The Flow Problem as LP problem

We let x_e be the flow on edge e. We have the constraints $0 \leq x_e \leq c(e)$ for all e. For each node x except s and t we have

$$\sum_{e \in In(x)} x_e = \sum_{e \in Out(x)} x_e$$

We set

$$v = \sum_{e \in Out(s)} x_e$$

The flow problem can be written as

Maximize v

when

$$\begin{cases} v = \sum_{e \in Out(s)} x_e & \\ \sum_{e \in In(x)} x_e = \sum_{e \in Out(x)} x_e & \text{for all x except s,t} \\ 0 \leq x_e \leq c(e) & \text{for all edges} \end{cases}$$

Another problem: A transport problem

A company produces milk in 4 different plants. The milk is delivered to 5 customers. The company has to consider three things:

1. The capacities of the plants.
2. The demands of the customers.
3. The costs of the transports between plants and customers.

Let us call the plants F1, F2, F3, F4.

Capacity:

F1	F2	F3	F4
30	40	30	40

(The numbers represent 1000 liters.)

Let us call the customers K1, K2, K3, K4, K5.

Demand:

K1	K2	K3	K4	K5
20	30	15	25	20

(The numbers represent 1000 liters.)

Transport costs:

	K1	K2	K3	K4	K4
F1	$2,80$	$2,55$	$3,25$	$4,30$	$4,35$
F2	$4,30$	$3,15$	$2,55$	$3,30$	$3,50$
F3	$3,00$	$3,30$	$2,90$	$4,30$	$3,40$
F4	$5,20$	$4,45$	$3,50$	$3,75$	$2,45$

Goal:

Decide how the ”flow” to the customers should be so that

1. The customers are satisfied.
2. The cost are minimal.

Mathematical model:

Use variables x_{ij} for the flow from plant i to customer j.

What demands do we have?

1. Capacities

Ex: For plant 1 we should have
$x_{11}+x_{12}+x_{13}+x_{14}+x_{15} \leq 30000$

2. Demand

Ex: For customer 1 we should have
$x_{11}+x_{21}+ x_{31}+x_{41} = 20000$

Cost:

z = 2,80 x_{11} + 2,55 x_{12} + ... + 2,45 x_{45}

We use the following definitions:

Let c_{ij} be the cost for transport from plant i to customer j.

Let s_i be the capacity for plant i.

Let d_j be the demand of customer j.

The problem can now be written as

Minimize $\sum_{i=1}^{4} \sum_{j=1}^{5} c_{\mathrm{ij}} x_{\mathrm{ij}}$

when
$$\begin{gathered} \sum_{j=1}^{5} x_{\mathrm{ij}} \leqslant s_i \;\; i = 1,2,3,4 \\ \sum_{i=1}^{4} x_{\mathrm{ij}} = d_j \;\; j = 1,2,3,4,5 \\ x_{\mathrm{ij}} \geqslant 0 \end{gathered}$$

Linear Programming

A Linear Programming problem is the following:

Input: We have n variables $x_1, x_2, ..., x_n$ and m linear equalities and/or inequalities in the variables. We can also have constraints that say that some (all) of the variables should be non-negative. We are given a linear function f in the variables.

Goal: We want to find values for the variables so that the constraints are fulfilled and the function f is optimized (maximized/minimized).

Different forms

We can express an LP-problem on different forms. We have

1. General form: That is the one described above.

2. Canonical form: Essentially a form with just inequalities. This form is suitable for analyzing mathematical properties of solutions.

3. Standard form: Essentially a forma with just equalities. This form is used when actually finding solutions.

The general form covers all LP-problems. But all problems can in a certain way be translated to equivalent problems on canonical and standard forms.

Linear Programming

A linear programming problem on canonical form is

Minimize $\sum_{j=1}^{n} c_j x_j$

when
$\sum_{j=1}^{n} a_{\mathrm{ij}} x_j \leqslant b_i \ i = 1, 2, \ldots, m$
$x_{j \geqslant 0}$

In some texts the authors use maximization instead of minimization. This doesn't matter much since we can always translate one form to the other by changing the sign of the c_i :s.

