Subgrid modeling for convection-diffusion-reaction in 1 space dimension using a Haar Multiresolution analysis

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Abstract

In this paper we propose and study a subgrid model for linear convection-diffusion-reaction equations with fractal rough coefficients. The subgrid model is based on scale extrapolation of a modeling residual from coarser scales using a computed solution on a finest scale as reference. We show in experiments that a solution with subgrid model on a scale \( h \) in most cases corresponds to a solution without subgrid model on a scale less than \( h/4 \). We also present error estimates for the modeling error in terms of modeling residuals.

Key words: convection-diffusion-reaction, dynamic subgrid modeling, Haar multiresolution analysis, modeling errors
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1 Introduction

A fundamental problem in science and engineering concerns the mathematical modeling of phenomena involving small scales. This problem arises in molecular dynamics, turbulent flow and flow in heterogeneous porous media, for example. Basic models for such phenomena, such as the Schrödinger equation or the Navier-Stokes equations, may be very accurate models of the real phenomena but may be so computationally intensive, because of the large number of degrees of freedom needed to represent the small scales, that even computers with power way beyond that presently available may be insufficient...
for accurate numerical solutions of the given equations. The traditional approach to get around this difficulty is to seek to find simplified models with computationally resolvable scales, whose solutions are sufficiently close to the solutions of the original full equations. Such simplified models, without the too small scales, build on mathematical modeling of the computationally unresolved scales of the full equations, which is referred to as subgrid modeling.

To find suitable simplified models including subgrid modeling, is the central activity in modeling of turbulence, molecular dynamics and heterogeneous porous media.

The problem of subgrid modeling may naturally be approached by seeking to find the simplified model by suitably averaging the full equations over the resolvable scales. In a Large Eddy Simulation LES the idea is to compute the larger scales of motion of the turbulent Navier-Stokes equations while approximating the smaller ones, for an overview see [15,8]. By suitably averaging the Navier-Stokes equations over a certain spatial scale one obtains a simplified set of equations involving additional stresses \( \tau_{ij} = u_iu_j - \overline{u_iu_j} \), where \( \overline{u} \) represents a local average of \( u_i \) and \( u = (u_i) \) is the velocity, called the subgrid scale Reynolds stresses SGSRS, representing the action of subgrid scales on the resolved scales.

The most common of subgrid models in turbulence are the Eddy viscosity models, where the SGSRS are modeled as viscous stresses \( \tau_{ij} \approx \nu \varepsilon_{ij}(\overline{u}) \), related to a certain turbulent viscosity (or eddy viscosity) \( \nu \) of the form \( \nu = Ch^\mu |\varepsilon(\overline{u})| \), where \( C = C(x) \) and \( \mu = \mu(x) \) are positive numbers in general depending on the spatial coordinate \( x \), \( h = h(x) \) represents the smallest resolvable scale at \( x \), and \( \varepsilon(u) = (\varepsilon_{ij}(u)) \) is the strain of the velocity \( u \). The subgrid modeling problem in this case is to find the functions \( C(x) \) and \( \mu(x) \). Attempts have been made to determine these functions analytically, or experimentally by finding best fit to given measured data. In both cases serious difficulties arise and the obtained simplified models do not seem to be useful over a range of problems with different data. Of course, the difficulties may stem from both the fact that the assumed form of the subgrid scale Reynolds stresses is not a reasonable one, and from the fact that the coefficients \( C(x) \) and \( \mu(x) \) depend on the particular problem being solved, and thus fitting the coefficients to one set of data may be of no value for other data.

In recent years, new approaches to the subgrid modeling problem have been taken based on dynamic subgrid modeling, an idea first introduced by Germano et al. [9]. The basic assumption here is that a particular model applies on different scales with the same value on the model parameters. Using this assumption, one seeks to find a subgrid model, for each set of data, by computing approximations of the subgrid model on coarser scales using a fine scale computed solution without subgrid model as reference, and then finally extrapolating the so obtained model to the finest computational scale, with
the hope of being able to extrapolate from the finest resolvable scales to unresolved scales. In the simplest case, this may come down to seeking to determine, for a given set of data, the coefficients $C(x)$ and $\mu(x)$ in an Eddy viscosity model by best fit. In this approach, at least the dependence of the coefficients on the data may be taken into account, but still the Ansatz with a turbulent viscosity is kept. More generally, it is natural to seek to extend this approach to different forms of the Ansatz. In order for such a dynamic modeling process based on extrapolation to work, it is necessary that the underlying problem has some ‘scale regularity’, so that the experience gained by fitting the model on a coarse scale with a fine scale solution as reference, may be extrapolated to the finer scale. It is conceivable that many problems involving a range of scales from large to small, such as fluid flow at larger Reynolds numbers, in fact does have such a regularity, once the larger scales related to the geometry of the particular problem have been resolved. The purpose of this paper is to study the feasibility of the indicated dynamic computational subgrid modeling in the context of some simple model problems related to linear convection-diffusion-reaction with irregular or non-smooth coefficients with features on many scales. The scale regularity in this case appears to be close to assuming that the coefficients of the equations are scale similar, and that the solution inherits this structure to some degree. In this paper we use a Haar Multiresolution analysis MRA for the scale extrapolation.

The problem of computational mathematical modeling has two basic aspects: numerical computation and modeling. The basic idea in dynamic subgrid modeling is to seek to extrapolate into unresolvable scales by comparing averaged fine scale computed solutions of the original model (without subgrid modeling) on different coarser scales. To make the extrapolation possible at all, the numerical errors in the computations underlying the extrapolation have to be small enough. If the numerical errors in the fine scale computation without subgrid model are not sufficiently small, then the whole extrapolation procedure from coarser scales may be meaningless. Thus, it will be of central importance to accurately balance the errors from numerical computation and subgrid modeling. In recent years the techniques for adaptive error control based on a posteriori error estimates have been considerably advanced, see e.g. [6,1,13,11]. Thus, today we have techniques available that allow the desired balance of numerical and modeling errors.

This is the first paper in this series and it is focused on modeling only, assuming the numerical errors can be made negligible compared to the modeling errors by computing on a very fine computational mesh. In the continuations [12,10] the model problem in two dimensions is considered, as well as the time dependent case, and the balancing of numerical errors and modeling errors is discussed.

An outline of this paper is as follows: In Section 2 the linear convection-diffusion-reaction model problem is introduced and different approaches to
the subgrid modeling problem is discussed. In Section 3 the basic features of MRA using the Haar basis is recalled, which is used to motivate an Ansatz on the form of the subgrid model, which is then used to extrapolate an approximation of the subgrid model. In Sections 6-8 simplified forms of the model problem with zero diffusion are studied, and it is shown that in these cases the solution inherits the fractal structure of the coefficients, and thus extrapolation is possible. In Section 9 the extension to non-zero diffusion is studied, and in Section 10 error estimates for the modeling errors are presented. Numerical experiments with subgrid modeling are presented along the lines.

2 Problem formulation

We consider a scalar linear convection-diffusion-reaction problem of the form

$$Lu(x) \equiv -D(\epsilon(x)Du(x)) + \beta(x)Du(x) + \alpha(x)u(x) = f(x),$$

(1)

$x \in I = (0, 1)$, together with suitable boundary conditions, where $\epsilon(x)$, $\beta(x)$ and $\alpha(x)$ are given coefficients depending on $x$, $f(x)$ is a given source, $D = \frac{d}{dx}$, and $u(x)$ is the solution. We assume that the coefficients $\epsilon$, $\beta$ and $\alpha$ are piecewise continuous, and we seek a solution $u(x)$ which is continuous on $I$ with $\epsilon Du$ piecewise continuous, and which satisfies (1) for all $x \in I$ which are not points of discontinuity of the coefficients. In the case $\epsilon = 0$, assuming that $\beta$ does not vanish on $I$ and $u(0) = 0$, the solution $u(x)$ is given by the formula

$$u(x) = \int_0^x \exp(A(y) - A(x)) \frac{f(y)}{\beta(y)} dy, \quad \text{for } x \in I,$$

(2)

where $A(x)$ is a primitive function of $\alpha/\beta$ (satisfying $DA = \alpha/\beta$, $A(0) = 0$). If also $\beta = 0$, then the solution is simply

$$u(x) = \frac{f(x)}{\alpha(x)}, \quad \forall x \in I,$$

assuming now that $\alpha$ does not vanish on $I$.

We assume that the coefficients $\epsilon$, $\beta$ and $\alpha$, and the given function $f$ vary on a range of scales from very fine to coarse scales, and we expect the exact solution $u$ in general to vary on a related range of scales. We denote by $h$ the finest possible scale in a computation of a solution, and we denote the corresponding approximate solution by $u_h$. We denote by $u_h$ the solution to the following simplified problem

$$L_hu_h(x) \equiv -D([\epsilon]^h(x)Du_h(x)) + [\beta]^h(x)Du_h(x) + [\alpha]^h(x)u_h(x) = [f]^h(x),$$

(3)
for $x \in I$, together with boundary conditions where $[\epsilon]^h$, $[\beta]^h$, $[\alpha]^h$, and $[f]^h$ are approximations of the corresponding functions on the scale $h$, with finer scales left out. The corresponding solution formula for $u_h$ in the case $\epsilon = 0$, reads

$$u_h(x) = \int_0^x \exp(A_h(y) - A_h(x)) \frac{[f]^h(y)}{[\beta]^h(y)} dy, \quad \text{for } x \in I,$$

(4)

where $A_h$ is a primitive function of $[\alpha]^h/[\beta]^h$ satisfying $A_h(0) = 0$, and $u_h(0) = 0$. We may think of $u_h$ as an approximation of the exact solution $u$ obtained by simplifying the model by simplifying the coefficients removing scales finer than $h$. Typically, the coefficient $[\beta]^h$ is some local average of $\beta$ on the scale $h$, etc. The difference $u - u_h$ thus represents a modeling error resulting from averaging the coefficients on the scale $h$.

