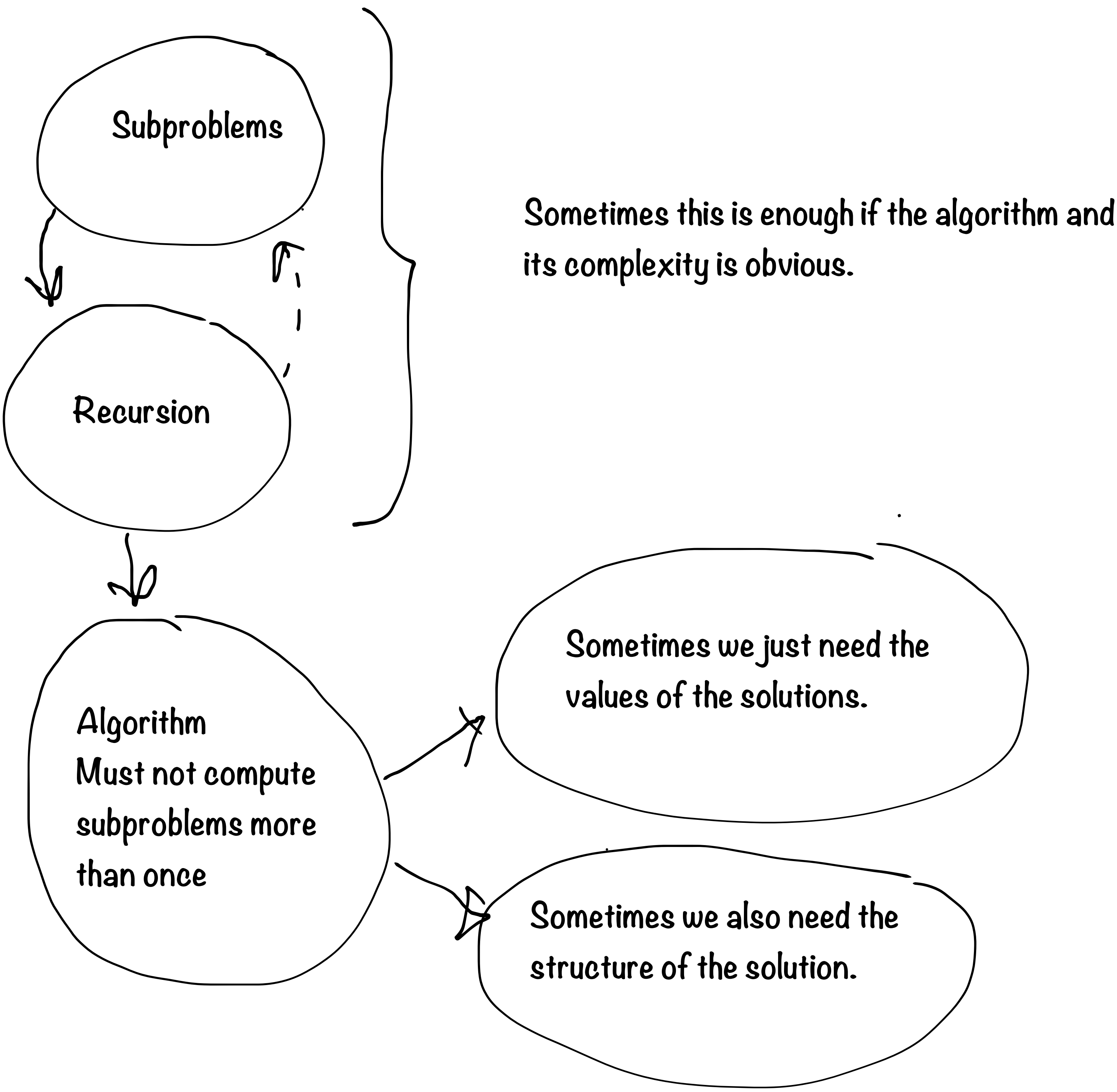
Dynamic Programming cont.

We repeat: The Dynamic Programming Template has three parts.



Let us return to the shortest path problem. Is Dijkstra's algorithm a DP-algorithm? We have subproblems
$d[u]$ = length of shortest path from s to u.
We have a type of recursion
$d[v] = d[u] + w[u,v]$
The problem is that we don't have a simple way of ordering the subproblems. In that sense, Dijkstra's algorithm isn't a true DP-algorithm.

If we have a directed graph with no cycles (A DAG = Directed Acyclic Graph) things are simpler. In a DAG we can find a so called Topological Ordering.