Translations

If we have a problem that is not on canonical form we can rewrite it on form. We show how it can be done by looking at some examples:

Example:

Minimize

$$x_1 + 2x_2 - x_3$$

when

$$\begin{cases} x_1 + x_3 = 1 \\ x_2 - x_3 \geq 3 \end{cases}$$

Inequalities "in the wrong direction"can be turned right by a sign change.

Equalities can be turned into inequalities by using two using two inequalities for each equality.

In our problem we get

Minimize

$$x_1 + 2x_2 - x_3$$

when

$$\begin{cases} x_1 + x_3 \leq 1 \\ -x_1 - x_3 \leq -1 \\ x_3 - x_2 \leq -3 \end{cases}$$

Example: A company ISC

The company ISC is an ice sport company that makes bandy sticks and hockey sticks.
There are two steps in the production: Sawing and glueing.

There are times needed for the two steps

	Sawing	Gluing	
Hockey	7	16	(minutes)
Bandy	10	12	

The firm has capacity for 3600 minutes of sawing and 5400 minutes of gluing per week.

The sticks can be sold for
Hockey: 125 kr but has a production cost of 105 kr.
Bandy: 115 kr but has a production cost of 97 kr.

Let x_1 be the number of produced hockey x_2 the number of produced bandy sticks.

We get the following problem

Maximize z $= 20x_1 + 18x_2$

when

$x_1 + 10x_2 \leq 3600$

$16x_1 + 12x_2 \leq 5400$

$x_1, x_2 \geq 0$

Obs: We can transform the problem to canonical form if we say that we want to minimize -20x_1-18x_2

Towards solutions: Standard forms

Preparation: We transform the problem to so called standard form

Standard form: We have equalities instead of inequalities.

Ex:

Minimize z = $3x_1$ + $5x_2$ - x_3

when

$x_1 - x_2 + 2x_3 = 5$
$x_1 + 2x_2 + 4x_3 = 12$
$x_1, x_2, x_3 \geqslant 0$

We get equalities by introducing Slack Variables.

Ex: Let us assume that we have the inequality $x_1 + 3x_2 \leqslant 10$

We set $x_3 =$ 10 - (x$_1 + 3x_2$)

x_3 is a new slack variable.

We get the equality $x_1 + 3x_2 + x_3 = 10$

The ISC problem put on standard form will be:

$7x_1 + 10x_2 \leqslant 3600$ reduces to $7x_1 + 10x_2 + x_3 = 3600$

$16x_1 + 12x_2 \leqslant 5400$ reduces to $16x_1 + 12x_2 + x_4 = 5400$

We get

Maximize $\quad$ z $= 20x_1 + 18x_2$

when
$7x_1 + 10x_2 + x_3 = 3600$
$16x_1 + 12x_2 + x_4 = 5400$
$x_1,\ x_2,\ x_3,\ x_4 \geqslant 0$

Standard form

Minimize z = $\sum_{j=1}^{n} c_j x_j$

when

$\sum_{j=1}^{n} a_{\mathrm{ij}} x_j = b_i \quad$ i = 1,... ,m

$x_j \geqslant 0 \quad$ j = 1, ... ,n

We can use matrix notation

Minimize $\bar{c}^T \bar{x}$

when

$A\bar{x} = \bar{b}$

$\bar{x} \geqslant \bar{0}$

ICS will look like:

$$A = \begin{pmatrix} 7 & 10 & 1 & 0 \\ 16 & 2 & 0 & 1 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \ \bar{c} = \begin{pmatrix} -20 \\ -18 \\ 0 \\ 0 \end{pmatrix} \ \bar{b} = \begin{pmatrix} 3600 \\ 5400 \end{pmatrix}$$

Minimize (-20, -18, 0, 0) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

when

$$\begin{pmatrix} 7 & 10 & 1 & 0 \\ 16 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3600 \\ 5400 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

How to find a solution

Maximize z = $20x_1 + 18x_2$.
When

$7x_1 + 10x_2 + x_3 = 3600$
$16x_1 + 12x_2 + x_4 = 5400$

How do we find the best solution?