2.1 Subgrid modeling

We now consider a situation where $u_h$ is not a sufficiently good approximation of $u$, and we would like to improve the quality of $u_h$ without computing using finer scales than $h$. The equation $Lu = f$ satisfied by the exact solution can be written in the form

$$L_h u = [f]^h + F_h(u),$$

(5)

where

$$F_h(u) = f - [f]^h - (L - L_h)u$$

acts as a modeling residual. The subgrid modeling problem is to model $F_h(u)$ on the scale $h$. There is a variety of possibilities to approach this problem. We may use $F_h(u)$ as a correction on the force and replace the model $L_h u_h = [f]^h$ by the model

$$L_h \tilde{u}_h = [f]^h + \tilde{F}_h,$$

with solution $\tilde{u}_h$, where $\tilde{F}_h$ is an approximation of $F_h(u)$. Alternatively, we may seek to model $L - L_h$ as a correction $\tilde{L}_h$ of the operator $L_h$ and solve a modified problem of the form

$$(L_h + \tilde{L}_h) \tilde{u}_h = f,$$

(7)

where thus the correction $\tilde{L}_h$ acts as a model of $L - L_h$. In the first approach the subgrid model takes the form of a corrective force $\tilde{F}_h$ independent of $\tilde{u}_h$, and in the second approach the subgrid model also contains a correction $\tilde{L}_h \tilde{u}_h$ depending on $\tilde{u}_h$.

To find the corrected (or effective) operator is a classical problem in homogenization theory, see e.g. Bensoussan et al. [2]. Analytical homogenization techniques based on asymptotics have been used to derive effective operators, but these techniques rely on the essential assumptions of periodicity of the coefficients, well separated scales, and an a priori knowledge of the number

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of scales, which may be serious restrictions. Another approach to this problem was proposed by Nielsen and Tveito [19] who studied Poisson’s equation with an irregular permeability on a fine scale, where the effective (or upscaled) permeability was defined as the solution to an optimization problem, where the difference between the fine scale and the coarse scale velocity fields were minimized. Brewster and Beylkin [3] used a numerical homogenization strategy based on a MRA, where the homogenized (or reduced) operator was constructed by recursively taking the equation at one scale and construct the effective equation on the next coarser scale. These ideas were then further developed by Dorobantu et.al. [5]. For this approach to be practical, two problems have to be solved. First, the transition between two scales has to be computationally efficient. Secondly, the form of the equations must be preserved for a recursive use of the reduction step to be possible, which is not the case in general. The Variational multiscale method, by Hughes et.al. [14], uses an idea based on a hierarchical FEM space divided into coarse scales and fine scales, where typically the linear basis functions on each element represents the coarse scales, and the bubble, edge (and face) basis functions represents the fine scales. For an overview of subgrid models in LES we refer to [15,8].

2.2 Dynamic models

Germano et.al. [9] first introduced the concept of a dynamic model, in the setting of LES. The dynamic model is not a model in itself, but rather a procedure taking a subgrid model as its basis. The basic assumption is that a particular model applies on different scales, with the same value of the model parameters. In this paper we propose methods for computing approximations of the modeling residual $F_h(u)$ using the idea of a dynamic model, in the case of the model problem (1). We show that the local average of $F_h(u)$ on the scale $h$ is equal to a sum of covariances of the form $[vw]^h - [v]^h[w]^h$. Based on a Haar MRA, an Ansatz of the form $[vw]^h - [v]^h[w]^h \approx Ch^\mu$ is proposed. The two functions $C(x)$ and $\mu(x)$ are then estimated by extrapolation from computing approximations $F_H(u_h) \approx F_H(u)$ on two coarser scales $H$, where the solution of the simplified problem $u_h$ is used as a substitute for the solution $u$ of the exact problem.

3 Subgrid modeling by scale extrapolation using MRA

The notion of Multiresolution Analysis (MRA) was introduced in the early 90’s by Meyer [18] and Mallat [17] as a general framework for construction of wavelet bases. An orthonormal MRA of $L_2([0,1])$ is a decomposition of
\[ L_2([0, 1]) \] into a chain of closed subspaces
\[ V_0 \subset V_1 \subset ... \subset V_j ... \]
such that
\[ \bigcup_{j \geq 0} V_j = L_2([0, 1]). \]
Each \( V_j \) is spanned by dilates and integer translates of a scale function \( \varphi \in V_0 \):
\[ V_j = \text{span}\{\varphi_{j,k}(x) = 2^{j/2}\varphi(2^jx - k)\}, \]
and the functions \( \varphi_{j,k} \) form an \( L_2 \)-orthonormal basis in \( V_j \). We denote the orthogonal complement of \( V_j \) in \( V_{j+1} \) by \( W_j \), which is generated by another orthonormal basis (the wavelets) \( \psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k) \), where \( \psi \in W_0 \) is called the mother wavelet. The space \( L_2([0, 1]) \) can now be represented as a direct sum
\[ L_2([0, 1]) = V_0 \oplus W_0 \oplus ... \oplus W_j \oplus ... \]
For a more detailed presentation of the MRA concept we refer to [4,16].

### 3.1 The Haar MRA

In the case of the Haar basis in \( L_2(I) \), with \( I = [0, 1] \), the space \( V_0 \) is spanned by the scale function
\[ \varphi(x) = \begin{cases} 
1 & x \in I \\
0 & x \notin I
\end{cases} \]
and \( V_j = \text{span}\{\varphi_{j,k}(x) = 2^{j/2}\varphi(2^jx - k)\} \). The orthogonal complement of \( V_j \) in \( V_{j+1} \) is \( W_j \), spanned by the wavelets
\[ \psi_{j,k} = 2^{j/2}\psi(2^jx - k), \quad \text{for } k = 0, 1, 2, ... < 2^j, j = 0, 1, 2, ... \]
where the mother wavelet \( \psi \) is defined by
\[ \psi(x) = \begin{cases} 
1 & 0 < x < 1/2 \\
-1 & 1/2 < x < 1 \\
0 & \text{otherwise}.
\end{cases} \]
Each \( f \in L_2(I) \) has a unique decomposition
\[ f = f_\varphi \varphi + \sum_{i,k} f_{i,k} \psi_{i,k} = f_\varphi + \sum_i f_i, \]
where the \( f_i \) represent the contributions on the different scales \( 2^{-i} \) corresponding to subdivisions \( S_i \) of \( I \) with mesh points \( x_{i,k} = k2^{-i}, k = 0, 1, ..., 2^i, \) and
Fig. 1. The “mother wavelet” \( \psi(x) \).

Subintervals \( I_{i,k} = (k2^{-i}, (k + 1)2^{-i}) \). The coefficients \( f_{i,k} \) are given as the \( L_2 \)-inner product of the function \( f \) and the corresponding Haar basis function:

\[
f_{i,k} = \int_0^1 f(x)\psi_{i,k}(x) \, dx,
\]

and \( f_\varphi = \int_0^1 f(x) \, dx \).

3.2 Scale extrapolation using a Haar MRA

For \( f \in L_2(I) \), where \( I = [0,1] \), we define \( [f]^h \) to be the piecewise constant function on \( S_i \), given by

\[
[f]^h = f_\varphi + \sum_{j<i} f_j,
\]

where we let \( h = 2^{-i} \) in the rest of this paper. Further, we recall the definition of the running average \( f^h \) of a function \( f \in L_2(I) \) on the scale \( h \) as

\[
f^h(x) = 2^i \int_{-h/2}^{h/2} f(x+y) \, dy,
\]

where \( x \in I \), and we extend \( f \) smoothly outside \( I \). We denote by \( \tilde{f}^h \) the piecewise constant function on the scale \( h \) which coincides with \( f^h \) at the midpoints of the subintervals, and by this definition \( \tilde{f}^h \) is independent of the extension of \( f \) outside \( I \). We shall use the following lemma:

**Lemma 1** \( f \in L_2(I) \Rightarrow [f]^h = \tilde{f}^h \).

**Proof.** We have

\[
f = f_\varphi + \sum_j f_j \Rightarrow \bar{f}^h = \tilde{f}^h + \sum_j \tilde{f}^h_j = \tilde{f}^h + \sum_{j<i} \tilde{f}^h_j = \tilde{f}^h + \sum_{j<i} f_\varphi + \sum f_j = [f]^h. \]

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We recall that $V_i$ is the space of piecewise constant functions on $S_i$, and the linear mapping $L_2 \ni f \mapsto [f]^h \in V_i$ can then be identified with the $L_2$-projection of $f$ onto $V_i$. Assuming for the moment that $\epsilon$ is constant, so that $[\epsilon]^h = \epsilon$, the modeling residual $F_h(u)$ is given by

$$F_h(u) = f - [f]^h - (L - L_h)u = f - [f]^h - (\beta Du + \alpha u - [\beta]^h Du - [\alpha]^h u).$$

From the definition we have that $[[f]^H g]^h = [f]^H g]^h$ whenever $H \geq h$ ($H = 2^{-j}$, $h = 2^{-i}$ with $j < i$). This gives that


