Pontryagin Approximations for Optimal Design of Elastic Structures

Jesper Carlsson
NADA, KTH
jesperc@nada.kth.se

Collaborators: Anders Szepessy, Mattias Sandberg

October 5, 2005

Optimal Design

A typical optimal design problem

$$\inf_{\rho:\Omega\to\{0,1\}} l(u), \qquad a_\rho(u,v) = l(v), \quad \forall v \in V, \qquad \int_\Omega \rho dx = C,$$

with compliance

$$l(u) \equiv \int_\Omega f_b \cdot u dx + \int_{\Gamma_N} f_s \cdot u ds,$$

and bilinear energy functional

$$a_\rho(u,v) \equiv \int_\Omega \rho \varepsilon_{ij}(u) E_{ijkl} \varepsilon_{kl}(v) dx.$$

An alternative formulation

$$\inf_{\rho:\Omega\to\{0,1\}} \left(l(u) + \eta \int_\Omega \rho dx \right), \qquad a_\rho(u,v) = l(v), \quad \forall v \in V.$$

Problem Optimal design problems are typically ill-posed i.e. no existence of solution.

Cure To relax the admissible set of controls $\mathscr{A}$, $\rho \in \mathscr{A}$.

Optimal design, continued

A harder inverse problem

$$\inf_{\rho} \int_{\Gamma} (u - u_{given})^2 ds$$

$$a_{\rho}(u,v) = l(v), \quad \forall v \in V.$$

Ill posed due to non-continuous dependence of error in measured data.

Hamilton-Jacobi-Bellman and the Pontryagin Principle

$$\inf_{\alpha\in\mathscr{A}}\{g(X_T)+\int_0^T h(X_t,\alpha_t)dt\},\quad X_t'=f(X_t,\alpha_t),\quad X(0)=X_0$$

Value function

$$u(x,t)=\inf_{\alpha\in\mathscr{A},X(t)=x}\int_t^T h(X_s,\alpha_s)ds+g(X_T)$$

solution to the HJB-equation

$$u_t+\overbrace{\min_{\alpha\in\mathscr{A}}\{u_x\cdot f(x,\alpha)+h(x,\alpha)\}}^{H(u_x,x)}=0,\quad u(x,T)=g(x)$$

Differentiation along optimal paths α_t, X_t, $\lambda_t\equiv u_x(X_t,t)$ gives Pontryagin's Principle

$$\lambda_t'=-\lambda_t f_X(X_t,\alpha_t)+h_X(X_t,\alpha_t),\quad \lambda(T)=g'(X_T)$$

Also

$$\alpha_t=\text{argmin}_{a\in\mathscr{A}}\{\lambda_t\cdot f(X_t,a)+h(X_t,a)\}$$

Assume H, f, h, g differentiable, $\lambda \in C^1$.

Lagrange principle

$$
\begin{aligned}
X_t' &= H_\lambda(\lambda_t, X_t) \\
-\lambda_t' &= H_X(\lambda_t, X_t)
\end{aligned}
$$

with control determined by Pontryagin principle

$$\alpha_t = \text{argmin}_{a \in \mathscr{A}} \{\lambda_t \cdot f(X_t, a) + h(X_t, a)\}$$

Two reasons for non-smooth control:

1. Only Lipschitz continuous Hamiltonian
2. Backward characteristic paths $X(t)$ may collide

Remedy for 1: construct a $\mathscr{C}^2$ concave approximation H^δ of the Hamiltonian H.
Error estimate [Sandberg, Szepessy 2005]:

$$\|\bar{u} - u\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} = \mathscr{O}(\delta)$$

Concave Maximization in Electric Conduction

Minimize power-loss in conductive medium [Pironneau 1984]