One possibility is $x_3 = x_4 = 0$

$7x_1 + 10x_2 = 3600$
$16x_1 + 12x_2 = 5400$

If we solve the system we get $x_1 \approx 142$ $x_2 \approx 260$

It gives us z ≈ 7520

But instead, we can put $x_2 = x_4 = 0$

We get the equations

$7x_1 + x_3 = 3600$
$16x_1 = 5400$

They give us $x_1 \approx 337$ $x_3 \approx 1237$

Then z $\approx$ 2362.

Are there more solutions?

Basic solutions:

Let us assume that we have n variables and m equations. We also assume that all equations are linearly independent. We assume that we have set n-m of the variables to 0.
Then the other m variables have unique values. This gives us a basic solution .

Feasible basic solution:

If all variables are $\geqslant 0$ we have a feasible basic solution.

The solution to a LP-problem is always a feasible basic solution (FBS).

But which FBS?

Method:

Variables which are 0 (at a certain stage) are called non-basic variables. The other variables are called basic variables.

We test different FBS:s by changing the basic variables one at a time.

Ex: Minimize $z = 2x_1 + x_2$

when $3x_1 + x_2 = 10$
$x_1, x_2 \geqslant 0$

Set $x_1 = 0$.

Then $x_2 = 10$ and $z = 10$.

We now change basic variables so that $x_2 = 0$.

Then $x_1 \approx 3{,}33$

we get z $\approx$ 6,67.

So we have found a better solution.

How do you know if you have found the best solution?

Ex: ISC

$x_1 = 142 \quad x_2 = 260 \quad z = 7520$

Is that the best solution?

We can write

x_3 = 3600 - 7x_1 -10x_2
x_4 = 5400 - 16x_1 - 12x_2

x_1 = 0,158x_3 - 0,132x_4 + 142,1
x_2 = -0,2x_3 + 0,092x_4 + 260,5

That gives us z = 20x_1 + 18x_2 = 20(0,15x_3 - 0,13x_4 + 142,1) + 18(-0,21x_3 + 0,09x_4 + 260,5) =
7520 - 0,62x_3 - 0,98x_4

Now we see that we would gain nothing by increasing x_3 or x_4.
We see that any change from this solution must end in a worse solution.

General description of the Simplex Method

Let's say that we have a maximization problem and a FBS with basic variables y_1, y_2, ... , y_m and non-basic variables v_1, v_2 , ... , v_{n-m}.

This means that $v_1 = v_2 = ... = v_{n-m} = 0$

We can then write y_1, y_2, ... , y_m as functions of v_1, v_2 , ... , v_{n-m}

$y_1 = f_1\,(v_1, ..., v_{n-m})\, y_2 = f_2\,(v_1, ..., v_{n-m})$
...
In the same way we can write z as

z $= c_1 v_1 + c_2 v_2 + ... + c_{n-m} v_{n-m} + z_0$

If all c_i are < 0 we must have an optimal solution.

If any $c_i > 0$, say $c_1 > 0$, we can increase z by increasing v_1. But then the values of the y:s must change. How much do they change?

We can increase v_1 until $f_k(v_1, v_2, ...) = 0$ for some k. Then v_1 will be a new basic variable and y_k will be a new non-basic variable.
We go on like this until all $c_i \leq 0$. Then we have found the optimal solution.

If we have a minimization problem we must try to increase variables with $c_i < 0$. When all $c_i \geq 0$ we have a solution.

Ex:

Minimize $z = 2x_1 + 2x_2 + x_3$

when

$$x_1 + x_2 + x_3 = 5$$
$$x_1 - x_2 + 2x_3 = 8$$
$$x_1, x_2, x_3 \geqslant 0$$

One FBS is $x_2 = 0$ (non-basic variable).