We denote the projection $[F_h(u)]^h$ of $F_h(u)$ onto $V_i$ by $\bar{F}_h(u)$. In the simplest situation above with $\beta = 0$, the full model is $\alpha u = f$, the approximate model is $[\alpha]^h u_h = [f]^h$, and the corresponding solutions are

$$u = f/\alpha, \quad u_h = [f]^h/\alpha[^h].$$

The modeling residual $F_h(u)$ is given by

$$F_h(u) = f - [f]^h - (\alpha u - [\alpha]^h u),$$

and we have

$$\bar{F}_h(u) = [\alpha]^h [u]^h - [\alpha u]^h = [\alpha]^h [f/\alpha]^h - [f]^h.$$

We shall now seek to extrapolate $\bar{F}_h(u)$, and we are thus led to study in particular quantities of the form

$$E_h(v, w) = [vw]^h - [v]^h [w]^h,$$

for given functions $v$ and $w$, which has the form of a covariance. Using the Haar basis, the covariance $E_h(v, w)$ takes a simple form:

**Lemma 2** $v, w \in L_2(I) \Rightarrow E_h(v, w)(x) = \sum \sum \sum 2^i v_{j,l} w_{j,l}.$

**Proof**. $v, w \in L_2(I) \Rightarrow v = \sum v_j, \quad w = \sum w_k \Rightarrow vw = \sum \sum v_j w_k,$

$$[v]^h = \sum_{j < i} v_j, \quad [w]^h = \sum_{k < i} w_k \Rightarrow [v]^{h}[w]^{h} = \sum_{j,k < i} v_j w_k.$$

**Lemma 1** $\Rightarrow [vw]^{h} = \overline{vw}^{h} = \sum_{j,k} v_j w_k^{h}$

$$= \sum_{j,k < i} v_j w_k + \sum_{j \geq i} v_j w_j = [v]^{h}[w]^{h} + \sum_{j \geq i} v_j w_j.$$
Finally, for \( x \in I_{j,l} \), \( v_j w_j(x) = v_{j,l} \psi_{j,l}(x) \) \( w_{j,l} \psi_{j,l}(x) = 2^j v_{j,l} w_{j,l} \). □

Lemma 2 asserts that \( E_h(v, w) \) only depends on the scales finer than \( h \), and that there is no mixing between the scales. We may alternatively express \( E_h(v, w) \) in terms of an average of Haar coefficients over \( h \), by using the fact that \( E_h(v, w) \) is constant on the scale \( h \) and thus equal to its mean value over \( h \):

**Lemma 3** \( v, w \in L_2(I) \) and \( x \in I_{i,k} \) \( \Rightarrow \) \( E_h(v,w)(x) = 2^i \sum_{j \geq i \atop l: I_{j,l} \subset I_{i,k}} v_{j,l} w_{j,l} \).

**Proof.** From the proof of Lemma 2 we have that \( E_h(v,w)(x) = \sum_{j \geq i \atop l: I_{j,l} \subset I_{i,k}} v_{j,l} w_{j,l} \psi_{j,l}(x) \).

We then use that \( E_h(v,w) \) is constant on each interval \( I_{i,k} \) and thus equal to its average over \( I_{i,k} \):

\[
E_h(v,w)(x) = 2^i \int_{I_{i,k}} E_h(v,w)(y) \, dy = 2^i \int_{I_{i,k}} \sum_{j \geq i \atop l: I_{j,l} \subset I_{i,k}} v_{j,l} w_{j,l} \psi_{j,l}^2(y) \, dy
\]

\[
= 2^i \sum_{j \geq i \atop l: I_{j,l} \subset I_{i,k}} v_{j,l} \int_{I_{j,l}} \psi_{j,l}^2(y) \, dy.
\]

which completes the proof since \( \int_{I_{j,l}} \psi_{j,l}^2(y) \, dy = 1 \). □

An interesting situation is when both \( v \) and \( w \) are *scale similar* in the sense that

\[ v_{j,k} = \alpha 2^{-j(1/2+\delta)} \quad \text{and} \quad w_{j,k} = \beta 2^{-j(1/2+\gamma)}, \quad (9) \]

where \( \alpha, \beta, \delta, \gamma \in V_i \), which corresponds to \( v \) and \( w \) having a simple fractal structure, \( v_{j+1,l}(x) = 2^{-\delta(x)} v_{j,k}(x) \) and \( w_{j+1,l}(x) = 2^{-\gamma(x)} w_{j,k}(x) \), for \( I_{j+1,l} \subset I_{j,k} \). In that particular situation we find that \( E_h(v,w) \) takes a certain form:

**Corollary 4** If \( v, w \in L_2(I) \) with \( v_{j,k} = \alpha 2^{-j(1/2+\delta)} \) and \( w_{j,k} = \beta 2^{-j(1/2+\gamma)} \), where \( \alpha, \beta, \delta, \gamma \in V_i \), then for \( x \in I \)

\[ E_h(v,w)(x) = C(x) h^\mu(x), \]

where \( C = (\alpha \beta)/(1 - 2^{-(\delta+\gamma)}) \) and \( \mu = \delta + \gamma \).

**Proof.** By Lemma 2 we have that for \( x \in I \),

...
\[ E_h(v, w)(x) = \sum_{j \geq i} 2^j v_{j,l} w_{j,l} = \sum_{j \geq i} 2^j \alpha 2^{-j(1/2+\delta)} \beta 2^{-j(1/2+\gamma)} \]

\[ = \sum_{j \geq i} \alpha \beta 2^{-j(\delta+\gamma)} = \alpha \beta h^{\delta+\gamma} \sum_{j \geq i} 2^{-(j-i)(\delta+\gamma)} \]

\[ = \alpha \beta h^{\delta+\gamma} \sum_{j=0}^{\infty} 2^{-j(\delta+\gamma)} = \frac{\alpha \beta}{1 - 2^{-(\delta+\gamma)}} \cdot h^{\delta+\gamma} \]

The asymptotic behaviour of the wavelet coefficients determine the micro scale structure of a function. Assuming that the coefficients in (1) are scale similar in the sense of Corollary 4, and that the solution inherits this local fractal structure to some degree, we formulate the following Ansatz: For given \( x \in I \),

\[ E_h(v, w)(x) = C(x)h^{\mu(x)}, \quad (10) \]

where \( C, \mu \in V_{i-n} \), with \( \tilde{H} = 2^{-i+n} \). If \( E_h(v, w) \) has this form, then extrapolation of \( E_h(v, w) \) will be possible from knowledge of \( E_H(v, w) \) and \( E_{\tilde{H}}(v, w) \) with \( h < H < \tilde{H} \), from which the coefficients \( C(x) \) and \( \mu(x) \) may be determined. Typically, we will assume that the coefficients have a local fractal structure. We then expect the solution \( u \) to inherit this structure to some degree, and we expect that extrapolation of the modeling residual \( \bar{F}_h(u) \) will be possible. This type of scale similarity with respect to a Haar MRA have, for example, been found in turbulent aerothermal data [2], and in computations of turbulent flow [13].

4 Numerics & fractal coefficients

In the next sections we investigate the indicated extrapolation procedure in the context of some numerical experiments. We use a Runge-Kutta method for the initial value problems in Section 8. For the boundary value problems in Section 9 we use a standard Finite Element Method (FEM) for the diffusion dominated problems, and a Streamline Diffusion (SD) method for the convection dominated problems (see Eriksson et. al. [7]). In all the experiments we use a cut-off scale \( h = 2^{-6} \), and we compute on a subdivision of \( I \) corresponding to a scale \( 2^{-12} \). We use coefficients of the type

\[ \alpha(x) = 1 + \sum_{j \geq 0} \gamma 2^{-j(1/2+\delta)} \psi_{j,l}(x) \equiv 1 + \alpha', \quad x \in I, \quad (11) \]

which are on the form (9) with \( \alpha_{j,l} = \gamma 2^{-j(1/2+\delta)} \) and \( \alpha_{\phi} = 1 \).
Fig. 2. $\alpha(x)$ for $\gamma = 0.05$ and $\delta = 0.5$.

5 Gain-factors & mesh-factors

Since we are computing on a very fine mesh (interval length $2^{-12}$) we will regard the numerical errors to be negligible. To measure the improvement of the corrected solution $\tilde{u}_h$ we introduce a gain-factor $GF$ defined by

$$GF = \frac{\| [u - u_h]^h \|}{\| [u - \tilde{u}_h]^h \|},$$

where the errors are projected onto the space $V$. We also introduce a mesh-factor $MF_p$, which measures the relative improvement in the corrected solution $\tilde{u}_h$ compared to a the improvement we get by a uniform refinement of $h$, defined by

$$MF_p = \frac{\| [u - u_{h/p}]^h \|}{\| [u - \tilde{u}_h]^h \|}.$$  

The exact solution $u$ is here approximated by a solution on the fine mesh corresponding to the scale $2^{-12}$. We also define corresponding gain-factors and mesh-factors for the derivatives by

$$GF^D = \frac{\| [Du - Du_h]^h \|}{\| [Du - D\tilde{u}_h]^h \|}, \quad MF^D_p = \frac{\| [Du - Du_{h/p}]^h \|}{\| [Du - D\tilde{u}_h]^h \|}.$$  

6 The case $\epsilon(x) = \beta(x) = 0$

We first consider the problem $\alpha u = f$. We assume to start with that $f = 1$ and we consider the quantity

$$\tilde{F}_h = [\alpha]^h[1/\alpha]^h - 1,$$
where \( \alpha \) is a fractal function of the type (11). For \( \alpha' \) small (\( \gamma \) small), we have that

\[
1/\alpha = 1/(1 + \alpha') \approx 1 - \alpha',
\]

so that \( 1/\alpha \) is on a similar form as \( \alpha \), and we thus have from Corollary 4 that \( \bar{F}_h(x) \) is of the form \( C(x) h^{\mu(x)} \).