Topological Ordering: An ordering of the nodes such that
$(v[i], v[j])$ is an edge $\Rightarrow i < j$
A topological ordering can be found in time $O(|E|)$ (See textbook).

Let's assume that the start node is $v[1]$. Set $w[i,j] = \infty$ if there is no edge $(v[i], v[j])$. Then

$$\begin{cases} d[1] = 0 \\ d[k] = \min_i \left(d[i] + w[i,k] \right) \quad 1 \le i \le k \end{cases}$$

The algorithm runs in $O(n^2)$

Subset Sum

We assume that we have n positive integers a[1], a[2], ... , a[n]. We are given an integer M. We want to know if there is a subset of the integers with sum M.

What are the natural subproblems here? We can try to get the sum M by using fewer than n integers. Or we can try to get a smaller sum than M. In fact, we will combine these two ideas.

Set v[i,m] = 1 if there is a subset of a[1], a[2], ... , a[i] with sum m and v[i,m] = 0 otherwise.

If v[i,m] = 1 it must be either because we can get m just by using the numbers a[1], a[2], ... , a[i-1] or because we can get the sum m - a[i] by using the same numbers. We get the recursion

$v[1, 0] = 1$

For all i such that $2 \leq i \leq n$ and all m M such that $a[i] \leq m$

$v[i,m] = \max(v[i-1,m], v[i-1,m-a[i]])$

We now try to construct an algorithm. We have to order the subproblems. We compute all v[i,m] by running an outer loop over $1 \leq i \leq n$ and an inner loop over $1 \leq m \leq M$.

```
Set all v[i,j] = 0
For i ←1 to n
    v[i,0] ←1
For i ←2 to n
    For m ←1 to M
        If v[i-1, m] = 1
            v[i,m] ←1
        Else If m > a[i]  and v[i-1, m-a[i]] = 1
            v[i, m] ←1
Return v[n,M]
```

When the algorithm stops, the value of v[n,M] tells us the solution to the problem. (1 = " It's possible" , 0 = " It's not possible".) The complexity is O(n M).

In Dynamic Programming-problems we have some value that we want to optimize. We express this value with some array like v[n] and try to find a recursion formula and then use it to find all values v[i]. In some cases this is all we want. In other cases we might want to find the actual "choices" leading to these values. If we have an algorithm which solves the recursion equation we can often modify it so that it gives us the actual choices.

For instance, in the previous problem we wanted to to find the values v[i,m]. If we know that v[i,m] = 1 and also want to find the terms in the sum, we can modify our algorithm:

```
Set all v[i,j] = 0
For i ← 1 to n
    v[i,0] ← 1
    choose[i,0] ← FALSE
    p[i,0] ← NULL
For i ← 2 to n
    For m ← 1 to M
        If v[i-1, m] = 1
            v[i,m] ← 1
            choose[i,0] ← FALSE
            p[i,m] ← [i-1,m]
        Else If m > a[i]  and  v[i-1, m-a[i]] = 1
            v[i, m] ← 1
            choose[i,m] ← TRUE
            p[i,m] ← [i-1,m-a[i]]
Return v[n,M]
```

New

A simpler Subset Sum problem

One thing that makes the original Subset Sum problem hard is that we are allowed to use each number just once. If we can use the numbers multiple times we get a simpler DP-problem.

Set $v[m] = 1$ if we can get m as a subset sum and 0 otherwise.

Then we can compute the values by

$$\begin{cases} v[m] = 0 \text{ for all } m < 0 \\ v[0] = 1 \\ v[m] = \max_k \left(v[m-a[k]] \right) \quad 1 \le k \le n \end{cases}$$

We will return to the Subset Sum problem once more. Remember that we defined

v[i,m] = 1 if there is a subset of a[1], a[2], ..., a[i] with sum m and v[i,m] = 0 otherwise.

We got the recursion formula

v[1, 0] = 1
For all i such that $2 \le i \le n$ and all m M such that a[i] $\le$ m
v[i,m] = max (v[i-1,m], v[i-1,m-a[i]])

In lecture 5 we gave an algorithm that solved the problem. It's possible to give a recursive algorithm as well. A first try could look like:

```
vrek[i,m] =
    If m < 0
        Return 0
    If m = 0
        Return 1
    If i = 1 and m = a[1]
        Return 1
    If vrek[i-1, m] = 1
        Return 1
    If vrek[i-1, m-a[i]] = 1
        Return 1
    Return 0
```

We make the call vrek[n,M] to get the answer.