We get

$$x_1 + x_3 = 5 - x_2$$
$$x_1 + 2x_3 = 8 + x_2$$

$$x_1 = 2 - 3x_2$$
$$x_3 = 3 + 2x_2$$

$$z = 2(2-3x_2) + 2x_2 + (3+2x_2) = 7 - 2x_2$$

We can increase x_2. But how much?

x_1 and x_3 must be $\geqslant 0$.

$x_1 = 2 - 3x_2$

This means $x_2 \leqslant \frac{2}{3}$

$x_3 = 3 + 2x_2$

This gives us no bound on x_2.

So $x_2 = \frac{2}{3}$ and $x_1 = 0$.

$x_3 = \frac{13}{3}$

We now write x_2, x_3 as functions of x_1.

$$x_2 = \frac{2}{3} - \frac{x_1}{3}$$

$$x_3 = 3 + 2x_2 = 3 + 2(\frac{2}{3} - \frac{x_1}{3}) = \frac{13}{3} - \frac{2x_1}{3}$$

$$z = 7 - 2x_2 = 7 - 2(\frac{2}{3} - \frac{x_1}{3}) = \frac{17}{3} + \frac{2x_1}{3}$$

Since we gain nothing by increasing x_1, we are done.

This is however far from the full story. There is a problem called degeneracy that can occur. This happens when when we have no $c_i > 0$ and some $c_i = 0$ (if we assume that we have a minimization problem). In that case we will have to chose some i with $c_i = 0$. Then there is a chance that we could get into an infinite cycle. In practice, there are several ways to avoid this. Another problem is how to find a starting point for the algorithm. It turns out that we can use a modified variant of the simplex algorithm to solve this problem.

Actually, in worst case, the Simplex Algorithm is not a polynomial time algorithm. In practice, however, the Simplex algorithm is always considered efficient enough.

Do solutions always exist?

If we form an LP-problem three things can happen:

1. 1. There is a solution to the problem (x_i :s and a value of f). The solution is not necessarily unique. It can be seen that if the solution is not unique, then there is an infinite set of values for the x_i :s. However, the value of f is unique.

2. 2. It can happen that there is no bound for f so that it can get arbitrarily large/small. In this case, there are no solutions.

3. 3. It is possible that there are no points fulfilling all the constraints. In this case, there are no solutions. (The Simplex algorithm can not even get started.)

   We will, of course, focus on the first case.

Examples

Ex: Maximize x_1 - x_2

when
$x_1 + x_2 \geqslant 10$
$x_1, x_2 \geqslant 0$
The problem is that $x_1 - x_2$ can be arbitrarily large. There is no solution.

Ex: Minimize $x_1 + x_2$

when
x_1 - 2 x_2 $\leqslant$ -2
$x_1 + 3x_2 \leqslant 1$
x_1, x_2 $\geqslant$0

Here we can not find x:s that satisfies the constraints. Then, of course, there is no solution to the problem.

Dual Problems

For each LP problem we can give a so called dual problem.

Ex: We have the problem

Maximize $5x_1 + 2x_2$

when
$x_1 + x_2 \leqslant 10$
$2x_1 + 3x_2 \leqslant 20$
x_1 , x_2 , $x_3 \geqslant 0$

The dual problem is

Minimize $10v_1 + 20v_2$

when
$v_1 + 2v_2 \geqslant 5$

$v_1 + 3v_2 \geqslant 2$
$v_1,\, v_2 \geqslant 0$

How do we define the dual problem?

We write the problem on the form

Maximize $\quad \bar{c}^T \bar{x}$

when
$A\bar{x} \leqslant \bar{b}$
$\bar{x} \geqslant \bar{0}$

The dual problem is

Minimize $\quad \bar{b}^T \bar{v}$

when
$A^T\ \bar{v} \geqslant \bar{c}$
$\bar{v} \geqslant \bar{0}$

The Duality Theorem

Let P_1 and P_2 be two dual problems. If one of the problems has a solution with value M, then the other problem also has a solution with value M. If we solve one of the problems we also get a solution to the other.