We now report the result of some numerical experiments using different values for \( \gamma \) and \( \delta \) in (11) with \( \gamma \) small. We compare \( \bar{F}_h \), which we here can compute directly from the data, to \( \tilde{F}_h \), which is an extrapolated approximation of \( \bar{F}_h \).

In our first example we let \( \gamma = 0.05 \) and \( \delta = 0.5 \), and we set \( H = 2h \) and \( H = 4h \). In Figure 3 we plot \( \bar{F}_H, \bar{F}_{\hat{H}}, \bar{F}_h \) and \( \tilde{F}_h \) for this particular case.

We see that \( \tilde{F}_h \) approximates \( \bar{F}_h \) well, except that \( \tilde{F}_h \) is not able to reconstruct the finest scale in \( \bar{F}_h \) since we extrapolate from \( \bar{F}_H \) and \( \bar{F}_{\hat{H}} \). We have a relative error \( \| \bar{F}_h - \tilde{F}_h \| / \| \bar{F}_h \| = 3.2 \cdot 10^{-2} \), and in Tab. 1 we report the relative errors for different values of \( \gamma \) and \( \delta \). We notice that scale extrapolation works better as \( \gamma \) decreases, which seems natural since then the approximation (15) is more

![Fig. 3. \( \bar{F}_H, \bar{F}_{\hat{H}}, \bar{F}_h \) and \( \tilde{F}_h \)]

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( | \bar{F}_h - \tilde{F}_h | / | \bar{F}_h | )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.3</td>
<td>7.3 \cdot 10^{-2}</td>
</tr>
<tr>
<td>0.06</td>
<td>0.5</td>
<td>3.9 \cdot 10^{-2}</td>
</tr>
<tr>
<td>0.08</td>
<td>0.3</td>
<td>1.0 \cdot 10^{-1}</td>
</tr>
<tr>
<td>0.08</td>
<td>0.5</td>
<td>5.3 \cdot 10^{-2}</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3</td>
<td>1.3 \cdot 10^{-1}</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>6.9 \cdot 10^{-2}</td>
</tr>
</tbody>
</table>

Table 1

Relative errors \( \| \bar{F}_h - \tilde{F}_h \| / \| \bar{F}_h \| \) for different \( \gamma \) and \( \delta \).
accurate. It also works better for $\delta$ large, which seems natural since then the small scale features are less significant.

In Figure 4 we plot the modeling residual for some different functions $f \neq 1$. The extrapolation works well for all source terms tested, except for $f(x) = x(1 - x)$, close to where $\bar{F}_h$ change sign. The fact that $\bar{F}_H$ and $\bar{F}_\hat{H}$ live on different scales can cause problems near where $\bar{F}_\hat{H}$ and $\bar{F}_H$ change signs, because $\bar{F}_H$ can be on the 'wrong side' of $\bar{F}_\hat{H}$. This makes the extrapolation go in the wrong direction, and it might ruin the quality of the extrapolation. To avoid this problem we could for example simply let $\bar{F}_h$ be equal to $\bar{F}_H$ whenever $\bar{F}_H$ is on the 'wrong side' of $\bar{F}_\hat{H}$. We employ this correction and plot for

Fig. 4. $f(x) = x^2$, $f(x) = x(1 - x)$ and $f(x) = \sin(2\pi x)$. 
By solving  
\[ f(x) = x(1 - x) \]  in Figure 5. By solving

\[ f(x) = x(1 - x) \]

we get an improved solution \( \tilde{u}_h \) (Figure 6), where the error \( u - \tilde{u}_h \) is proportional to the difference \( F_h - \tilde{F}_h \), as \( u - \tilde{u}_h = (F_h - \tilde{F}_h)/[\alpha]^h \). The error in the non corrected solution \( u_h \) is proportional to \( F_h \), since \( u - u_h = F_h/[\alpha]^h \). Remembering that here \( \tilde{F}_h = \tilde{F}_h \) (since \( \tilde{F}_h \) is directly computable from the

\[ [\alpha]^h \tilde{u}_h = [f]^h + \tilde{F}_h, \]

data), the simple calculation

\[ \tilde{u}_h = ([f]^h + \tilde{F}_h)/[\alpha]^h = ([f]^h + [\alpha]^h[u]^h - [f]^h)/[\alpha]^h = [u]^h, \]

shows that here in fact \( \tilde{u}_h = [u]^h \), so \( \tilde{u}_h \) is the best solution we can obtain on the scale \( h \).
Now we consider the problem of finding a continuous function $u$ such that

$$
\beta Du = f \quad \text{in} \ I, \quad u(0) = 0,
$$

where $Du = \frac{d}{dx}$. The solution is given by the formula

$$
u(x) = \int_0^x \frac{f(y)}{\beta(y)} dy.
$$

In this case the simplified problem takes the form

$$
[\beta]^h Du_h = [f]^h \quad \text{in} \ I, \quad u_h(0) = 0,
$$

with solution

$$
u_h(x) = \int_0^x \frac{[f]^h(y)}{[\beta]^h(y)} dy.
$$

The modeling residual is given by

$$
F_h(u) = f^h - [f]^h - (\beta Du - [\beta]^h Du),
$$

and

$$
$$

Again, $\tilde{F}_h$ is computable from data. Assuming now $f$ and $\beta$ to be fractal, extrapolation should be possible by the same reasoning as in the previous section. We compute a corrected solution $\tilde{u}_h$ by solving

$$
[\beta]^h D\tilde{u}_h = [f]^h + \tilde{F}_h,
$$

where $\tilde{F}_h$ is an extrapolated approximation of $\tilde{F}_h$. Since $\tilde{F}_h$ is computable directly from data, and thereby $\tilde{F}_h = \tilde{F}_h$, again a simple calculation shows that $D\tilde{u}_h = [Du]^h$, but we are also interested in the quality of the solution $\tilde{u}_h$ ($D\tilde{u}_h = [Du]^h \Rightarrow \tilde{u}_h = [u]^h$). In Table 2 we present the result of some numerical experiments for $f(x) = 1$. We extrapolate from $H = 2h$ and $\tilde{H} = 4h$, and we find in Table 2 that the corrected solution on $h$ is better than a non corrected solution on $h/4$.

There is a difference in structure between the error in the solution and the error in the derivative. The error $u - u_h$ is here integrated, whereas the error $Du - Du_h$ has the same structure as in the previous section:

$$
(u - u_h)(x) = \int_0^x \frac{F_h(y)}{[\beta]^h(y)} dy,
$$

$$
(Du - Du_h)(x) = F_h(x)/[\beta]^h(x).
$$
The error in the corrected solution depends on the difference $F_h - \tilde{F}_h$:

\[
(u - \tilde{u}_h)(x) = \int_0^x (F_h(y) - \tilde{F}_h(y))/[\beta^h(y)] \, dy,
\]

\[
(Du - D\tilde{u}_h)(x) = (F_h(x) - \tilde{F}_h(x))/[\beta^h(x)].
\]

The error in the solution at $x$ is equal to the integrated error from 0 to $x$, whereas the error in the derivative at $x$ only depends on the value of the relevant functions at $x$. This means that that the effect of adding the modeling residual in the $L_2$-norm might be 'averaged out' if $\gamma$ is not great enough. The same is true for $\delta$ being too great, since then again the small scale features are not sufficiently significant ($\tilde{F}_h$ is too small). This we can see in Table 2 where $GF$ for the corrected solution $\tilde{u}_h$ is better for $\gamma$ large and $\delta$ small.

In Table 3 we present numerical experiments for the case when $f \neq 1$ ($\gamma = 0.08$), and we find that also for $f \neq 1$ the corrected solutions on $h$ are more accurate than a non corrected solution on $h/4$. Here, fractal denotes a fractal function according to (11), with $\gamma = 0.04$ and $\delta = 0.4$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$GF$</th>
<th>$MF_2$</th>
<th>$MF_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.1</td>
<td>11</td>
<td>8.5</td>
<td>6.5</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>2.6</td>
<td>1.9</td>
<td>1.2</td>
</tr>
<tr>
<td>0.08</td>
<td>0.1</td>
<td>13</td>
<td>10</td>
<td>7.6</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>3.5</td>
<td>2.3</td>
<td>1.5</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>15</td>
<td>12</td>
<td>9.4</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>4.3</td>
<td>2.9</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Table 2
$GF$ and mesh-factors for different $\gamma$ and $\delta$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\delta$</th>
<th>$GF$</th>
<th>$MF_2$</th>
<th>$MF_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(1-x)$</td>
<td>0.1</td>
<td>16</td>
<td>12</td>
<td>9.4</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>4.5</td>
<td>2.8</td>
<td>1.7</td>
</tr>
<tr>
<td>fractal</td>
<td>0.1</td>
<td>14</td>
<td>12</td>
<td>8.8</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>3.7</td>
<td>2.6</td>
<td>1.7</td>
</tr>
</tbody>
</table>

Table 3
$GF$ and mesh-factors for different $\delta$ ($\gamma = 0.08$).
Now we consider the initial value problem
\[ \beta Du + \alpha u = f \text{ on } I, \ u(0) = 0. \]

In this case we have
\[
\tilde{F}_h(u) = [\beta]^h[Du]^h - [\beta Du]^h + [\alpha]^h[u]^h - [\alpha u]^h,
\]
\[
\]
where now the solution \( u \) appears, and thus \( \tilde{F}_h(u) \) is not directly computable from data. Of course, using the solution formula (2) above, we can eliminate \( u \), but at any rate global effects enter.

