be shown to be appropriate in Atkinson & Wilmott [1] among others. An expansion in fractions of less than $\lambda^{1/4}$ will cancel out terms leaving us with the same expansion as here. One explanation for this is that the width of the hedging interval around the ideal amount of shares held in absence of transaction costs is of order $O(\lambda^{1/4})$.

3.1.1 Change of Variables

Since, as mentioned above, the hedging interval is of order $O(\lambda^{1/4})$ it is suitable to make the transformation

$$y(S,t) = y^*(S,t) + \lambda^{1/4}Q \tag{3.4}$$

around the ideal amount of stocks, y^*, in our portfolio. The variable Q is a dimensionless and measures the difference between the number of shares actually held in the portfolio and the ideal amount held in the absence of transaction costs. We also rescale our transaction cost function as

$$k(S,\lambda^{1/4}Q) = \lambda K(S,Q) \tag{3.5}$$

so that the transaction cost function $K(S,Q) = \kappa SQ$ is of size $O(1)$.

As in [14, 15] the change of variables $V(S,t,y,Y) \rightarrow V(S,t,Q,Y)$ implies that

$$\begin{aligned}
\frac{\partial}{\partial y} &\rightarrow \lambda^{-1/4}\frac{\partial}{\partial Q} \\
\frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x} - \lambda^{-1/4}y_x^*\frac{\partial}{\partial Q} \qquad x = t, S \\
\frac{\partial^2}{\partial S^2} &\rightarrow \frac{\partial^2}{\partial S^2} - \lambda^{-1/4}\left(y_{SS}^*\frac{\partial}{\partial Q} + 2y_S^*\frac{\partial^2}{\partial Q\partial S}\right) + \lambda^{-2/4}(y_S^*)^2\frac{\partial^2}{\partial Q^2} \\
\frac{\partial^2}{\partial S\partial Y} &\rightarrow \frac{\partial^2}{\partial S\partial Y} - \lambda^{-1/4}\left(y_{SY}^*\frac{\partial}{\partial Q} + y_S^*\frac{\partial^2}{\partial Q\partial S}\right)
\end{aligned} \tag{3.6}$$

which transforms (3.2) to

$$\begin{aligned}
V(S,t,Q,Y) = {} & (y^* + \lambda^{1/4}Q)S + V_0(S,t,Q,Y) \\
& + \lambda^{1/4}V_1(S,t,Q,Y) + \ldots
\end{aligned} \tag{3.7}$$

Denoting the boundary between the sell region and non-transaction region as Q^+ and similarly the boundary between the non-transaction region and the buy region as Q^-, one sees from (2.13) and the change of variables that

$$V_Q(S,t,Q^+,Y) = (\lambda^{1/4} + \lambda\kappa)S \tag{3.8}$$

and

$$V_Q(s,t,Q^-,Y) = (\lambda^{1/4} - \lambda\kappa)S. \tag{3.9}$$

One also has a smoothness condition at the boundary

$$V_{QQ}(S,t,Q^+,Y) = V_{QQ}(S,t,Q^-,Y) = 0. \tag{3.10}$$

3.1.2 Comparing Orders of λ

Changing variables in (3.1) according to (3.6), and inserting (3.7) followed by comparing orders of λ yields the following main results (the complete derivation is found in appendix B):

1. From the $O(\lambda^{-2/4})$ equation we see that V_0, V_1, V_2 and V_3 are independent of Q and thus

$$\begin{aligned} V(t,S,Q,Y) = & \ (y^* + \lambda^{1/4}Q)S + V_0(S,t,Y) \\ & + \lambda^{1/4}V_1(S,t,Y) + \lambda^{1/2}V_2(S,t,Y) \\ & + \lambda^{3/4}V_3(S,t,Y) + \lambda V_4(S,t,Q,Y) + \ldots \end{aligned} \tag{3.11}$$

2. From the $O(\lambda^{1/4})$ equation we get an expression for the ideal number of stocks

$$y^*(S,t) = -\langle V_{0S}(S,t,Y)\rangle + \frac{\delta(t,T)(\mu - r)}{\gamma S\langle f(Y)^2\rangle}. \tag{3.12}$$

   We have here used the averaged volatility with respect to the invariant distribution Φ of the OU-process

$$\langle f(Y)\rangle = \int_{-\infty}^{\infty} f(Y)\Phi(Y)dY = 0.$$

   The $O(\lambda^{1/4})$ equation also states that $V_1 = 0$.

3. The $O(\lambda^{2/4})$ combined with $V_1 = 0$, can be solved for V_4. The boundary conditions (3.8), (3.9) and (3.10) will then reveal the boundaries between the solution regions

$$Q^+ = -Q^- = \left(\frac{3\kappa S\delta(t,T)(y_S^*)^2}{2\gamma}\right)^{1/3}. \tag{3.13}$$

   Thus if $y^* + Q^- \leq y(S,t) \leq y^* + Q^+$, we are in the non-transaction region.

3.2 Asymptotic Expansion in $\sqrt{\epsilon}$

We now focus only on the V_0 term to achieve a complete expression for the hedging interval, and expand V_0 in $\epsilon = 1/\alpha$ as

$$V_0 = V_{00} + \sqrt{\epsilon} V_{01} + \epsilon V_{02} + \epsilon^{3/2} V_{03} + \ldots \quad (3.14)$$

By inserting (3.14) in the equation derived from comparing terms of $O(1)$ in our λ-expansion we reach the following conclusions (see Section B.2 for details):

1. V_{00} and V_{01} are independent of Y.
2. V_{00} is the solution to Black-Scholes equation plus the particular solution

$$V_{00} = -V_{BS} + (T-t)\frac{\delta(t,T)(\mu-r)^2}{2\gamma}\langle\frac{1}{f(Y)^2}\rangle \quad (3.15)$$

   with boundary values

$$V_{00}^i(T,S,y_i,Y) = \begin{cases} 0 & ,i = wo \\ -\max\{S-K,0\} & ,i = w \end{cases} \quad (3.16)$$

   derived from Equation 2.16.

3. Defining $\bar{V}_{01} = \sqrt{\epsilon} V_{01}$, the solution to $\bar{V}_{01}$ is given as

$$\begin{aligned} \mathcal{L}_2(\bar{\sigma})\bar{V}_{01} = \; & A_1(2S^2 V_{00SS} + S^3 V_{00SSS}) + A_2(\mu-r)S^2 V_{00SS} \\ & + A_3\tfrac{\delta(t,T)(\mu-r)^3}{\gamma} - \tfrac{\delta(t,T)(\mu-r)^2}{2\gamma\sqrt{\alpha}}\langle\tfrac{1}{f(Y)^2}\rangle \end{aligned} \quad (3.17)$$

   with boundary value $\bar{V}_{01}(T,S,y_i,Y) = 0$, and

$$\begin{aligned} A_1 &= \tfrac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle \\ A_2 &= \tfrac{\rho\nu}{\sqrt{2\alpha}}\langle\tfrac{\phi'}{f}\rangle \\ A_3 &= \tfrac{\rho\nu}{\sqrt{2\alpha}}\langle\tfrac{\psi'}{f}\rangle. \end{aligned} \quad (3.18)$$

The definition of $\bar{V}_{01}$ may seem unclear, but it simply moves $\epsilon = 1/\alpha$ to the terms A_1, A_2 and A_3 on right side of (3.17). In Section 4.2 we shall see that this step has a big advantage: It is no longer necessary to give an estimate of the stochastic volatility parameter ϵ to calculate the first two terms in V_0. Instead one can estimate the parameters A_1, A_2 and A_3 from implied volatility data.