Ex: ISC again

We want to

Maximize $20x_1 + 18x_2$

when

$7x_1 + 10x_2 \leqslant 3600$

$16x_1 + 12x_2 \leqslant 5400$

$x_1, x_2 \geqslant 0$

The corresponding dual problem is

Minimize $3600v_1 + 5400v_2$

when
$7v_1 + 16v_2 \geqslant 20$
$10v_1 + 12v_2 \geqslant 18$
$v_1,\, v_2 \geqslant 0$

Both problems have the same value as solution.

But what does the dual problem mean?

Let us assume that ISC want to rent out its production facilities. What rent would the marketbe willing to pay? We can suppose that the market will pay v_1 kr/minute for sawing and v_2 kr/minute for gluing.

What prices v_1 and v_2 should the market set?

The market will want to minimize $3600v_1 + 5400v_2$

The market must also consider the following requirements: ISC must want to rent out. This means that ISC must make at least as much money as it would if it run the production itself.
A hockey stick can be sold with a profit of 20 kr. It will take 7 minutes of sawing and 10 minutes of gluing to make it. When ISC rents out it would get $7v_1 + 16v_2$ kr . This number must be at least 20.

$$7v_1 + 16v_2 \geqslant 20$$

In the same way we get
$10v_1 + 12v_2 \geqslant 18.$

This gives us

Minimize $3600v_1 + 5400v_2$

when

$7v_1 + 16v_2 \geqslant 20$

$10v_1 + 12v_2 \geqslant 18$

$v_1,\ v_2 \geqslant 0$

Classical example: The diet problem

Suppose that you want to buy food. You have the choice of n types of food which can have some of m different nutrients.

Let

a_{ij} be the amount of ith nutritient in a unit of the jth food.
r_i be the yearly requirement of ith nutritient.
x_j be the yearly consumption of the jth food.
$c_j$be the cost per unit of the jth food.

Then you (probably) face the problem:

Minimize $\overline{c}\overline{x}$

when
$A\bar{x} \geq \bar{r}$
$\bar{x} \geq 0$

The dual problem

The dual problem is then

Maximize $\bar{r}\bar{w}$

when $A^T\bar{w} \leq \bar{c}$
$\bar{w} \geq 0$

What does this mean? A possible application is that we have a store who wants to sell m pills containing all the nutrients. Let w_i be the price he sets for pill i (containing nutrient i). In order for the pills to be competitive with real food he must make sure that $A^T\bar{w} \leq \bar{c}$. He also wants to maximize his profit, i.e. $\bar{r}\bar{w}$.

Reduction of a problem to a LP problem

Example: Find the shortest path $s \to t$ in a weighted graph G.

There are several ways of doing this.

Maximize d_t

when

$$\begin{cases} d_v \leq d_u + w(u,v) & \text{for all edges}(u,v) \\ d_s = 0 \end{cases}$$

Dual form

A translation of the shortest path problem to dual form gives us:

Minimize

$$\sum_e x_e w(e)$$

when

$$\begin{cases} 1 = \sum_{e \in Ut(s)} x_e & \\ \sum_{e \in In(x)} x_e = \sum_{e \in Ut(x)} x_e & \text{for all x except s,t} \\ 0 \leq x_e \leq 1 & \text{for all edges} \end{cases}$$

Another example of dual problems

The flow problem can be put on dual form:

The vector $\bar{y}$ contains $|V|+|E|$ numbers. They are g_i for each node v_i and γ_j for each edge e_j.

Minimize

$$\sum_j \gamma_j c_j$$

when

$$\begin{cases} g_i - g_j + \gamma_k \geq 0 & \text{om} \quad e_k = (v_i, v_j) \\ g_n - g_1 \geq 1, \quad \gamma_j \geq 0 & \text{for all} \quad j \end{cases}$$

The solution to this problem generates a minimal cut $(S, V-S)$ and an assignment of values $g_i = 0$ if $v_i \in S$, $g_i = 0$ otherwise. $\gamma_j = 1$ if e_j goes from S to $V-S$, $\gamma_j = 0$ otherwise.