### 8.1 The case \( \alpha(x) = f(x) = 1 \)

First we consider the case when \( \alpha(x) = f(x) = 1 \), which gives
\[
\tilde{F}_h(u) = ([\beta]^h[1/\beta]^h - 1) + ([u]^h - [\beta]^h[u/\beta]^h) = \tilde{F}_h^1 + \tilde{F}_h^2(u).
\]

Here we cannot compute \( \tilde{F}_h(u) \) directly from data, since \( \tilde{F}_h(u) \) is a function of \( u \). Instead we use \( u_h \) as a substitute for the exact solution \( u \). Clearly we should be able to extrapolate \( \tilde{F}_h^1 \) by the discussion in Section 6, and \( \tilde{F}_h^2(u) \) is also a quantity of the form (8). We are thus led to an Ansatz of the form
\[
\tilde{F}_h^1(x) \approx C_1(x)h^{\mu_1(x)}, \quad \tilde{F}_h^2(u)(x) \approx C_2(x)h^{\mu_2(x)},
\]
where the coefficients \( C_i(x) \) and \( \mu_i(x) \) are computed by extrapolating \( (\tilde{F}_{h_i}^1, \tilde{F}_{h_i}^2(u)) \) separately by computing the corresponding quantities \( (\tilde{F}_H^1, \tilde{F}_H^1) \) and \( (\tilde{F}_H^2(u_h), \tilde{F}_H^2(u_h)) \). We start by computing \( u_h \), then we compute \( (\tilde{F}_H^1, \tilde{F}_H^1) \) and \( (\tilde{F}_H^2(u_h), \tilde{F}_H^2(u_h)) \) from which we compute the coefficients \( C_1(x), C_2(x), \mu_1(x) \) and \( \mu_2(x) \). This gives \( \tilde{F}_h \), by which we can compute \( \tilde{u}_h \) from the equation \( [\beta]^hD\tilde{u}_h + [\alpha]^h\tilde{u}_h = [f]^h + \tilde{F}_h \).

In this section \( D\tilde{u}_h \neq [Du]^h \), and we are interested in the quality of \( D\tilde{u}_h \). In Table 4, we present the result of some numerical experiments. Again, we find that a corrected derivative \( D\tilde{u}_h \) on the scale \( h \) is a better solution than a non corrected derivative \( Du_h \) on the scale \( h/4 \). In Table 5 we present the result of similar numerical experiments for the corrected solution \( \tilde{u}_h \). Here we see that a corrected solution \( \tilde{u}_h \) on the scale \( h \) corresponds to a non corrected solution on a finer scale than \( h/4 \) when \( \delta \) is small, and a scale \( h_f \) such that \( h/4 < h_f < h/2 \) when \( \delta \) is larger. The dependence on \( \gamma \) and \( \delta \) seems natural, since the equation
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$GF^D$</th>
<th>$MF_2^D$</th>
<th>$MF_4^D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.1</td>
<td>3.1</td>
<td>2.2</td>
<td>1.6</td>
</tr>
<tr>
<td>0.3</td>
<td>3.6</td>
<td>2.1</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>0.08</td>
<td>0.1</td>
<td>2.4</td>
<td>1.7</td>
<td>1.2</td>
</tr>
<tr>
<td>0.3</td>
<td>4.5</td>
<td>2.6</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.9</td>
<td>1.5</td>
<td>1.0</td>
</tr>
<tr>
<td>0.3</td>
<td>4.3</td>
<td>2.5</td>
<td>1.4</td>
<td></td>
</tr>
</tbody>
</table>

Table 4
$GF^D$ and mesh-factors for the derivative $D\bar{u}_h$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$GF$</th>
<th>$MF_2$</th>
<th>$MF_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.1</td>
<td>2.2</td>
<td>1.7</td>
<td>1.2</td>
</tr>
<tr>
<td>0.3</td>
<td>2.2</td>
<td>1.2</td>
<td>0.63</td>
<td></td>
</tr>
<tr>
<td>0.08</td>
<td>0.1</td>
<td>3.1</td>
<td>2.2</td>
<td>1.6</td>
</tr>
<tr>
<td>0.3</td>
<td>2.9</td>
<td>1.6</td>
<td>0.91</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>3.6</td>
<td>2.8</td>
<td>1.9</td>
</tr>
<tr>
<td>0.3</td>
<td>2.0</td>
<td>1.1</td>
<td>0.56</td>
<td></td>
</tr>
</tbody>
</table>

Table 5
$GF$ and mesh-factors for $\bar{u}_h$.

in this section is very similar to the one considered in Section 7 (apart from the forcing term being changed from $f(x) = 1$ to the non linear $f(x) = 1 - u(x)$), and the discussion on how the effect of adding the modeling residual in the $L_2$-norm might be averaged out in some cases applies also here.

8.2 The case $\beta(x) = f(x) = 1$

We now consider the problem

$$Du + \alpha u = 1 \text{ on } I, \ u(0) = 0.$$ 

In this case we have

$$\bar{F}_h(u) = [\alpha]^h[u]^h - [\alpha u]^h,$$

so by the previous discussion the extrapolation of $\bar{F}_h(u)$ should be possible if $\alpha$ and $u$ have a fractal structure. However, the result of adding the extrapolated modeling residual is not that significant in this case, there is in fact no improvement in the corrected solution $\tilde{u}_h$, and the improvement in the corrected derivative $D\tilde{u}_h$ is not as significant as in the previous cases. The lack of improvement of $\tilde{u}_h$ could be explained by the connection between $\alpha$
and \( u \) through the solution formula (2), where the primitive function of \( \alpha \) is integrated to get \( u \). In Table 6 we present the result of some numerical experiments for the derivative \((H = 2h, \tilde{H} = 4h)\). We see that in this problem the

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( GF^D )</th>
<th>( MF^D_2 )</th>
<th>( MF^D_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>5.0</td>
<td>2.2</td>
<td>0.85</td>
</tr>
<tr>
<td>0.3</td>
<td>3.2</td>
<td>1.2</td>
<td>0.45</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>5.0</td>
<td>2.2</td>
<td>0.85</td>
</tr>
<tr>
<td>0.3</td>
<td>3.2</td>
<td>1.2</td>
<td>0.44</td>
<td></td>
</tr>
</tbody>
</table>

Table 6

\( GF^D \) and mesh-factors for \( D\tilde{u}_h \).

quality of the corrected derivative on the scale \( h \) corresponds to the quality of a non corrected derivative on a scale \( h_f \) such that \( h/4 < h_f < h/2 \).

8.3 The case \( f(x) = 1 \)

Now we consider the case when both \( \alpha \) and \( \beta \) are fractal test functions (not necessary the same), and \( f(x) = 1 \). We have

\[
\bar{F}_h(u) = \left( [\beta]^h[1/\beta]^h - 1 \right) + \left( [\alpha]^h[u]^h - [\beta]^h[\alpha u/\beta]^h \right) = \bar{F}^1_h + \bar{F}^2_h(u),
\]

where \( \bar{F}_h(u) \) is split into the two parts \( \bar{F}^1_h \) and \( \bar{F}^2_h(u) \). The first term \( \bar{F}^1_h \) is of the type described in Section 6, and by (15) we have that extrapolation is possible. The second term \( \bar{F}^2_h(u) \) should be possible to extrapolate if \( \alpha \), \( \beta \) and \( u \) have a fractal structure. So again we are led to an Ansatz of the form

\[
\bar{F}^1_h(x) \approx C_1(x)h^{\mu_1(x)}, \quad \bar{F}^2_h(u)(x) \approx C_2(x)h^{\mu_2(x)}.
\]

We present the result of some numerical experiments \((H = 2h, \tilde{H} = 4h)\) in Table 7, where we find that a corrected solution \( \tilde{u}_h \) on the scale \( h \) corresponds a non corrected solution on a finer scale than \( h/4 \) when \( \delta \) is small, and a scale \( h_f \) such that \( h/4 < h_f < h/2 \) when \( \delta \) is larger. The quality of a corrected derivative on the scale \( h \) is better than a non corrected derivative on the scale \( h/4 \), and in Figure 7 we plot \( \bar{F}_H(u_h), \bar{F}_{\tilde{H}}(u_h), \bar{F}_h(u) \) and \( \bar{F}_h \).

8.4 Different force terms

We conclude this section by presenting the result of some numerical experiments when \( f \neq 1 \). In Table 8 we present numerical results for \( \gamma_\alpha = 0.5 \), \( \delta_\alpha = 0.3 \), \( \gamma_\beta = 0.1 \), \( \delta_\beta = 0.1 \), and \( H = 2h, \tilde{H} = 4h \). We find that both the
Table 7  
Gain-factors and mesh-factors for $\tilde{u}_h$ and $D\tilde{u}_h$.

<table>
<thead>
<tr>
<th>$\gamma_\alpha$</th>
<th>$\delta_\alpha$</th>
<th>$\gamma_\beta$</th>
<th>$\delta_\beta$</th>
<th>$GF$</th>
<th>$MF_2$</th>
<th>$MF_4$</th>
<th>$GF^D$</th>
<th>$MF_2^D$</th>
<th>$MF_4^D$</th>
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</thead>
<tbody>
<tr>
<td>0.5 0.3 0.08 0.1</td>
<td>2.2 1.6 1.1 2.5</td>
<td>1.8 0.93</td>
<td>0.1 0.3 2.8 2.0</td>
<td>1.4 2.7</td>
<td>1.9 1.4</td>
<td>0.3 3.0 1.9 1.0</td>
<td>4.4 2.6 1.5</td>
<td>0.5 0.08 0.1 2.1</td>
<td>1.4 2.8</td>
</tr>
</tbody>
</table>

Fig. 7. $\bar{F}_H(u_h)$, $\bar{F}_{\hat{H}}(u_h)$, $\bar{F}_h(u)$ and $\tilde{F}_h$

quality of the corrected solution and the corrected derivative on the scale $h$ is better than the quality of their non corrected counterparts on the scale $h/4$. Here, fractal denotes a fractal test function according to (11), with $\gamma = 0.04$ and $\delta = 0.4$.