3.3 Summary

In conclusion, from the asymptotic expansions, we derived the optimal number of shares to hold in our portfolios to be

$$y_i^*(S,t) = \begin{cases} -V_{0S}^i + \frac{\delta(t,T)(\mu-r)}{\gamma S\bar{\sigma}^2} & , i = w \\ \\ \frac{\delta(t,T)(\mu-r)}{\gamma S\bar{\sigma}^2} & , i = wo \end{cases} \tag{3.19}$$

where $\bar{\sigma} = \sqrt{\langle f(Y)^2 \rangle}$ is the effective volatility, and where

$$V_{0S}^i = V_{00S}^i + \bar{V}_{01S}^i + \ldots \tag{3.20}$$

From the asymptotic expansion in ϵ we saw that

$$V_{00}^i = -V_{BS} + (T-t)\frac{\delta(t,T)(\mu-r)^2}{2\gamma}\langle\frac{1}{f(Y)^2}\rangle \tag{3.21}$$

with boundary value

$$V_{00}^i(T,S,y_i) = \begin{cases} 0 & , i = wo \\ -\max\{S-K,0\} & , i = w \end{cases} \tag{3.22}$$

$\bar{V}_{01}^i$ is given by the equation

$$\mathcal{L}_2(\bar{\sigma})\bar{V}_{01}^i = H(S,t) + A_3\frac{\delta(t,T)(\mu-r)^3}{\gamma} - \frac{\delta(t,T)(\mu-r)^2}{2\gamma\sqrt{\alpha}}\langle\frac{1}{f(Y)^2}\rangle \tag{3.23}$$

with boundary values $\bar{V}_{01}^i(T,S,y_i) = 0$, and

$$H(S,t) = A_1(2S^2V_{00SS}^i + S^3V_{00SSS}^i) + A_2(\mu-r)S^2V_{00SS}^i. \tag{3.24}$$

This is a perturbed Black-Scholes equation, and the solution is simply

$$\bar{V}_{01}^i = -(T-t)\Big(H(S,t) + A_3\frac{\delta(t,T)(\mu-r)^3}{\gamma} - \frac{\delta(t,T)(\mu-r)^2}{2\gamma\sqrt{\alpha}}\langle\frac{1}{f(Y)^2}\rangle\Big). \tag{3.25}$$

Chapter 4

The Complete Model

We will here summarize the results from previous chapters.

4.1 The Optimal Strategy

For the writer of an European call option we have derived an optimal hedging strategy based on the price moves of the underlying asset. The strategy is based on the assumption that the volatility of the underlying stock price is mean-reverting. This is shown to be the case in studies on both the S&P500-index (see [6, 7]) and the Swedish OMX-index (see [12]).

Assuming that the writer has a risk aversion factor γ and invests in y shares, it is optimal to trade so that

$$y^* + Q^- \leq y(S,t) \leq y^* + Q^+. \tag{4.1}$$

Thus, at each moment, we have to calculate the boundaries and the ideal amount to be held in our portfolio in the case of no transaction costs. If y does not satisfy the above criteria we change y, via selling or buying shares, until we reach the boundaries and condition above is satisfied. This procedure has to be carried out throughout all times $[0, T]$.

Now, remember that [4] and [14, 15] derived different hedging strategies (see Section 1.2.2). In [14, 15] a hedge according to the difference between the two portfolios was proposed, while [4] only uses the hedge for the portfolio with an option obligation.

The ideal amount y^*, proposed in [14, 15], using the difference between portfolios, is then given by

$$y^*(S,t) = V_S^{BS}(S,t) - \bar{V}_S(S,t) + \ldots \tag{4.2}$$

while the ideal amount, proposed in [4], given by the portfolio with an option obligation is

$$y^*(S,t) = V_S^{BS}(S,t) - \bar{V}_S(S,t) + \frac{\delta(t,T)(\mu - r)}{\gamma S \bar{\sigma}^2} + \dots \quad . \tag{4.3}$$

The first term in the two equations is the Black-Scholes $\Delta(S,t)$ calculated from the Black-Scholes equation with averaged volatility $\bar{\sigma}$

$$\frac{\partial V^{BS}}{\partial t} + \frac{1}{2}\bar{\sigma}^2 S^2 \frac{\partial^2 V^{BS}}{\partial S^2} + rS\frac{\partial V^{BS}}{\partial S} = rV^{BS}, \tag{4.4}$$

with boundary value

$$V^{BS}(S,T) = \max\{S - K, 0\}. \tag{4.5}$$

The second term is given by

$$\bar{V}_S = -(T-t)\frac{\partial H(S,t)}{\partial S} \tag{4.6}$$

where

$$H(S,t) = -A_1\left(2S^2 V_{SS}^{BS} + S^3 V_{SSS}^{BS}\right) - A_2(\mu - r)S^2 V_{SS}^{BS} \tag{4.7}$$

and

$$\begin{aligned} A_1 &= \tfrac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle, \\ A_2 &= \tfrac{\rho\nu}{\sqrt{2\alpha}}\langle \tfrac{\phi'}{f}\rangle. \end{aligned} \tag{4.8}$$

Using only the first two terms in (4.2) and the first three terms in (4.3) , the respective boundaries are given by

$$\begin{aligned} Q^+ &= -Q^- = \left(\tfrac{3\kappa S\delta(t,T)(y_S^*)^2}{2\gamma}\right)^{1/3} \\ &= \left(\tfrac{3\kappa S\delta(t,T)}{2\gamma}\right)^{1/3}\left(|V_{SS}^{BS} - \bar{V}_{SS} - \tfrac{\delta(t,T)(\mu-r)}{\gamma S^2\bar{\sigma}^2}|^{2/3} - |\tfrac{\delta(t,T)(\mu-r)}{\gamma S^2\bar{\sigma}^2}|^{2/3}\right) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} Q^+ &= -Q^- = \left(\tfrac{3\kappa S\delta(t,T)(y_S^*)^2}{2\gamma}\right)^{1/3} \\ &= \left(\tfrac{3\kappa S\delta(t,T)}{2\gamma}\right)^{1/3}\left(|V_{SS}^{BS} - \bar{V}_{SS} - \tfrac{\delta(t,T)(\mu-r)}{\gamma S^2\bar{\sigma}^2}|\right)^{2/3}. \end{aligned} \tag{4.10}$$

In reality one will have to use $|Q^+|$ and $|Q^-|$ in (4.1) since the boundaries in (4.9) occasionally may change place with each other.

4.2 Using Implied Volatilities

To calculate a strategy for a given level of risk aversion we would, according to (4.7) and (4.8) need to estimate the parameters ρ, ν and α in the stochastic volatility process, and calculate $\bar{\sigma}$, $\langle f\phi' \rangle$ and $\langle \frac{\phi'}{f} \rangle$. This is a cumbersome procedure which will result in inaccurate and unstable estimates.

Luckily, in real life, we only need to estimate the parameters $\bar{\sigma}$, A_1 and A_2 from market data. To see this we follow a method from [6, 7] where using implied volatility yields estimates for both A_1 and A_2.