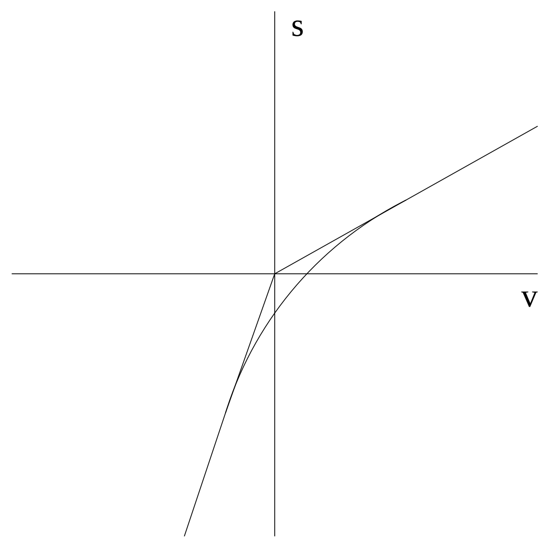
$$\min_{\sigma}\left(\int_{\partial\Omega} q\varphi ds + \eta\int_{\Omega}\sigma dx\right) \qquad \mathrm{div}\left(\sigma\nabla\varphi\right) = 0,\ x\in\Omega \qquad \sigma\frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega} = q$$

Lagrangian

$$\int_{\Omega}\sigma\underbrace{(\eta - \nabla\varphi\cdot\nabla\lambda)}_{v}dx + \int_{\partial\Omega} q(\varphi+\lambda)ds$$

Hamiltonian

$$H = \min_{\sigma}\int_{\Omega}\sigma v dx + \int_{\partial\Omega} q(\varphi+\lambda)ds = \int_{\Omega}\underbrace{\min_{\sigma}\sigma v}_{s(v)}dx + \int_{\partial\Omega} q(\varphi+\lambda)ds$$



Concave regularization

$$H^{\delta} = \int_{\Omega} s_{\delta}(\eta - \nabla\varphi \cdot \nabla\lambda)dx + \int_{\partial\Omega} q(\varphi + \lambda)ds,$$

By symmetry $\lambda = \varphi$ the Hamiltonian system reduces to

$$\int_{\Omega} s'_{\delta}(\eta - |\nabla\varphi|^2)\nabla\varphi \cdot \nabla w dx = \int_{\partial\Omega} qwds$$

or

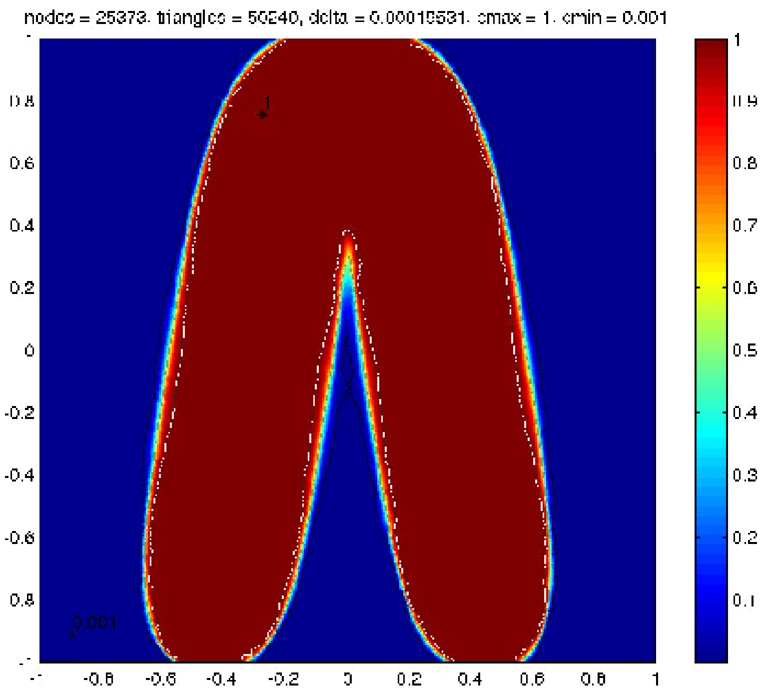
$$\mathrm{div}\Big(s'_{\delta}(\eta - |\nabla\varphi|^2)\nabla\varphi(x)\Big) = 0, \qquad x \in \Omega$$