Table 8  
Gain-factors and mesh-factors for different source terms.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$GF$</th>
<th>$MF_2$</th>
<th>$MF_4$</th>
<th>$GF^D$</th>
<th>$MF_2^D$</th>
<th>$MF_4^D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>2.0</td>
<td>1.5</td>
<td>1.1</td>
<td>1.9</td>
<td>1.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$x(1-x)$</td>
<td>3.0</td>
<td>2.1</td>
<td>1.5</td>
<td>2.2</td>
<td>1.6</td>
<td>1.1</td>
</tr>
<tr>
<td>$\sin(2\pi x)$</td>
<td>2.8</td>
<td>2.0</td>
<td>1.4</td>
<td>2.7</td>
<td>2.1</td>
<td>1.4</td>
</tr>
<tr>
<td>fractal</td>
<td>2.6</td>
<td>1.9</td>
<td>1.2</td>
<td>2.0</td>
<td>1.4</td>
<td>1.1</td>
</tr>
</tbody>
</table>
We now consider the boundary value problem
\[-D(\epsilon Du) + \beta Du + \alpha u = f, \quad x \in I,\] (16)
with a Dirichlet boundary condition \(u(0) = 0\) at inflow, and a Neumann boundary condition \(Du(1) = 0\) at outflow (assuming \(\beta > 0\)). Now we have an even more complex connection between data and solution. The simplified problem is now to find \(u_h\) such that
\[-D([\epsilon]^h Du_h) + [\beta]^h Du_h + [\alpha]^h u_h = [f]^h, \quad x \in I.\] (17)
The solution to the exact problem \(u\) satisfies
\[-D([\epsilon]^h Du) + [\beta]^h Du + [\alpha]^h u = [f]^h + F_h(u), \quad x \in I,\] (18)
where
\[F_h(u) = f - [f]^h - (-D(\epsilon Du) + \beta Du + \alpha u + D([\epsilon]^h Du) - [\beta]^h Du - [\alpha]^h u),\]
which gives
\[\tilde{F}_h(u) = \left(\left[D(\epsilon Du)^h - D([\epsilon]^h Du)^h\right] + \left(\left[\beta]^h Du^h - [\beta Du]^h\right) + \left(\left[\alpha]^h u^h - [\alpha u]^h\right) \right) = \tilde{F}_h^1(u) + \tilde{F}_h^2(u) + \tilde{F}_h^3(u).\]

As in the previous sections we make a separate Ansatz for \(\tilde{F}_h^1(u), \tilde{F}_h^2(u),\) and \(\tilde{F}_h^3(u)\). Using this Ansatz, we can extrapolate to obtain \(\tilde{F}_h\), an approximation of \(\tilde{F}_h(u)\), which we use to compute the corrected solution \(\tilde{u}_h\) from the equation
\[-D([\epsilon]^h D\tilde{u}_h) + [\beta]^h D\tilde{u}_h + [\alpha]^h \tilde{u}_h = [f]^h + \tilde{F}_h, \quad x \in I.\] (19)

9.1 The elliptic case with \(\alpha = \beta = 0\)

We now consider the case when \(\beta = \alpha = 0\), that is the problem
\[-D(\epsilon Du) = f, \quad x \in I,\]
with boundary conditions \(u(0) = Du(1) = 0\). Now \(u_h\) is defined to be the solution to the simplified problem
\[-D([\epsilon]^h Du_h) = [f]^h, \quad x \in I,\] (20)
and the solution to the exact problem \( u \) satisfies
\[
-D([\varepsilon]^h Du) = [f]^h + F_{h}(u), \quad x \in I,
\]
with the modeling residual

\[
F_{h}(u) = f - [f]^h - (-D(\varepsilon Du) - D([\varepsilon]^h Du)),
\]
and
\[
\bar{F}_{h}(u) = [D(\varepsilon Du)]^h - [D([\varepsilon]^h Du)]^h.
\]

In this case \( \bar{F}_{h}(u) \) does not appear to be on the form (8), and consequently we cannot motivate the Ansatz on \( \bar{F}_{h}(u) \) which we have used in the previous sections. We therefore turn to the following variational formulation of (9.1):

Find \( u \in V = \{ v : v, Dv \in L_2(I), v(0) = 0 \} \) such that

\[
\int_0^1 \epsilon Du Dv \, dy = \int_0^1 f v \, dy, \quad \forall v \in V.
\] (21)

We have that \( u_h \) as defined in (20) satisfies

\[
\int_0^1 [\varepsilon]^h Du_h Dv \, dy = \int_0^1 [f]^h v \, dy, \quad \forall v \in V,
\]
and the solution to the exact problem \( u \) satisfies

\[
\int_0^1 [\varepsilon]^h Du Dv \, dy = \int_0^1 [f]^h v \, dy + \int_0^1 F_{h}^\nabla (u) Dv \, dy, \quad \forall v \in V,
\]
where we have introduced a new modeling residual \( F_{h}^\nabla (u) \), defined by

\[
F_{h}^\nabla (u) = [\varepsilon]^h Du - \epsilon Du,
\]
and

\[
\bar{F}_{h}^\nabla (u) = [\varepsilon]^h [Du]^h - [\epsilon Du]^h,
\]
which is of the form (8). We can therefore again use the Ansatz

\[
\bar{F}_{h}^\nabla (u)(x) \approx C(x) h^{u(x)},
\]
and we expect extrapolation of \( \bar{F}_{h}^\nabla (u) \) to be possible if \( \varepsilon \) and \( Du \) have a fractal structure. We denote the extrapolated approximation of \( \bar{F}_{h}^\nabla (u) \) by \( \tilde{F}_{h}^\nabla \) and we denote the corresponding corrected solution \( \tilde{u}_h \), where \( \tilde{u}_h \) is the solution to the problem

\[
\int_0^1 [\varepsilon]^h D\tilde{u}_h Dv \, dy = \int_0^1 [f]^h v \, dy + \int_0^1 \tilde{F}_{h}^\nabla Dv \, dy, \quad \forall v \in V.
\]

For the periodic version of this problem, with a smooth force, it is well known that the homogenized operator is obtained by using the harmonic average
\[ \tilde{u}_h(h,4h) = (2h,4h) \]
\[ \tilde{u}_h(4h,8h) = 1.7 \]
\[ \tilde{u}_h(8h,16h) = 3.1 \]

<table>
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<th>( \tilde{u}_h )</th>
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</table>

Table 9

\[ \tilde{u}^{harm}_h \]
\[ (h,4h) \]
\[ 2h \]
\[ 2.8 \]
\[ 2.0 \]
\| \| \| \| \|

\[ \tilde{u}^{harm}_h \]
\[ (h,8h) \]
\[ 3.2 \]
\[ 2.2 \]
\| \| \| \| \|

\[ \tilde{u}^{harm}_h \]
\[ (h,16h) \]
\[ 3.1 \]
\| \| \| \|

\[ \tilde{u}^{harm}_h \]
\[ (h/2,4h) \]
\[ 2.8 \]
\| \| \| \|

Table 10

Fractal force \( (\gamma = 0.1, \delta = 0.2) \) and \( (\gamma_c = 0.1, \delta_c = 0.3) \).
If the diffusion coefficient $\epsilon$ in the problem (16) is constant, then

$$\bar{F}_h(u) = \left( [\beta]^h [Du]^h - [\beta Du]^h \right) + \left( [\alpha]^h [u]^h - [\alpha u]^h \right).$$

(22)

$\bar{F}_h(u)$ consists of two parts, each of the type (8), which can be extrapolated if $\beta$, $Du$, and $\alpha$, $u$ respectively have a fractal structure. By choosing fractal test functions $\alpha$ and $\beta$ we expect extrapolation to work when $\epsilon$ is small, since then we have a convection dominated problem as in Section 8. In Figure 8 we plot

Fig. 8. $\bar{F}_{H}(u_h)$, $\bar{F}_{\hat{H}}(u_h)$, $\bar{F}_h(u)$ and $\tilde{F}_h$ for $H = 2h$ and $\hat{H} = 4h$.

the two parts of $\bar{F}_h(u)$, and their extrapolated approximations for $\epsilon = 10^{-6}$ $(H = 2h$ and $\hat{H} = 4h)$. We find that the extrapolation of the second part of $\bar{F}_h(u)$, based on the solution $u$, works fine. But in the case of the first part of $\bar{F}_h(u)$, which is based on the derivative $Du$, the extrapolation is not as good.
This is the same problem as we met in the previous section, that is, \( Du_h \) does not approximate \( Du \) well enough. We can then either extrapolate from scales further from \( h \), or we can avoid to base the extrapolation on the derivative at all by eliminating \( Du \) from (22) by using (16):

\[
\bar{F}_h(u) = [\beta]^h[(f - \alpha u + \epsilon D^2 u)/\beta]^h - [f - \alpha u + \epsilon D^2 u]^h + [\alpha]^h[u]^h - [\alpha u]^h \\
= F^1_h + F^2_h(u) + F^3_h(u).
\]

Here we recognise \( F^1_h \) and \( F^2_h(u) \) as being the same corrective forces as in Section 8, and they should be possible to extrapolate by the discussion there. This seems natural, since then if the diffusion is small the corrective force is close to the case when the diffusion is zero. Now in the case of \( \epsilon \) being small, we approximate \( \bar{F}_h(u) \approx F^1_h + F^2_h(u) \). In Table 11 we present the result of numerical experiments for \( \epsilon = 10^{-6} \) (\( H = 2h \) and \( \hat{H} = 4h \)). We find that the quality of the corrected solution \( \tilde{u}_h \) on the scale \( h \) is better than the quality of a non corrected solution on the scale \( h/4 \), and the same is true for the corrected derivative \( D\tilde{u}_h \).