From (1.6) and (1.7) we can derive the following expressions

$$V_S^{BS} = N(d_1) \tag{4.11}$$

$$V_{SS}^{BS} = \frac{e^{-d_1^2/2}}{S\bar{\sigma}\sqrt{2\pi(T-t)}} \tag{4.12}$$

$$V_{SSS}^{BS} = \frac{-e^{-d_1^2/2}}{S^2\bar{\sigma}\sqrt{2\pi(T-t)}}\Big(1+\frac{d_1}{\bar{\sigma}\sqrt{T-t}}\Big) \tag{4.13}$$

where

$$d_1(S,t) = \frac{1}{\bar{\sigma}\sqrt{T-t}}\Big\{ln(\frac{S}{K}) + (r+\frac{\bar{\sigma}^2}{2})(T-t)\Big\}. \tag{4.14}$$

Inserted into (4.7) this gives

$$H(S,t) = \frac{-Se^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi(T-t)}}\Big(A_1\big(1-\frac{d_1}{\bar{\sigma}\sqrt{T-t}}\big) + A_2(\mu - r)\Big) \tag{4.15}$$

and

$$\bar{V}(S,t) = \frac{-Se^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi}}\Big(A_1\frac{d_1}{\bar{\sigma}} - \sqrt{T-t}\big(A_1 + A_2(\mu-r)\big)\Big). \tag{4.16}$$

Now, remember that the implied volatility, I, was derived from the equation

$$V(S,t,I,T,K) = V_{obs} \tag{4.17}$$

and expand $I = \bar{\sigma} + \sqrt{\epsilon}I_1 + \ldots$ in the left-hand side and use the approximated price $V = V^{BS} + \bar{V}$ on the right-hand side to get

$$V^{BS} + \sqrt{\epsilon}I_1\frac{\partial V^{BS}}{\partial\bar{\sigma}} + \ldots = V_{BS} + \bar{V} + \ldots \tag{4.18}$$

which leads to

$$\sqrt{\epsilon} I_1 = \bar{V} \Big(\frac{\partial V^{BS}}{\partial \bar{\sigma}} \Big)^{-1} \tag{4.19}$$

and the expansion

$$I = \bar{\sigma} + \bar{V} \Big(\frac{\partial V^{BS}}{\partial \bar{\sigma}} \Big)^{-1} + \dots \quad . \tag{4.20}$$

Using (4.16) and

$$\frac{\partial V^{BS}}{\partial \bar{\sigma}} = \frac{S e^{-d_1^2/2} \sqrt{T-t}}{\sqrt{2\pi}} \tag{4.21}$$

leads to

$$\begin{aligned} I &= \bar{\sigma} - \tfrac{2A_1 ln(\frac{S}{K})}{\bar{\sigma}^3 (T-t)} - \tfrac{2A_1}{\bar{\sigma}^3} \Big(r + \tfrac{\bar{\sigma}^2}{2} \Big) + \tfrac{2\big(A_1 + A_2(\mu - r)\big)}{\bar{\sigma}^2} + O(\epsilon) \\ &= a \tfrac{ln(\frac{S}{K})}{(T-t)} + b + O(\epsilon) \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} a &= -\tfrac{2A_1}{\bar{\sigma}^3} \\ b &= \bar{\sigma} - \tfrac{2A_1}{\bar{\sigma}^3} \Big(r + \tfrac{\bar{\sigma}^2}{2} \Big) + \tfrac{2}{\bar{\sigma}^2} \big(A_1 + A_2(\mu - r) \big). \end{aligned} \tag{4.23}$$

Up to order $O(\epsilon)$, I is thus an affine function of the log-moneyness-to-maturity-ratio (LMMR), and a and b can be fitted to implied volatility data. A_1 and A_2 can then easily be calculated from (4.23).

4.3 Estimating $\bar{\sigma}$ and μ

In previous section a procedure for estimating A_1 and A_2 was presented. The only parameter left to estimate is now the averaged volatility $\bar{\sigma}$ and the diffusion factor μ.

From (1.11) the stock price values can be approximated as realizations of the Euler discretization

$$\Delta S_n := S_{n+1} - S_n = S_n(\mu \Delta t_n + f(Y_n)\eta_n^1 \sqrt{\Delta t_n}) \tag{4.24}$$

and the volatility process as realizations of the backwards Euler discretization

$$\Delta Y_n := Y_{n+1} - Y_n = \alpha(m - Y_{n+1})\Delta t_n + \beta(\rho \eta_n^1 + \sqrt{1-\rho^2}\eta_n^2)\sqrt{\Delta t_n} \tag{4.25}$$

where η_n^1 and η_n^2 are independent sequences of independent $N(0,1)$ random variables. The timesteps of the data are t_n, where $n = 0, 1, 2 \dots N$ with uniform spacings $\Delta t_n = t_{n+1} - t_n$.

The diffusion factor μ can the be estimated by

$$\hat{\mu} = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\Delta S_n}{S_n \Delta t_n} \tag{4.26}$$

and the averaged historical value $\bar{\sigma}$ by

$$\hat{\bar{\sigma}} = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} D_n^2} \tag{4.27}$$

where D_n is the normalized fluctuation sequence

$$D_n := \left(\frac{\Delta S_n}{S_n} - \hat{\mu} \Delta t_n \right) \frac{1}{\sqrt{\Delta t_n}}. \tag{4.28}$$

Chapter 5

Numerical Simulations

To determine the efficiency of the different methods, they were tested against each other in a mean-variance framework.

5.1 The Mean-Variance Framework

The methods were tested in a mean-variance framework in the following manner:

1. The stochastic volatility parameters α, ν, ρ, m and $Y(0)$, the stock process and option parameters μ, r, $S(0)$, K and the cost parameters κ and γ were set.

2. N time steps were simulated for $S(t)$ and $Y(t)$ according to (4.24) and (4.25)

3. An exponential volatility function $f(Y) = e^Y$ was assumed. This is a common assumption, used in [6, 7, 9], but other positive volatility functions can be used as well.

4. The premium for writing an option was set to the Black-Scholes value using the averaged volatility $\sigma = \bar{\sigma}$. In fact, the premium depends on the particular hedging strategy, but using the same premium for all methods is better for comparison.

5. For all times, $t \in [0, T]$, the claim was hedged according to the chosen strategy and the total hedging cost was calculated for each path.

Monte Carlo simulation of step 1-5 was then performed to calculate mean and variance of the total hedging cost over I realizations.

5.2 Exponential Volatility

From assuming exponential volatility $f(Y) = e^Y$ in Equation 1.11 some useful properties from the mean-reverting Ohrstein-Uhlenbeck process emerge:

1. The average volatility can be derived as $\bar{\sigma} = e^{m+\nu^2}$
2. Our implied volatility parameters are given as

$$\begin{aligned} A_1 &= \frac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle = \frac{\rho\nu}{\sqrt{2\alpha}}\left(e^{3m+5\nu^2/2} - e^{3m+9\nu^2/2}\right) \\ A_2 &= \frac{\rho\nu}{\sqrt{2\alpha}}\langle \frac{\phi'}{f}\rangle \quad = \frac{\rho\nu}{\sqrt{2\alpha}}\left(e^{m+\nu^2/2} - e^{m+5\nu^2/2}\right) \end{aligned} \tag{5.1}$$

Thus, to make things simpler, we use theoretical expressions for $\bar{\sigma}$, A_1 and A_2 when calculating correction terms in the simulations.

5.3 Results

For all tests, if not otherwise mentioned, the parameters were set according to Table 5.1 and 5.2. We are thus hedging a short position in a European at-the-money call option with three months to maturity.

Parameter	Value
α	200
ν	0.25
ρ	-0.5
$\bar{\sigma}$	0.1
m	$log(\bar{\sigma}) - \nu^2$
$Y(0)$	m

Table 5.1. Stochastic volatility parameters: The parameters are similar to observed values from OMX-data (see [12]). Note that we have a negative correlation factor ρ.

5.3.1 Constant Volatility and No Transaction Costs

First we compared the methods proposed by Whalley & Wilmott [14, 15] and Davis, Panas & Zariphopoulou [4] for constant volatility ($\nu = 0$). We refer to the method in [14, 15] as the WW-method (see Equation 1.34 and 1.35), and to the method in [4] as the DPZ-method (see Equation 1.32 and 1.33).