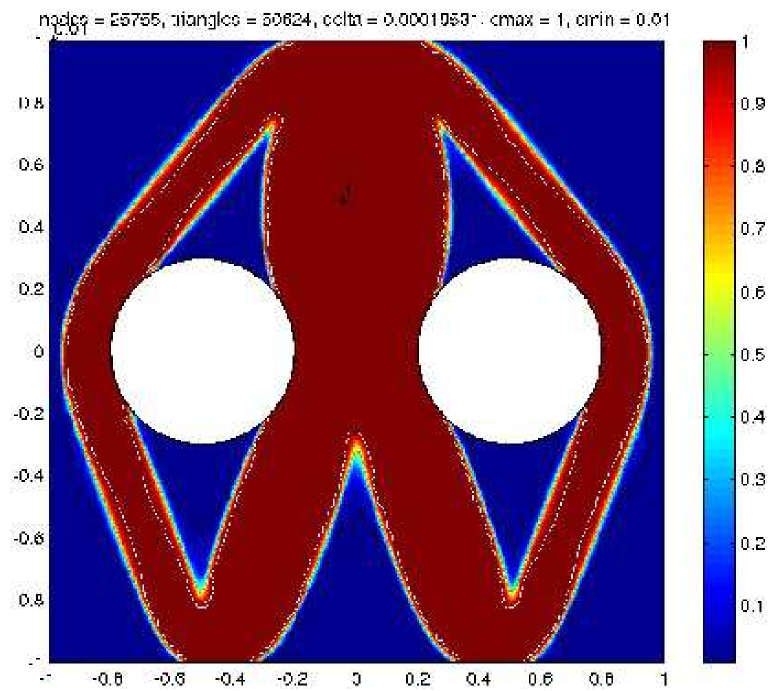
$$s'_{\delta}\frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega} = q,$$

Concave maximization problem: $\varphi \in V$ is the unique maximizer of

$$H^{\delta}(\varphi) = \int_{\Omega} s_{\delta}(\eta - |\nabla\varphi(x)|^2)dx + 2\int_{\partial\Omega} q\varphi ds.$$

Numerical Examples of Electric Conduction





Optimal Design of an Elastic Structure

Lagrangian for compliance minimization

$$l(u) + l(\lambda) + \int_\Omega \rho\big(\underbrace{\eta - \varepsilon_{ij}(u) E_{ijkl} \varepsilon_{kl}(\lambda)}_{v}\big)\, dx,$$

Hamiltonian

$$H = l(u) + l(\lambda) + \int_\Omega \underbrace{\min_\rho \rho v}_{s(v)}\, dx.$$

Regularization gives Hamiltonian system which by symmetry $u = \lambda$ reduces to

$$\int_\Omega s'_\delta\big(\eta - \varepsilon_{ij}(u) E_{ijkl} \varepsilon_{kl}(u)\big) \varepsilon_{ij}(u) E_{ijkl} \varepsilon_{kl}(v)\, dx = l(v)$$

Concave maximization problem: u is the unique maximizer of

$$H^\delta = 2l(u) + \int_\Omega s_\delta\big(\eta - \varepsilon_{ij}(u) E_{ijkl} \varepsilon_{kl}(u)\big)\, dx.$$