<table>
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<td>0.1</td>
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</tr>
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</table>

Table 11
Gain-factors and mesh-factors for \( \epsilon = 10^{-6} \).

### 9.3 The full problem

We consider the problem (16) where we now all coefficients \( \epsilon, \alpha \) and \( \beta \) are fractal. Following the discussion in Section 9.1 we consider the following variational formulation of (16): Find \( u \in V = \{ v : v, Dv \in L^2(I), v(0) = 0 \} \) such that

\[
\int_0^1 \epsilon DuDv \, dy + \int_0^1 \beta Duv \, dy \int_0^1 \alpha uv \, dy = \int_0^1 fv \, dy, \quad \forall v \in V. \tag{23}
\]
The solution \( u_h \) to the simplified problem (17) satisfies

\[
\int_0^1 [\epsilon]^h Du_h dv + \int_0^1 [\beta]^h Du_h v dv \int_0^1 [\alpha]^h u_h v dv = \int_0^1 [f]^h v dv, \quad \forall v \in V,
\]

and the exact solution \( u \) satisfies

\[
\int_0^1 [\epsilon]^h Du dv + \int_0^1 [\beta]^h Du v dv \int_0^1 [\alpha]^h u v dv
\]

\[
= \int_0^1 [f]^h v dv + \int_0^1 F_h(u)v dv + \int_0^1 F_h^\nabla(u)Dv dv, \quad \forall v \in V,
\]

where the two modeling residuals \( F_h(u) \) and \( F_h^\nabla(u) \) are given by

\[
F_h^\nabla(u) = [\epsilon]^h Du - \epsilon Du,
\]

\[
F_h(u) = [\beta]^h Du - \beta Du + [\alpha]^h u - \alpha u + f - [f]^h,
\]

and thus

\[
F_h^\nabla(u) = [\epsilon]^h [Du]^h - [\epsilon Du]^h,
\]

\[
F_h(u) = ([\beta]^h [Du]^h - [\beta Du]^h) + ([\alpha]^h [u]^h - [\alpha u]^h) \equiv F_h^1(u) + F_h^2(u).
\]

We see that \( F_h^\nabla(u) \), \( F_h^1(u) \) and \( F_h^2(u) \) are of the form (8) and we expect extrapolation to be possible by using an Ansatz of the form

\[
F_h^1(u)(x) = C_1(x)h^{\mu_1(x)}, \quad F_h^2(u)(x) = C_2(x)h^{\mu_2(x)}, \quad F_h^\nabla(u)(x) = C_3(x)h^{\mu_3(x)}.
\]

We obtain the improved solution \( \tilde{u}_h \in V \) from the variational problem

\[
\int_0^1 [\epsilon]^h D\tilde{u}_h dv + \int_0^1 [\beta]^h D\tilde{u}_h v dv \int_0^1 [\alpha]^h \tilde{u}_h v dv
\]

\[
= \int_0^1 [f]^h v dv + \int_0^1 F_h v dv + \int_0^1 F_h^\nabla dv dv, \quad \forall v \in V.
\]

The relative importance of \( F_h^\nabla(u) \) compared to \( F_h(u) \) depends on the size of \( \epsilon \). If \( \epsilon \) is small, then the problem is convection dominated and the results are similar to the results in Section 9.2 (by using (23) and neglecting the diffusion term). On the other hand, if the problem is diffusion dominated (\( \epsilon \) large) the modeling residual \( F_h^\nabla(u) \) based on \( \epsilon \) is the dominant one.

In Table 12 we present the result of numerical experiments for \( \epsilon \phi = 10^{-3} \), where both \( F_h(u) \) and \( F_h^\nabla(u) \) contributes to the correction of \( \tilde{u}_h \). As in Section 9.1 we cannot avoid using the derivative as a base for the extrapolation. To compensate for the lack of accuracy of \( Du_h \) as an approximation for \( Du \),
again we extrapolate further away from $h$. We use $H = 4h$, $\tilde{H} = 8h$ and we let $f(x) = 1$. In this case a corrected solution $\tilde{u}_h$ on the scale $h$ is better than a non corrected solution on a scale $h/2$. For the derivative the improvement is not that great.

Finally, in Table 13 we present the numerical results for a fractal force $f$, with $\gamma = 0.05$ and $\delta = 0.5$, using the scale extrapolated modeling residual as a correction on the force.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\delta_\epsilon$ & $\delta_\alpha$ & $\delta_\beta$ & $GF$ & $MF_2$ & $MF_2^D$ \\
\hline
0.3 & 0.1 & 0.1 & 2.6 & 1.4 & 1.6 & 0.90 \\
0.3 & 11 & 5.8 & 1.6 & 0.93 \\
0.3 & 0.1 & 2.1 & 1.3 & 1.6 & 0.87 \\
0.3 & 5.1 & 2.8 & 1.8 & 0.94 \\
\hline
\end{tabular}
\caption{Gain-factors and mesh-factors ($\gamma_\epsilon = 0.1$, $\gamma_\alpha = 0.3$, $\gamma_\beta = 0.1$).}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\delta_\epsilon$ & $\delta_\alpha$ & $\delta_\beta$ & $GF$ & $MF_2$ & $MF_2^D$ \\
\hline
0.3 & 0.1 & 0.1 & 2.5 & 1.4 & 1.6 & 0.90 \\
0.3 & 11 & 5.8 & 1.6 & 0.92 \\
0.3 & 0.1 & 2.1 & 1.3 & 1.6 & 0.88 \\
0.3 & 4.8 & 2.7 & 1.8 & 0.94 \\
\hline
\end{tabular}
\caption{Fractal force, with $\gamma_\epsilon = 0.1$, $\gamma_\alpha = 0.3$, and $\gamma_\beta = 0.1$.}
\end{table}

10 Error analysis

We now turn to the issue of developing adaptive algorithms, including quantitative error control based on a posteriori error estimates, using the extrapolation technique described above. We recall that we formally have

\begin{align}
  u - u_h &= L_h^{-1}F_h(u), \quad (27) \\
  u - \tilde{u}_h &= L_h^{-1}(F_h(u) - \tilde{F}_h), \quad (28)
\end{align}

which shows the connection between $F_h(u)$ and $u - u_h$, and $F_h(u) - \tilde{F}_h$ and $u - \tilde{u}_h$, via the inverse $L_h^{-1}$ of $L_h$. In order for $u - \tilde{u}_h$ to be smaller than $u - u_h$, that is, for $\tilde{u}_h$ to be an improvement of $u_h$, we anticipate that $F_h(u) - \tilde{F}_h$ should be smaller than $F_h(u)$. 

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In the next sections we derive explicit estimates corresponding to the abstract relations (27)-(28), and we first fix some notation for this section. The space $C(I)$, consisting of continuous functions on $I$, is equipped with the uniform norm $\| \cdot \|_\infty$, defined by

$$\|u\|_\infty = \sup_{x \in I} |u(x)|.$$  

The space $L_2(I)$, consisting of Lebesgue square integrable functions on $I$, is a Hilbert space with inner product and norm defined by

$$(v, w) = \int_I v(x)w(x) \, dx, \quad \|w\| = \sqrt{(w, w)}.$$  

We also use the $\rho$-weighted inner product $(\cdot, \cdot)_\rho$ and norm, defined by

$$(v, w)_\rho = \int_I \rho(x)v(x)w(x) \, dx, \quad \|w\|_\rho = \sqrt{(w, w)_\rho},$$  

for $v, w \in L_2(I)$, and $\rho : I \to \mathbb{R}_+$ locally integrable in $I$. We define $L_{2,\rho}$ to be the Hilbert space where the norm $\| \cdot \|_\rho$ is finite.

**Remark 5** In this paper we only consider modeling errors. We assume the numerical errors to be zero, and thus the total residuals are equal to the modeling residuals, that is

$$(u - u_h)(x) = \int_0^x \exp(A_h(y) - A_h(x)) \frac{F_h(u)(y)}{[\beta]^h(y)} \, dy,$$  

where $A_h(x)$ is a primitive function of $[\alpha]^h/[\beta]^h$ (satisfying $DA_h = [\alpha]^h/[\beta]^h$, $A_h(0) = 0$). We take absolute values of both sides, then estimate the right
hand side to get
\[ |u - u_h|(x) \leq \frac{1}{c_1} \|F_h(u)\|_\infty \int_0^1 |\exp(A_h(y) - A_h(x))|dy. \]

By the mean value theorem for \( x, y \in I \) we get that
\[ |A_h(x) - A_h(y)| \leq \max_{t \in I} |DA_h(t)| |y - x| = \max_{t \in I} \frac{\alpha^h(t)}{\beta^h(t)} |y - x|, \]
from which the desired result follows. \( \square \)

**Theorem 7** Assume that \( 0 < \alpha^h \leq c_0, 0 < c_1 \leq \beta^h \), and \( \epsilon = 0 \), then
\[ \|u - \tilde{u}_h\|_\infty \leq C\|F_h(u) - \tilde{F}_h\|_\infty, \]
where \( C = (1/c_1) \exp(c_0/c_1) \).