In Figure 5.1 a comparison of the two methods when $\kappa = 0$ is shown. When no transaction costs are present the WW-method simply equals the Black-Scholes

Parameter	Value
T	0.25
N	250
μ	0.1
r	0.05
$S(0)$	100
K	100
κ	0.005

Table 5.2. Stock and option parameters for a European in-the-money call with three months to maturity. 250 time steps were used in all simulations.

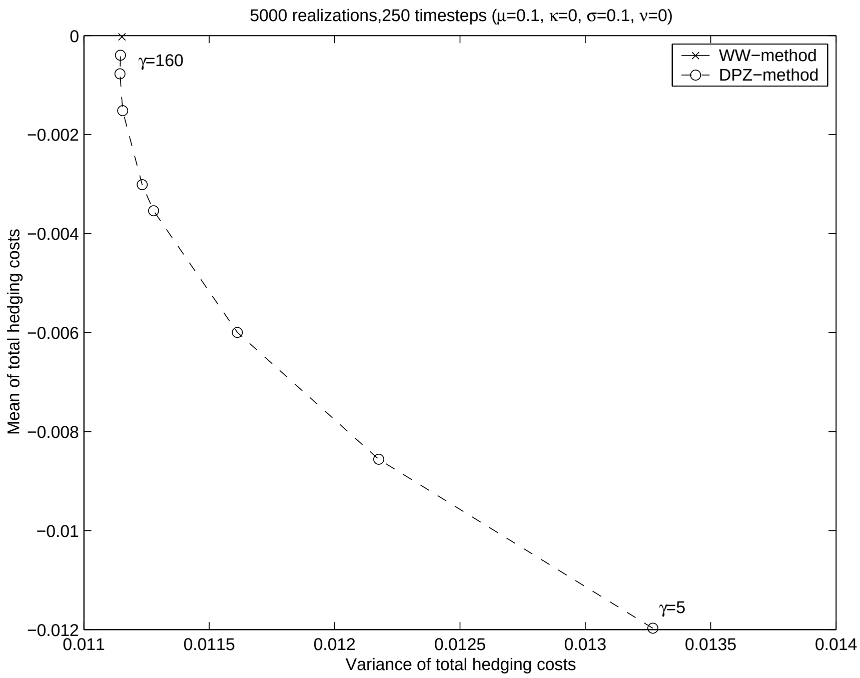

Figure 5.1. The effective frontier when $\nu = \kappa = 0$ and $\bar{\sigma} = 0.1$. The DPZ-method converges to the WW-method when γ goes to infinity.

method, and is thus independent of the risk-aversion γ. The DPZ-method, on the other hand, is sensitive to changes in γ and will converge to Black-Scholes method as γ goes to infinity.

It is clear that the DPZ-method allows us to choose what risk we are willing to take. A small γ-value leads to an increased risk, but also a greater average profit.

5.3.2 Constant Volatility and Transaction Costs

In Figure 5.2 the effective frontiers for the two methods, when transaction costs are present ($\kappa = 0.005$) and the volatility is constant ($\nu = 0$), are shown. One can clearly see that the DPZ-method produces lower average costs for a given level of variance than the WW-method, and that the difference between the methods decreases with an increasing degree of risk aversion γ. Although the DPZ-method has lower variance it is not necessarily the best method to use practically as we shall see later on.

We also see that there is a clear trade-off between average cost and risk for both methods. An increasing γ leads to reduced risk but greater average cost. Note that when transaction costs are present we can never make an average profit as in Figure 5.1. It always comes down to reducing our costs instead of increasing our profit.

In Figure 5.3 the skew (third moment) and curtosis (fourth moment) for the two methods are plotted. A positive skew indicates that the distribution has longer tails to the right of the mean value and the curtosis measures the size of the tails. The symmetric normal distribution has curtosis equal to three. The DPZ-method has both longer and heavier tails to the right than the WW-method. This means that, even though a smaller variance, the expected value of the largest, say 5%, losses (so called expected shortfall) can be higher for the DPZ-method. Which method to use therefore depends on what kind of risk one wants to minimize. We here focus primarily on minimizing variance, but one should be aware that there are also other risk measures to consider.

Sensitivity to Drift and Volatility

For both methods, only two parameters, drift μ and volatility $\bar{\sigma}$, need to be estimated. It is thus necessary for us to know something about how important it is to use good estimates. In Figure 5.4 effects on the methods due to ±50% changes in μ can be observed. The WW-method is, contrary to the DPZ-method, quite insensitive to changes in μ, and since producing stable and accurate estimates for μ is usually quite difficult, the WW-method should be preferred over the DPZ-method, even though the latter is better from a mean-variance point of view.

In Figure 5.5 it can be seen that estimating volatility significantly effects the results, but no method is better than the other. A 5% change in $\bar{\sigma}$ here produces more or less the same changes in the results.

5.3.3 Stochastic Volatility and Transaction Costs

Figure 5.6 shows a mean-variance plot of the methods with correction terms presented in Section 4.1 versus the original WW-method and DPZ-method, for the case of stochastic volatility ($\nu = 0.25$). All curves in Figure 5.6 have higher vari-

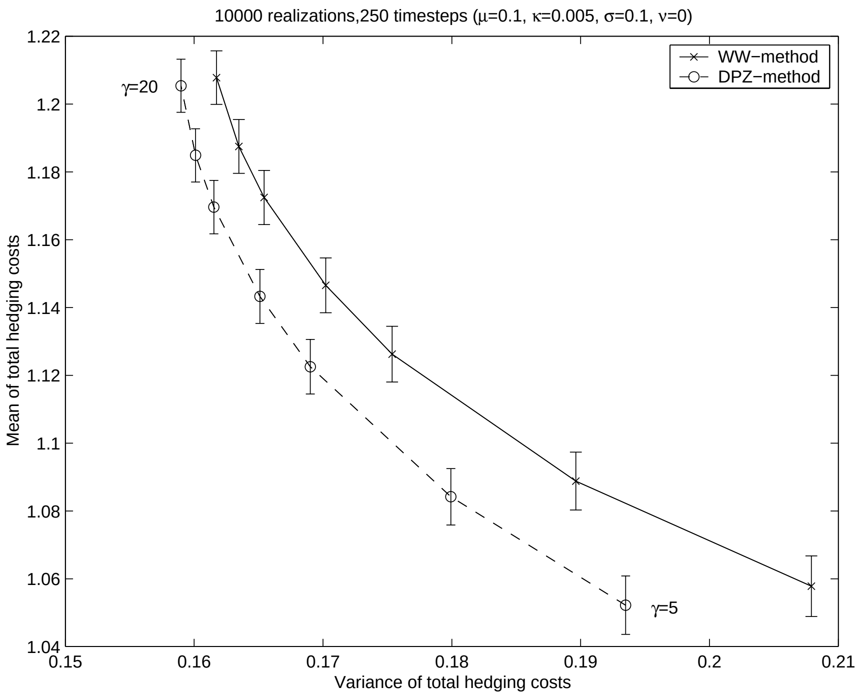

Figure 5.2. The effective frontier when $\nu = 0$, $\kappa = 0.005$ and $\bar{\sigma} = 0.1$ (Bars show 95% confidence intervals): The DPZ-method gets closer to the WW-method as γ increases.

ance than for the case of constant volatility (see Figure 5.2). We refer to the method presented in (4.2) and (4.9) as the Corrected WW-method, and the method in (4.3) and (4.10) as the Corrected DPZ-method.