But this solution is no good. The problem is that the algorithm uses repeated calls to subproblems that already have been solved.

To get a better algorithm will have to keep track of all computed values of subproblems. To do this, we use an array comp[i,m],

```
Set all comp[i,j] to FALSE
Set all v[i,j] to 0
vrek[n,M]


vrek[i,m] =
    If comp[i,m]
        Return v[i,m]
    If m < 0
        Return 0
    If m = 0
        Return 1
    If vrek[i-1, m] = 1
        comp[i,m] ← TRUE
        v[i,m] ←1
        Return 1
    If vrek[i-1, m-a[i]] = 1
        comp[i,m] ← TRUE
        v[i,m] ←1
        Return 1
    comp[i,m] ←TRUE
    v[i,m] ←0
```

This technique of remembering already computed values is called Memoization. Sometimes it can be useful, but in most cases the bottom-up method should be preferred.

Matrix Chain Multiplication

We want to compute the product of two matrices A and B. A is a $p \times q$- matrix and B is a $q \times r$-matrix. The cost (number of products of elements) is pqr.

Let us assume that we want to compute a chain of matrices. We want to find the best way to multiply them. If we have three matrices A, B, C then we know from the associative law of multiplication that (AB)C = A(BC). But the costs of computing the product will normally differ!

If we have a chain of matrices M[1] M[2] ... M[n] what is the best way of computing the product?

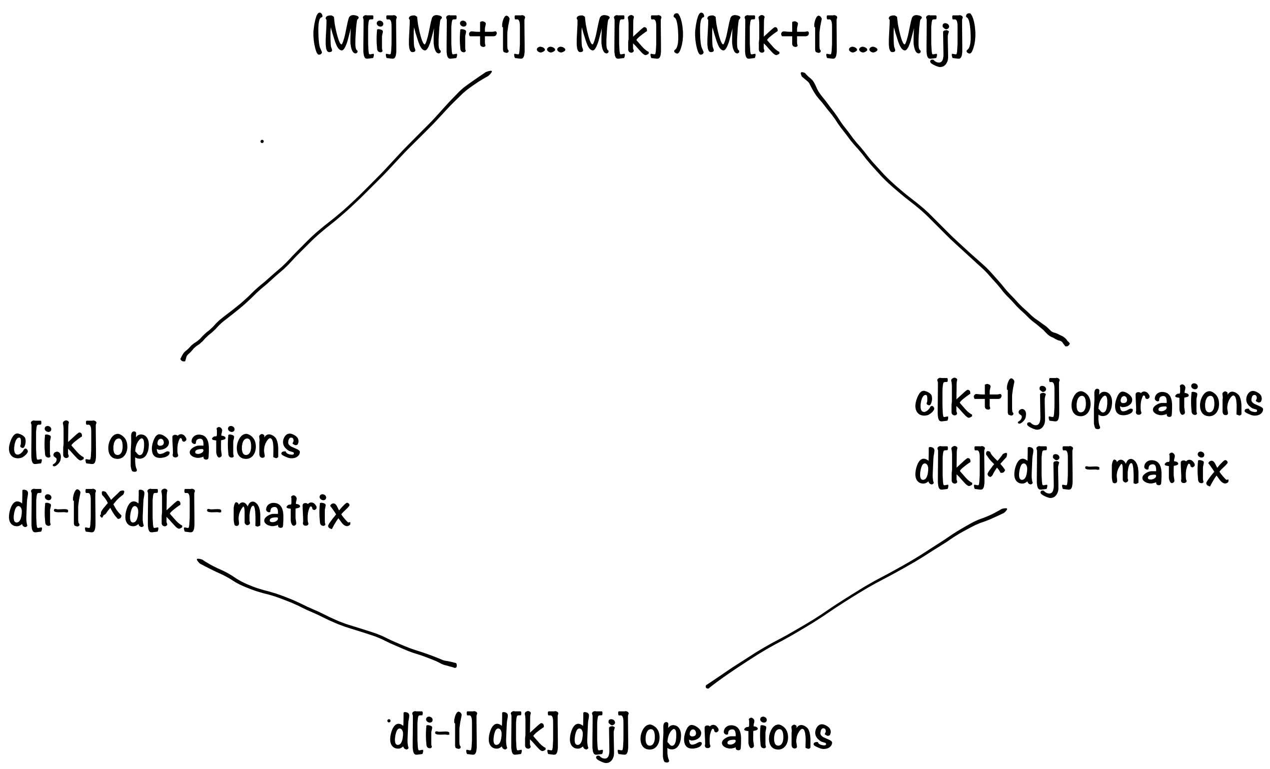
Subproblems:
Set c[i,j] = smallest possible cost of computing M[i] M[i+1] ... M[j].