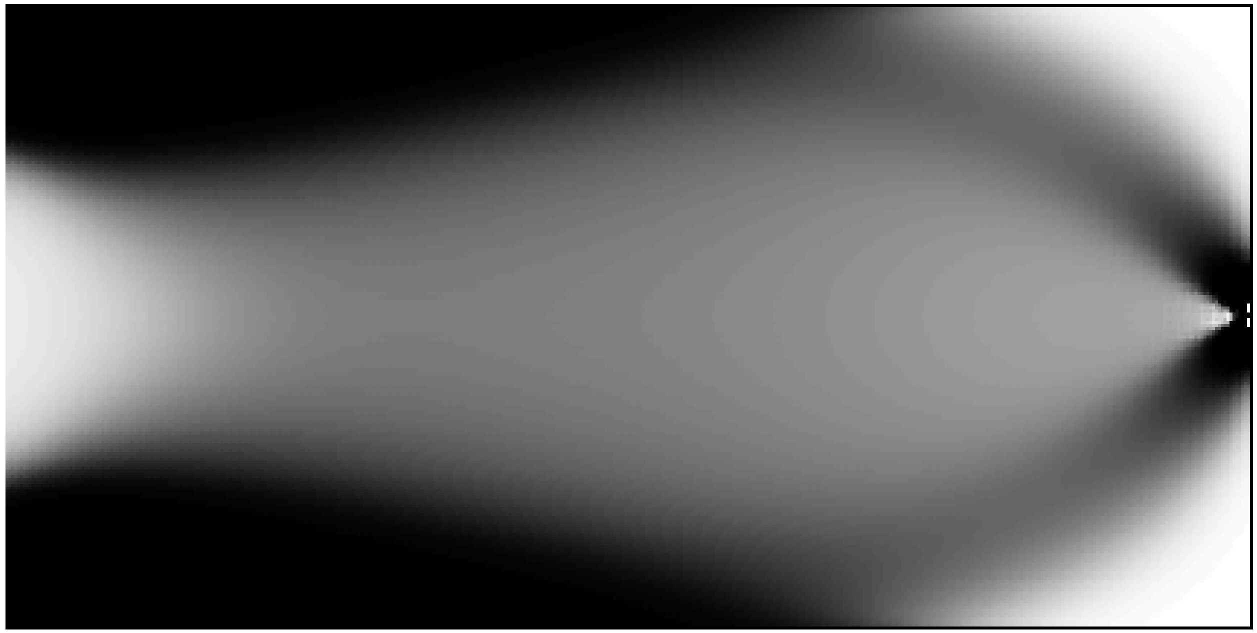


Figure 1: Plot of s'_δ as approximation of $\rho \in [0.001, 1]$. Compliance = 0.45.