It is here evident that the corrected methods will a have lower average total hedging cost than the the original methods for any given variance. Also, one can see from Figure 5.7 that the corrected methods have slightly larger tails to the right. We once again have to think of what kind risk we want to minimize.

Sensitivity to Drift and Volatility

The question is now how the corrected methods behave when giving poor estimates of volatility and drift. As shown in Section 5.3.2 both the WW and DPZ methods are sensitive to the volatility estimate, but only the DPZ-method is sensitive to the estimated drift. To see if the correction terms effect the sensitivity, the Corrected WW-method was compared with the original one, for different estimates of drift and volatility.

The sensitivity when changing the drift ±50% for the Corrected WW-method was

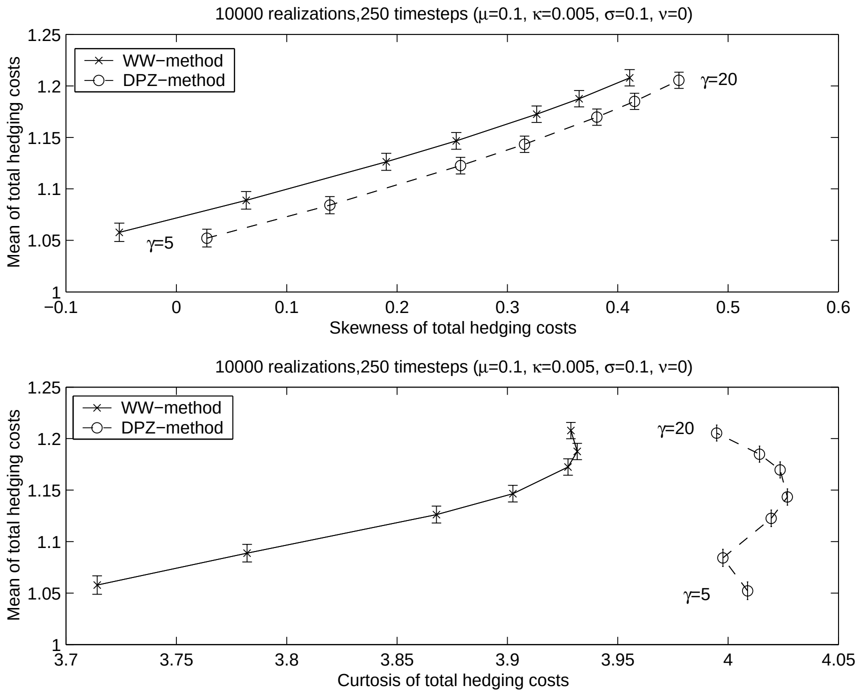

Figure 5.3. Skew and curtosis when $\nu = 0$, $\kappa = 0.005$ and $\bar{\sigma} = 0.1$ (Bars show 95% confidence intervals). Positive skew indicates that the distribution has longer tails to the right of the mean value. Curtosis measures the size of the tails (the normal distribution has curtosis equal to three).

tested to be of the same order as the ordinary WW-method in Figure 5.4. The correction terms do not significantly effect the sensitivity to drift. Also, the sensitivity when changing volatility was observed to be of the same order as in Figure 5.5. In Figure 5.8 an example of the sensitivity to changes in volatility and drift can be seen.

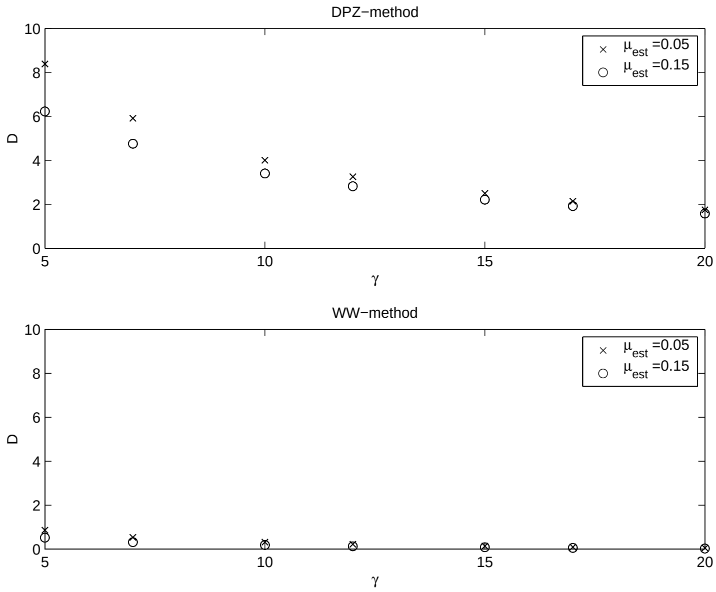

Figure 5.4. *D* indicates the absolute percentage change in variance of the total costs from a ±50% change in μ for the WW-method and the DPZ-method. DPZ is more sensitive than WW to changes in μ.

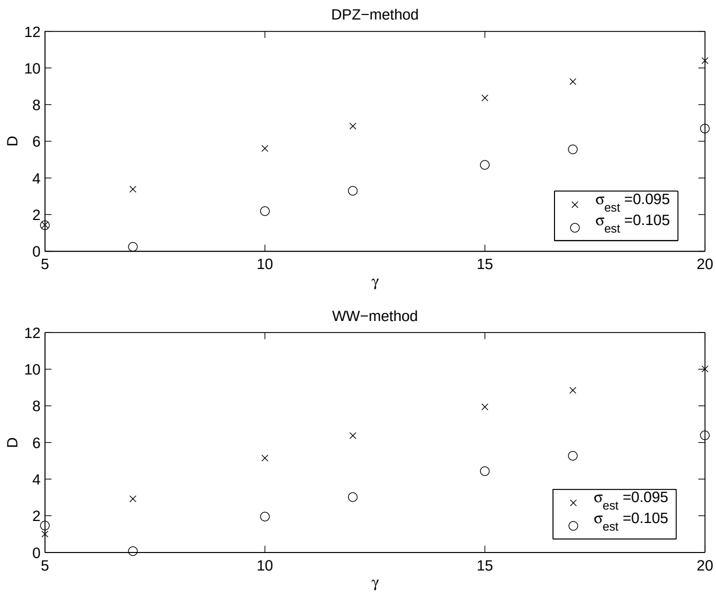

Figure 5.5. *D* indicates the absolute percentage change in variance of the total costs from a ±5% change in $\bar{\sigma}$ for the DPZ-method and the WW-method. Changes are of the same magnitude for both methods.

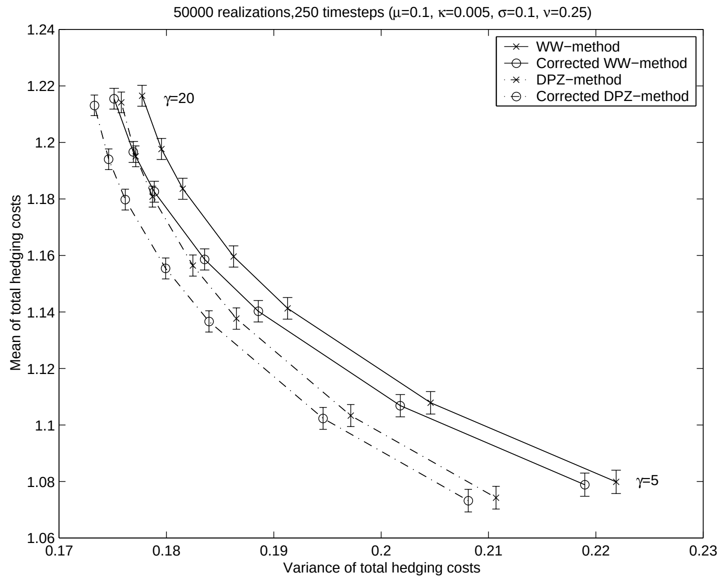

Figure 5.6. The effective frontier when $\bar{\sigma} = 0.1$, $\nu = 0.25$ and $\kappa = 0.005$ (95% confidence intervals). The corrected methods outperform the original WW-method and DPZ-method.