Recursion:

Let us first assume that the matrices have dimensions $d[0] \times d[1], d[1] \times d[2], \ldots, d[n-1] \times d[n]$.

$$\begin{cases} c[i,i] = 0 \text{ for all } 1 \le i \le n \\ c[i,j] = \min_k \left(c[i,k] + c[k+1,j] + d[i-1]d[k]d[j] \right) \text{ where } i \le k < j \end{cases}$$

Why?



All together: c[i,k] + c[k+1] + d[i-1] d[k] d[j] operations

Now we have to find an algorithm using the recursion. Essentially we have to find suitable loops. We can try to first compute all c[i,j] with $|j-i| = 1$, then with $|j-i| = 2$ and so on. If we do this we are able to use the recursion formula.

```
For i ← 1 to n
    c[i,i] ← 0
For diff ← 1 to n-1
    For i ← 1 to n - diff
        j ← i + diff
        min ← c[i+1,j] + d[i-1] d[i+1] d[j]
        best_k ← i
        For k ← i+1 to j - 1
            If min > c[i,k] + c[k+1,j] + d[i-1] d[k] d[j]
                min ← c[i,k] + c[k+1,j] + d[i-1] d[k] d[j]
                best_k ← k
        c[i,j] ← min
        break[i,j] ← best_k
```

The value of c[1,n] gives the minimum number of operations.

The values of break[i,j] tells how the split should be done. The complexity is $O(n^3)$

Pretty Print

We have a set of n words. They have lengths l[i] (number of characters). We want to print them on a page. Each line on the page contains space for M characters. There must be a space 1 between each pair of words.

Set $s[i,j] = \sum_{k=i}^{j} l[k] + j - i$.

This will be the number if characters left on the line if the words i to j are put on the line. Let E = M - s[i,j] be the excess of space on the line. We want to put the words (in correct order) on lines so that the excesses are as small as possible. We can use a penalty function f( ) and try to make a split of the words such that f(E1) + f(E2) + ... i.e. the sum of the penalties from the lines is as small as possible.

It's natural to use the Last Line Excluded rule (LLE), i.e. we give no penalty for excess on the last line.

We now want to find the best way to arrange the words. It's simplest to first ignore LLE.

Let w[k] = least penalty when using the first k words and not using LLE.

Recursion:

$w[0] = 0$

$w[k] = \min_i (w[i-1] + f(M - s[i,k]))$ where the min is taken over all $1 \le i \le k$ such that $s[i,k] \le M$

To get the solution with LLE we compute

$\min_j w[j]$ such that $s[j+1, n] \le M$

Editing distance and sequence alignment

We have two strings x[1], x[2], ... , x[m] and y[1], y[2], ... , y[n]. We want to align them so the number of positions where the alligned sequences are different is minimal. We are allowed to put gaps into the sequences.

Ex: The sequences EXPONENTIAL and POLYNOMIAL can be aligned as

EXPO _ _ NENTIAL
_ _ POLYNOM_IAL

Let
D[p,q] = distance of best alignment of a[1], ..., a[p] and b[1], ..., a[q]
We measure distance by adding a number α for each match between a character and a blank and adding β for a match between two different characters.

Then we get the recursion formula

D[p,0] = αp   D[0,q] = αq for all p,q

D[p,q] = min (D[p,q-1] + α , D [p-1,q] + α, D[p-1,q-1] + β diff[a[p],b[q]])
if p>1 and q > 1

TSP (Travelling Salesperson Problem)

Then input is a complete, weighted graph G. The goal is to find the lenght of a shortest cycle visisting every nod exactly once.

We can give a DynP-solution to this problem. Let S be a set of nodes such that 1 (node 1) is in S. Assume j is in S. We let C[S, j] be the length of a shortest path in S starting at 1 and ending at j anf visiting every node in S exactly once. We can the find the solution to TSP by:

C[{1},1] = 1

For k = 2 to n

For all subsets S of {1, ... , n} of size k such that 1 belongs to S

For all j ≠ 1 in S

C[S, j] = min { C[S-{j}, i] + w[i,j] : i in S and i ≠ 1, j}

Return min { C[S, i] + w[i,1] }

So the returned value is the length of a shortest cycle visiting every node exactly once. The time complexity is $O(n^2 2^n)$.