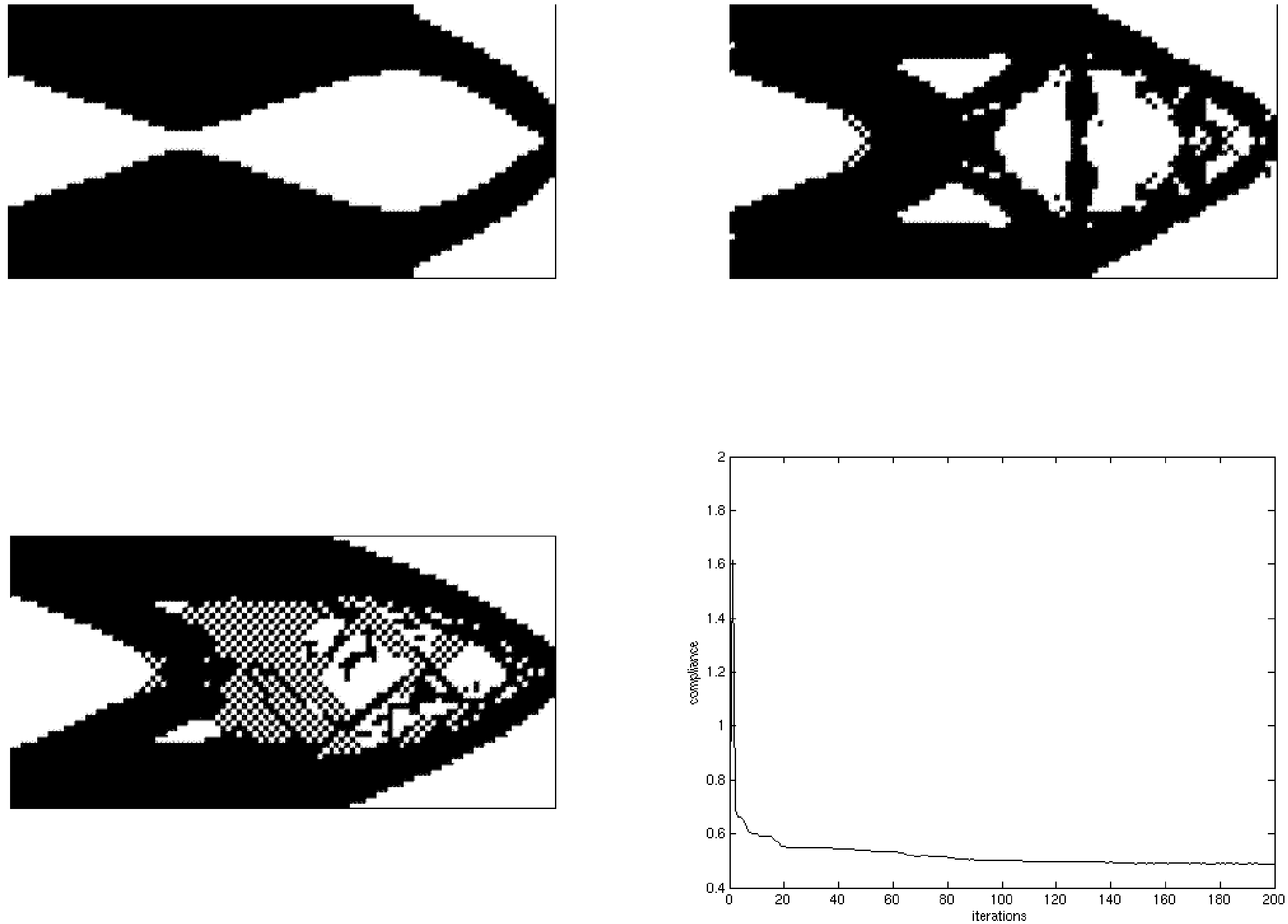


Figure 2: Plot of the density $\rho \in \{0.001, 1\}$ after 1, 5 and 200 iterations.

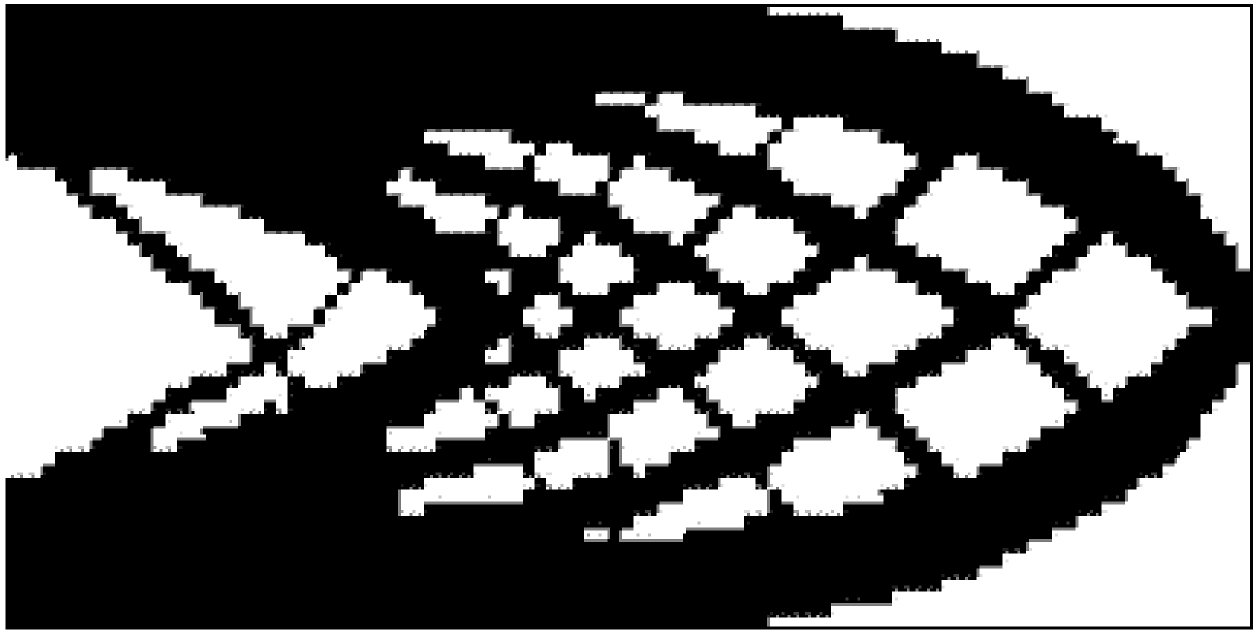


Figure 3: 10 iterations when only removing material. Compliance = 0.51.

References

[1] Allaire G., Shape Optimization by the Homogenization Method. Applied Mathematical Sciences, 146. Springer-Verlag, New York, 2002.

[2] Bendsø M. P. and Sigmund O., Topology optimization. Theory, methods and applications. Springer-Verlag, Berlin, 2003.

[3] Pironneau O., Optimal Shape Design for Elliptic Systems. Springer Series in Computational Physics. Springer-Verlag, New-York, 1984.

[4] Sandberg M. and Szepessy A., Convergence rates of Symplectic Pontryagin Approximations in Optimal Control Theory, to appear in M2AN Mathematical Modelling and Numerical Analysis. Preprint available at `http://www.nada.kth.se/~szepessy`.