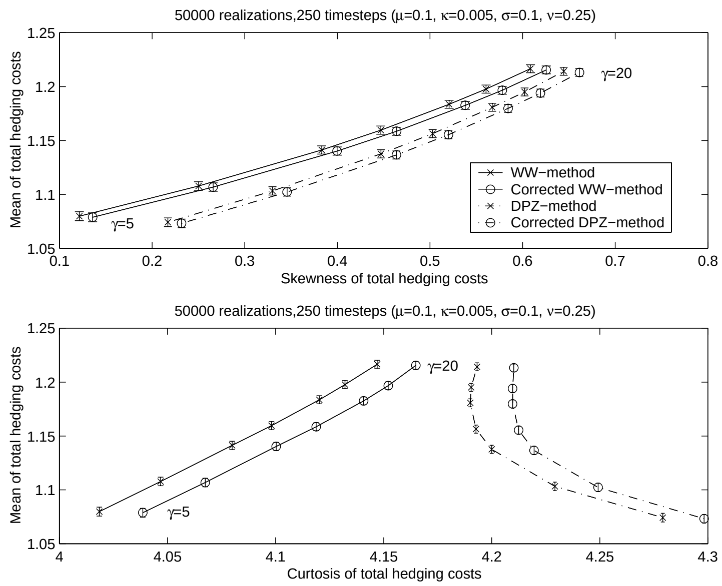

Figure 5.7. Skewness and curtosis when $\bar{\sigma} = 0.1$, $\nu = 0.25$ and $\kappa = 0.005$ (95% confidence intervals). The corrected methods have slightly heavier tails to the right.

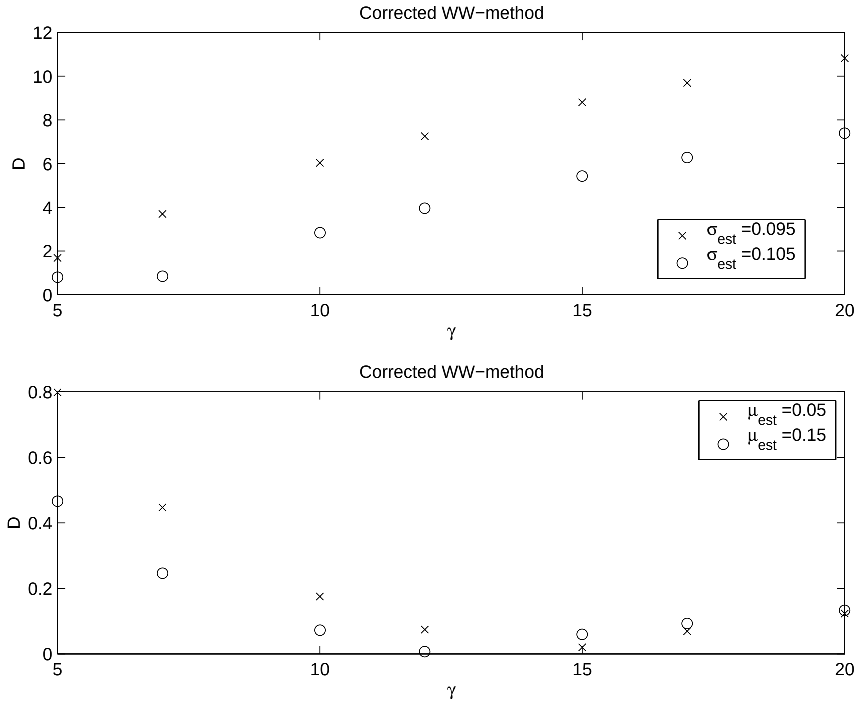

Figure 5.8. *D* indicates the absolute percentage change in variance of the total costs. In the top graph, a ±5% change in σ for the Corrected WW-method is shown. Changes are of the same magnitude as in Figure 5.5 (bottom graph). In the bottom graph, a ±50% change in μ is shown. Changes are of the same magnitude as in Figure 5.4 (bottom graph).

Chapter 6

Conclusions

We have in this thesis compared the delta-move-based theories presented by Whalley & Wilmott [14, 15] and Davis, Panas & Zariphopoulou [4] in a mean-variance framework. The performed tests concluded that the results achieved by the method presented in [4] gave lower variance for a given level of average total cost, but were highly sensitive to the estimated drift. The method presented by Whalley & Wilmott was almost insensitive to estimated drift. Since estimates of the drift usually are poor and unstable, the Whalley-Wilmott method is more appealing to use in practice.

Also, the delta-move-based theory presented in [14, 15] and [4] was extended, via correction terms, to include fast mean-reverting stochastic volatility for the underlying. Numerical simulations showed that the new corrected method, in a mean-variance framework, gave strategies with lower variance than the original methods in [4] and [14, 15].

Testing the strategies showed that estimating volatility is a crucial factor to produce good results, but the correction terms do not themselves add to the sensitivity to μ and $\bar{\sigma}$. Using the correction terms to adjust for stochastic volatility will thus give lower variance without affecting the sensitivity to the estimates.

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Appendix A

Black-Scholes Model

1. Assume that we have a self-financing portfolio $\Pi = \alpha f + \beta S$ of α options and β stocks where $d\Pi = \alpha df + \beta dS$. Self-financing is thus defined as $d\alpha f + d\beta S = 0$.
2. The market has no arbitrage opportunities. That is, given an interest rate r, there can be no risk-free investments with a higher rate of return. If $d\Pi$ is riskless we will have $d\Pi = r\Pi dt = r(\alpha f + \beta S)dt$.

We can from these assumptions derive a partial differential equation for the option price:

1. From Itô's lemma we have

$$df = \Big(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Big)dt + \frac{\partial f}{\partial S}\sigma S dW \tag{A.1}$$

2. Now construct a riskless portfolio where $d\Pi$ is deterministic:

$$\begin{aligned} d\Pi = \alpha df + \beta dS &= \alpha\Big(\big(\tfrac{\partial f}{\partial S}\mu S + \tfrac{\partial f}{\partial t} + \tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2\big)dt + \tfrac{\partial f}{\partial S}\sigma S dW\Big) \\ &+\beta\big(\mu S dt + \sigma S dW\big) \\ &= \alpha\big(\tfrac{\partial f}{\partial S}\mu S + \tfrac{\partial f}{\partial t} + \tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2\big)dt + \beta\mu S dt \\ &+\big(\alpha\tfrac{\partial f}{\partial S}\sigma S + \beta\sigma S\big)dW \end{aligned} \tag{A.2}$$

The only way to make $d\Pi$ deterministic is to let $\beta = -\alpha\frac{\partial f}{\partial S}$.
This combined with no arbitrage opportunities gives

$$\Big(\alpha\big(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\big) + \beta\mu S\Big)dt = r\alpha(f - \frac{\partial f}{\partial S})dt \tag{A.3}$$

or

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S} + \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \tag{A.4}$$

which is the Black-Scholes partial differential equation.