**Proof.** We recall that \( L_h(u - \tilde{u}_h) = F_h(u) - \tilde{F}_h \), and as in the proof of Theorem 6, we have
\[ (u - \tilde{u}_h)(x) = \int_0^x \exp(A_h(y) - A_h(x)) \frac{F_h(u)(y) - \tilde{F}_h(y)}{\beta^h(y)} dy, \]
from which the desired estimate follows. \( \square \)

**Remark 8** The errors in Theorem 6-7 are also bounded in the \( L_2 \)-norm, since
\[ \|w\| = \left( \int_0^1 w^2(y) \, dy \right)^{1/2} \leq \left( \int_0^1 \|w\|_\infty^2 \, dy \right)^{1/2} = \|w\|_\infty. \]
We can also obtain estimates in terms of \( L_2 \)-norms by using Cauchy-Schwarz inequality.

### 10.2 A priori error estimates for convection-diffusion-reaction problem

For the problem with non-zero diffusion, we do not have an explicit solution formula as we had for the problem in the previous section. Instead, we are going to use an energy argument to obtain error estimates. In this section, Theorem 9 and 10 corresponds to (27)-(28) when \( \epsilon \neq 0 \).

**Theorem 9** Assume that \( 0 < c_0 \leq \beta^h \leq C_0 \) and \( 0 < c_1 \leq \alpha^h \), then
\[ \|Du - Du_h\|_{\ell^h}^2 + \|u - u_h\|^2 + (u - u_h)^2(1) \leq C \left( \|F_h(u)\|^2 + \|F_{\nabla}^h(u)\|_{\ell^h}^2 \right), \]
where \( C = (c_0 \min(c_1, 1) \min(1, c_1/C_0, 1/C_0))^{-1} \exp(C_0/c_0) \).
Proof. First we subtract (17) from (18), where we set $e = u - u_h$. Then we multiply both sides with $e$ and integrate from 0 to $x_{i,k}$. By using partial integration and rewriting $e De$ as $De^2/2$, we get:

$$
2 \int_0^{x_{i,k}} [e]^h(De)^2 \, dy + \int_0^{x_{i,k}} [\beta]^h De^2 \, dy + 2 \int_0^{x_{i,k}} [\alpha]^h e^2 \, dy
$$
$$
= 2 \int_0^{x_{i,k}} F_h(u)e \, dy + 2 \int_0^{x_{i,k}} F_h^\n(u)De \, dy,
$$

where $F_h^\n(u)$ and $F_h(u)$ are defined by (24) and (25). Since $[\beta]^h$ is constant on each interval $I_{i,l}$ we have

$$
\int_0^{x_{i,k}} [\beta]^h De^2 \, dy = \sum_{l=1}^{k-1} \int_{I_{i,l}} [\beta]^h De^2 \, dy = [\beta]^h(x_{i,k}^-)e^2(x_{i,k}) - \sum_{l=1}^{k-1} e^2(x_{i,l})[\beta]^h|_l,
$$

where $|[\beta]^h|_l = [\beta]^h(x_{i,l}^-) - [\beta]^h(x_{i,l}^+)$ is the jump in $[\beta]^h$ at $x_{i,l}$. This gives

$$
2 \int_0^{x_{i,k}} [e]^h(De)^2 \, dy + \int_0^{x_{i,k}} [\alpha]^h e^2 \, dy + [\beta]^h(x_{i,k}^-)e^2(x_{i,k})
$$
$$
= 2 \int_0^{x_{i,k}} F_h(u)e \, dy + 2 \int_0^{x_{i,k}} F_h^\n(u)De \, dy + \sum_{l=1}^{k-1} e^2(x_{i,l})[\beta]|_l.
$$

Now we can split the first integral in the right hand side by using Cauchy-Schwarz inequality, and the inequality $2ab \leq a^2 + b^2$ ($a, b \in \mathbb{R}$):

$$
2 \int_0^{x_{i,k}} F_h(u)e \, dy \leq 2(\int_0^{x_{i,k}} F_h^2(u) [\alpha]^h \, dy)^{1/2}(\int_0^{x_{i,k}} [\alpha]^h e^2 \, dy)^{1/2}
$$
$$
\leq \int_0^{x_{i,k}} \frac{F_h^2(u)}{[\alpha]^h} \, dy + \int_0^{x_{i,k}} [\alpha]^h e^2 \, dy.
$$

Then we move the second of these new integrals to the left hand side. By the same operations on the second integral we get:

$$
\int_0^{x_{i,k}} [e]^h(De)^2 \, dy + \int_0^{x_{i,k}} [\alpha]^h e^2 \, dy + [\beta]^h(x_{i,k}^-)e^2(x_{i,k})
$$
$$
= \int_0^{x_{i,k}} \frac{F_h^2(u)}{[\alpha]^h} \, dy + \int_0^{x_{i,k}} (F_h^\n(u))^2 \, dy + \sum_{l=1}^{k-1} e^2(x_{i,l})[\beta]|_l.
$$

We divide the equation by $[\beta]^h(x_{i,k}^-)$ and use that $0 < c_0 \leq [\beta]^h$. By then using Grönwall’s lemma we get
\[
\frac{1}{[\beta]^h(x_{i,k})} \int_0^{x_{i,k}} [\epsilon]^h (De)^2 \, dy + \frac{1}{[\beta]^h(x_{i,k})} \int_0^{x_{i,k}} [\alpha]^h e^2 \, dy + e^2(x_{i,k}) \\
\leq \frac{1}{c_0} \left( \int_0^{x_{i,k}} F^2(u) \, dy + \int_0^{x_{i,k}} \frac{(F\nabla)^2(u)}{[\epsilon]^h} \, dy \right) \exp \left( \frac{1}{c_0} \sum_{l=1}^{k-1} |[\beta]^h|_l \right).
\]

This is true for all \(x_{i,k} \in I\), in particular it is true for \(x_{i,k} = x_{i,2^i} = 1\). If we then use the given bounds on \([\alpha]^h\) and \([\beta]^h\) we get

\[
\frac{1}{C_0} \int_0^1 [\epsilon]^h (De)^2 \, dy + \frac{1}{C_0} \int_0^1 c_1 e^2 \, dy + e^2(1) \\
\leq \frac{1}{c_0} \left( \int_0^1 F^2_h(u) \, dy + \int_0^1 \frac{(F\nabla)^2(u)}{[\epsilon]^h} \, dy \right) \exp \left( \frac{1}{c_0} \sum_{l=1}^{2^i-1} |[\beta]^h|_l \right).
\]

But now we observe that

\[
\sum_{l=1}^{2^i-1} |[\beta]^h|_l = [\beta]^h(1),
\]

from which the desired estimate follows. \(\blacksquare\)

**Theorem 10** Assume that \(0 < c_0 \leq [\beta]^h \leq C_0\) and \(0 < c_1 \leq [\alpha]^h\), then

\[
\| Du - D\tilde{u}_h\|_{[\epsilon]^h}^2 + \| u - \tilde{u}_h\|^2 + (u - \tilde{u}_h)^2(1) \\
\leq C \left( \| F_h(u) - \tilde{F}_h\|^2 + \| F\nabla_h(u) - \tilde{F}\nabla_h\|^2 \right),
\]

where \(C = (c_0 \min(c_1, 1) \min(1, c_1/C_0, 1/C_0))^{-1} \exp(C_0/c_0)\).

**Proof.** If we subtract (19) from (17) and set \(e = u - \tilde{u}_h\), then multiply both sides by \(e\) and integrate from 0 to \(x_{i,k}\), we get

\[
2 \int_0^{x_{i,k}} [\epsilon]^h (De)^2 \, dy + \int_0^{x_{i,k}} [\beta]^h De^2 \, dy + 2 \int_0^{x_{i,k}} [\alpha]^h e^2 \, dy \\
= 2 \int_0^{x_{i,k}} (F_h(u) - \tilde{F}_h) e \, dy + 2 \int_0^{x_{i,k}} (F\nabla_h(u) - \tilde{F}\nabla_h) De \, dy.
\]

Then by following the proof of Theorem 9, the desired estimate follows. \(\blacksquare\)

### 11 Conclusions

In this paper we investigated the performance of a subgrid model based on scale extrapolation of a modeling residual, in the case of a one dimensional
convection-diffusion-reaction problem with rough coefficients with features on a range of scales. Assuming a certain scale regularity, with respect to a Haar MRA, of the coefficients we motivated an Ansatz on the modeling residual of the form $C h^\mu$, where $h$ is the mesh size and $C = C(x)$ and $\mu = \mu(x)$ coefficients to be determined. We then showed that it was possible to determine the subgrid model on coarse scales with a fine scale computed solution without subgrid model as a reference, and then extrapolate the subgrid model to the computational scale. We presented error estimates for the modeling error of the corrected and the non corrected solutions in terms of modeling residuals. We showed in numerical experiments that by extrapolating from $2h$ and $4h$, where $h$ is the computational scale, the modeling errors in the corrected solution were typically less than in a non corrected solution on the scale $h/4$. It was further noted that the scale extrapolation procedure was more effective when the extrapolation was based on the computed solution $u_h$ itself and not on the derivative $Du_h$, since $Du_h$ is not a sufficiently good approximation of $Du$.

References


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