Appendix B

Asymptotic Expansion

B.1 Asymptotic Expansion in λ

Changing variables in (3.1) according to (3.6), and inserting (3.7) followed by comparing orders of λ yields the following calculations:

B.1.1 The $O(\lambda^{-2/4})$ Equation

For the $O(\lambda^{-2/4})$ term we get an ordinary differential equation for V_0 with independent variable Q

$$V_{0QQ} = \frac{\gamma}{\delta(t,T)} V_{0Q}^2. \tag{B.1}$$

Using our boundary conditions above we conclude that $V_{0Q} = 0$. As in Whalley-Wilmott we can arrive at similar expressions for V_1, V_2, and V_3 which together with the boundary conditions reveals $V_{1Q} = V_{2Q} = V_{3Q} = 0$ and thus

$$\begin{aligned} V(t,S,Q,Y) = &\ (y^* + \lambda^{1/4} Q)S + V_0(S,t,Y) \\ &+\lambda^{1/4} V_1(S,t,Y) + \lambda^{1/2} V_2(S,t,Y) \\ &+\lambda^{3/4} V_3(S,t,Y) + \lambda V_4(S,t,Q,Y) + \ldots \end{aligned} \tag{B.2}$$

B.1.2 The $O(1)$ Equation

The $O(1)$ equation is

$$
\begin{gathered}
V_{0t} - r(y^* S + V_0) + \mu S(y^* + V_{0S}) + \\
\tfrac{1}{2} f(Y)^2 S^2 (V_{0SS} - \tfrac{\gamma}{\delta(t,T)}(y^* + V_{0S})^2) + \alpha(m - Y)V_{0Y} \\
+ f(Y)\rho\beta S(V_{0SY} - \tfrac{\gamma}{\delta(t,T)}(y^* + V_{0S})V_{0Y}) + \tfrac{\beta^2}{2}(V_{0YY} - \tfrac{\gamma}{\delta(t,T)}V_{0Y}^2) = 0
\end{gathered}
\tag{B.3}
$$

B.1.3 The $O(\lambda^{1/4})$ Equation

Deriving our $O(\lambda^{1/4})$ equation we see that it contains both terms proportional to and independent of Q. Since these must be independent of each other the $O(\lambda^{1/4})$ equation splits up into the two equations

$$
\begin{gathered}
V_{1t} - rV_1 + \mu S V_{1S} + \tfrac{1}{2} f(Y)^2 S^2 \big(V_{1SS} - \tfrac{2\gamma}{\delta(t,T)}(y^* + V_{0S})V_{1S}\big) \\
+ \alpha(m - Y)V_{0Y} + f(Y)\rho\beta S\big(V_{1SY} - \tfrac{\gamma}{\delta(t,T)}((y^* + V_{0S})V_{1Y} + V_{1S}V_{0Y})\big) \\
+ \tfrac{\beta^2}{2}\big(V_{1YY} - \tfrac{2\gamma}{\delta(t,T)}V_{0Y}V_{1Y}\big) = 0
\end{gathered}
\tag{B.4}
$$

and

$$
(\mu - r)S - \frac{\gamma}{\delta(t,T)} f(Y)^2 S^2 (y^* + V_{0S}) = 0. \tag{B.5}
$$

From the term proportional to Q we have

$$
y^*(S,t) = -V_{0S}(S,t,Y) + \frac{\delta(t,T)(\mu - r)}{\gamma S f(Y)^2}. \tag{B.6}
$$

We have thus reached an expression for the ideal number of stocks we would hold in our portfolio in the absence of transaction costs. This expression is different from the one proposed by [14, 15] since it depends on our tracking volatility $f(Y)$ and a option gamma V_{0S} depending on Y. Note also that our controls y and y^* only can depend on the measurable process $S(t)$ and not $Y(t)$.

In practice we can not rely on the tracking volatility since $Y(t)$ is an unmeasurable quantity so we have to use the averaged volatility $\langle f(Y)\rangle$ [1].

[1] The average with respect to the invariant distribution Φ of the OU-process is given as

$$
\langle g(Y)\rangle = \int_{-\infty}^{\infty} g(Y)\Phi(Y)dY = 0
$$

From equation (B.5) we see that

$$(\mu - r)S - \frac{\gamma}{\delta(t,T)}\langle f(Y)^2\rangle S^2(y^* + \langle V_{0S}\rangle) = 0 \tag{B.7}$$

and

$$y^*(S,t) = -\langle V_{0S}(S,t,Y)\rangle + \frac{\delta(t,T)(\mu - r)}{\gamma S\langle f(Y)^2\rangle}. \tag{B.8}$$

We will use this property later on.

(B.6) inserted in our $O(1)$ equation gives us

$$\begin{aligned} &V_{0t} + rSV_{0S} - rV_0 + \tfrac{1}{2}f(Y)^2S^2V_{0SS} \\ &+\tfrac{\delta(t,T)(\mu-r)^2}{2\gamma f(Y)^2} + \alpha(m-Y)V_{0Y} + f(Y)\rho\beta SV_{0SY} \\ &+\tfrac{\rho\beta(\mu-r)}{f(Y)}V_{0Y} + \tfrac{\beta^2}{2}(V_{0YY} - \tfrac{\gamma}{\delta(t,T)}V_{0Y}^2) = 0 \end{aligned} \tag{B.9}$$

while the $O(\lambda^{1/4})$ term independent of Q is

$$\begin{aligned} &V_{1t} + rSV_{1S} - rV_1 + \tfrac{1}{2}f(Y)^2S^2V_{1SS} + \alpha(m-Y)V_{1Y} \\ &+f(Y)\rho\beta S(V_{1SY} - \tfrac{(\mu-r)}{Sf(Y)^2}V_{1Y} - \tfrac{\gamma}{\delta(t,T)}V_{1S}V_{0Y}) \\ &+\tfrac{\beta^2}{2}(V_{1YY} - 2\tfrac{\gamma}{\delta(t,T)}V_{0Y}V_{1Y}) = 0. \end{aligned} \tag{B.10}$$

After some tedious calculations one can see that $V_1 = 0$.

B.1.4 The $O(\lambda^{2/4})$ Equation

Defining $\epsilon = 1/\alpha$, the variance from the OU-process as $\nu^2 = \beta^2/2\alpha$, and using the operators

$$\begin{aligned} \mathcal{L}_0 &:= \nu^2\tfrac{\partial^2}{\partial Y^2} + (m-Y)\tfrac{\partial}{\partial Y} \\ \mathcal{L}_1 &:= \sqrt{2}\nu(\rho Sf(Y)\tfrac{\partial^2}{\partial S\partial Y} + \tfrac{\rho(\mu-r)}{f(Y)}\tfrac{\partial}{\partial Y}) \\ \mathcal{L}_2 &:= \tfrac{\partial}{\partial t} + \tfrac{1}{2}f(Y)^2S^2\tfrac{\partial^2}{\partial S^2} + r(S\tfrac{\partial}{\partial S} - \cdot) \end{aligned} \tag{B.11}$$

we get for the $O(\lambda^{2/4})$ equation ($V_1 = 0$)

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)V_2 = \frac{\nu^2}{\epsilon}\frac{2\gamma}{\delta(t,T)}V_{0Y}V_{2Y} + \frac{f(Y)\rho\sqrt{2}\nu S}{\sqrt{\epsilon}}\frac{\gamma}{\delta(t,T)}V_{0Y}V_{2S} - \frac{f(Y)^2S^2}{2}((y_S^*)^2V_{4QQ} - \frac{2\gamma}{\delta(t,T)}Q^2) \tag{B.12}$$

where $V_4 = V_4(S, t, Q, Y)$.

Solving this equation for V_4 and fitting it to the boundary conditions (3.8), (3.9) and (3.10), assuming a symmetric hedging interval $Q^+ = -Q^-$, reveals

$$Q^+ = -Q^- = \left(\frac{3\kappa S\delta(t,T)(y_S^*)^2}{2\gamma}\right)^{1/3}. \tag{B.13}$$

B.2 Asymptotic Expansion in ϵ

Equation (B.9) can using the operators defined above be written as

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)V_0 = \frac{\nu^2\gamma}{\epsilon\delta(t,T)}V_{0Y}^2 - \frac{\delta(t,T)(\mu - r)^2}{2\gamma f(Y)^2} \tag{B.14}$$

and correspondingly

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)V_1 = \frac{2\nu^2\gamma}{\epsilon\delta(t,T)}V_{0Y}V_{1Y} + \frac{\sqrt{2}\nu\rho S\gamma}{\sqrt{\epsilon}\delta(t,T)}V_{0Y}V_{1S} \tag{B.15}$$

for (B.10).

Now assume that each term V_k can be expanded in ϵ as

$$V_k = V_{k0} + \sqrt{\epsilon}V_{k1} + \epsilon V_{k2} + \epsilon^{3/2}V_{k3} + \dots \tag{B.16}$$

and insert (B.16) in (B.14) and compare different orders of ϵ.

B.2.1 The $O(\epsilon^{-1})$ Equation

For the O(ϵ^{-1}) equation we get

$$\mathcal{L}_0 V_{00} = \nu^2\frac{\gamma}{\delta(t,T)}V_{00Y}^2. \tag{B.17}$$

To see that V_{00} doesn't depend on Y we follow an argument in [9] where Y_0 is assumed to be smooth and of controlled growth.

Since $v^2 \frac{\gamma}{\delta(t,T)} V_{00Y}^2 \geq 0$ we have the ordinary differential inequality

$$\mathcal{L}_0 V_{00} \geq 0 \tag{B.18}$$

and by integrating with respect to Y we get

$$\begin{aligned} V_{00Y}(t,S,Y) &\geq V_{00Y}(t,S,m)e^{\frac{(Y-m)^2}{2v^2}} \quad ,Y \geq m \\ V_{00Y}(t,S,Y) &\leq V_{00Y}(t,S,m)e^{\frac{(Y-m)^2}{2v^2}} \quad ,Y \leq m. \end{aligned} \tag{B.19}$$

Since Y_0 is of controlled growth we conclude that $V_{00Y}(t,S,m) = 0$ and further on $V_{00Y}(t,S,Y) = 0$. Thus $V_{00} = V_{00}(t,S)$

B.2.2 The $O(\epsilon^{-1/2})$ equation

Using this our $O(\epsilon^{-1/2})$ equation is to $\mathcal{L}_0 V_{01} = 0$ and $V_1 = V_{01}(t,S)$.

B.2.3 The $O(1)$ Equation

The $O(1)$ equation becomes

$$\mathcal{L}_0 V_{02} + \mathcal{L}_2 V_{00} = -\frac{\delta(t,T)(\mu-r)^2}{2\gamma f(Y)^2}. \tag{B.20}$$

Following [6] we see that given $V_{00} = V_{00}(t,S,Q)$ this is a Poisson equation for V_{02} and a solution exists only if the so called centering condition is achieved

$$\langle \mathcal{L}_2 V_{00} + \frac{\delta(t,T)(\mu-r)^2}{2\gamma f(Y)^2} \rangle = 0 \tag{B.21}$$

or

$$\mathcal{L}_2(\bar{\sigma}) V_{00} = -\frac{\delta(t,T)(\mu-r)^2}{2\gamma} \langle \frac{1}{f(Y)^2} \rangle \tag{B.22}$$

where $\bar{\sigma} = \sqrt{\langle f(Y)^2 \rangle}$ is the effective volatility. V_{00} is thus simply the solution to Black-Scholes equation plus the particular solution

$$V_{00} = -V_{BS} + (T-t)\frac{\delta(t,T)(\mu-r)^2}{2\gamma} \langle \frac{1}{f(Y)^2} \rangle \tag{B.23}$$

with boundary value

$$V_{00}^i(T,S,y_i,Y) = \begin{cases} 0 & ,i = wo \\ -\max\{S-K,0\} & ,i = w \end{cases} \tag{B.24}$$

from equation 2.16.

Satisfying the centering condition we can write

$$\begin{aligned}
\mathcal{L}_0 V_{02} &= -(\mathcal{L}_2 V_{00} + \tfrac{\delta(t,T)(\mu-r)^2}{2\gamma f(Y)^2}) \\
&= -(\mathcal{L}_2 - \mathcal{L}_2(\overline{\sigma}))V_{00} - \tfrac{\delta(t,T)(\mu-r)^2}{2\gamma}(\tfrac{1}{f(Y)^2} - \langle \tfrac{1}{f(Y)^2} \rangle) \\
&= -\tfrac{1}{2}(f(Y)^2 - \overline{\sigma}^2)S^2 V_{00SS} - \tfrac{\delta(t,T)(\mu-r)^2}{2\gamma}(\tfrac{1}{f(Y)^2} - \langle \tfrac{1}{f(Y)^2} \rangle)
\end{aligned} \tag{B.25}$$

or

$$V_{02} = -\tfrac{1}{2}(\phi(Y) + c_1(S,t))S^2 V_{00SS} - \tfrac{\delta(t,T)(\mu-r)^2}{2\gamma}(\psi(Y) + c_2(S,t)) \tag{B.26}$$

where $\phi(Y)$ is given as the solution to the ordinary differential equation $\mathcal{L}_0\phi(Y) = f(Y)^2 - \overline{\sigma}^2$ and $\psi(Y)$ is the solution to $\mathcal{L}_0\psi(Y) = \frac{1}{f(Y)^2} - \langle \frac{1}{f(Y)^2} \rangle$.

B.2.4 The $O(\epsilon^{1/2})$ Equation

Since V_{00} and V_{01} are constant with respect to Y the O($\epsilon^{1/2}$) term reduces to

$$\mathcal{L}_0 V_{03} + \mathcal{L}_1 V_{02} + \mathcal{L}_2 V_{01} = -\frac{\delta(t,T)(\mu-r)^2}{2\gamma f(Y)^2}. \tag{B.27}$$

This is a Poisson equation for V_{03} with respect to $\mathcal{L}_0$ and the centering condition gives

$$\langle \mathcal{L}_1 V_{02} + \mathcal{L}_2 V_{01} + \frac{\delta(t,T)(\mu-r)^2}{2\gamma f(Y)^2} \rangle = 0. \tag{B.28}$$

Inserting (B.26) and defining $\overline{V}_{01} = \sqrt{\epsilon} V_{01}$ gives

$$\begin{aligned}
\mathcal{L}_2(\overline{\sigma})\overline{V}_{01} = &\ A_1(2S^2 V_{00SS} + S^3 V_{00SSS}) + A_2(\mu - r)S^2 V_{00SS} \\
&+ A_3 \tfrac{\delta(t,T)(\mu-r)^3}{\gamma} - \tfrac{\delta(t,T)(\mu-r)^2}{2\gamma\sqrt{\alpha}} \langle \tfrac{1}{f(Y)^2} \rangle
\end{aligned} \tag{B.29}$$

where

$$\begin{aligned}
A_1 &= \tfrac{\rho\nu}{\sqrt{2\alpha}} \langle f\phi' \rangle \\
A_2 &= \tfrac{\rho\nu}{\sqrt{2\alpha}} \langle \tfrac{\phi'}{f} \rangle \\
A_3 &= \tfrac{\rho\nu}{\sqrt{2\alpha}} \langle \tfrac{\psi'}{f} \rangle.
\end{aligned} \tag{B.30}$$