Plucker Line Coordinates

$$A = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \end{bmatrix} \qquad L_{ij} = A_i B_j - B_i A_j$$

$$L = \begin{bmatrix} 0 & 1 & 8 & 1 \\ -1 & 0 & 9 & 1 \\ -8 & -9 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

Quadric (Klein Quadric) in IP^5

6-Vector An element of IP^5

$$Det(L) = 0 \Rightarrow l_{12} l_{34} - l_{13} l_{24} + l_{14} l_{23} = 0$$

Corresponds to a line if the above condition is met

$$\Lambda = \{1, 8, 1, 9, 1, -1\}$$

$$1.-1-8.1+1.9 = -1-8+9 = 0$$

Coplanarity and Plucker Line Coordinates

$$\begin{matrix} \Lambda : [A, B] \\ \hat{\Lambda} : [\hat{A}, \hat{B}] \end{matrix} \xrightarrow{Coplanarity} Det(A, B, \hat{A}, \hat{B}) = 0$$

$$Det(A, B, \hat{A}, \hat{B}) = l_{12}\hat{l}_{34} + \hat{l}_{12}l_{34} - \left(l_{13}\hat{l}_{24} + \hat{l}_{13}l_{24}\right) + l_{14}\hat{l}_{23} + \hat{l}_{14}l_{23}$$

$$= \left(\Lambda \middle| \hat{\Lambda}\right)$$

→ BiLinear Product

→ Independent of the points

Two lines are coplanar if and only if $\left(\Lambda \middle| \hat{\Lambda}\right) = 0$

Coplanarity and Plucker Line Coordinates

$$\begin{matrix}\Lambda : [P, Q] \\ \hat{\Lambda} : [\hat{P}, \hat{Q}]\end{matrix} \xrightarrow{Coplanarity} Det(P, Q, \hat{P}, \hat{Q}) = 0$$

$$Det(P, Q, \hat{P}, \hat{Q}) = \left(\Lambda \middle| \hat{\Lambda}\right)$$

Independent of the planes

Two lines are coplanar if and only if $\left(\Lambda \middle| \hat{\Lambda}\right) = 0$

Coplanarity and Plucker Line Coordinates

$$\Lambda : [A, B]$$
$$\hat{\Lambda} : [\hat{P}, \hat{Q}]$$

$$\left(\Lambda \middle| \hat{\Lambda}\right) = \left(P^T A\right)\left(Q^T B\right) - \left(Q^T A\right)\left(P^T B\right)$$

Plucker coordinates are useful in algebraic derivations. They are used in defining the map from a line in 3-space to its image.

Quadrics

A quadric is a surface in IP^3 defined by the equation

$$X^T Q X = 0$$

where Q is a symmetric 4x4 matrix.

Quadrics and Conics

- A quadric has 9 degrees of freedom.
- Nine points in general position define a quadric.
- If the matrix Q is singular, then the quadric is *degenerate, and may be defined by fewer points.*

Quadrics and Conics

- A quadric defines a polarity between a point and a plane. The plane $\pi = QX$ *is the polar plane of X with respect to Q.*

Quadrics and Conics

- The intersection of a plane π *with a quadric Q is a conic C.*

Quadrics and Conics

- Under the point transformation X' = HX, a (point) quadric transforms as

$$Q' = H^{T} Q H^{-1}$$

Categories of Quadrics

Two parameters are used to categorize quadrics:

- Rank
- Signature

Signature of Quadrics

Since the quadric matrix Q is symmetric, it can be decomposed into an orthogonal and a diagonal matrix as follows:

$$Q = U^T D U$$

By appropriate scaling, the matrix Dcan be re-written to exclusively contain zeros, +1 s and -1 s:

$$Q = H^T D H$$

Finally one can rearrange the matrix so that zeros are located at the end and +1 s are in the beginning.

Signature of Quadrics

Since the quadric matrix Q is symmetric, it can be decomposed into an orthogonal and a diagonal matrix as follows:

$$Q = U^T D U$$

By appropriate scaling, the matrix Dcan be re-written to exclusively contain zeros, +1 s and -1 s:

$$\sigma(Q) = \sigma(D) \qquad Q = H^T D H$$

Signature=
# of diagonal +1 s
– # of diagonal -1 s

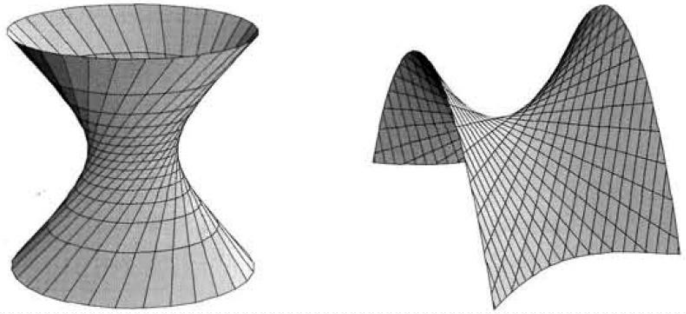
Finally one can rearrange the matrix so that zeros are located at the end and +1 s are in the beginning.

Categories of Quadrics

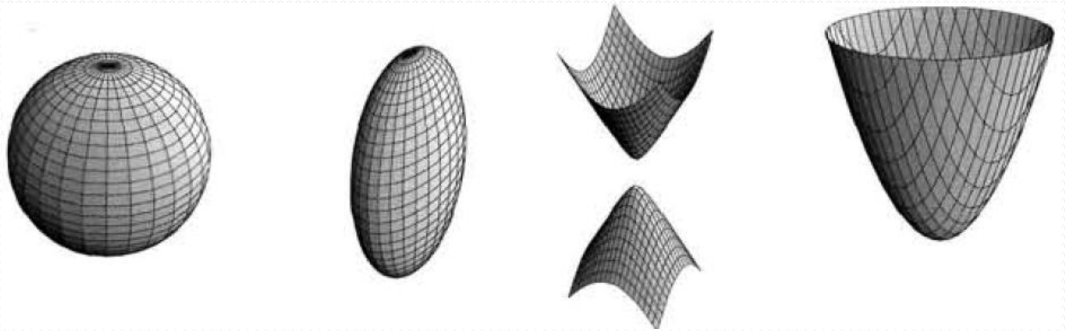
Rank	σ	Diagonal	Equation	Realization
4	4	$(1,1,1,1)$	$X^2+Y^2+Z^2+1=0$	No real points
	2	$(1,1,1,-1)$	$X^2+Y^2+Z^2=1$	Sphere
	0	$(1,1,-1,-1)$	$X^2+Y^2=Z^2+1$	Hyperboloid of one sheet
3	3	$(1,1,1,0)$	$X^2+Y^2+Z^2=0$	One point $(0,0,0,1)^T$
	1	$(1,1,-1,0)$	$X^2+Y^2=Z^2$	Cone at the origin
2	2	$(1,1,0,0)$	$X^2+Y^2=0$	Single line (Z-axis)
	0	$(1,-1,0,0)$	$X^2=Y^2$	Two planes $X=\pm Y$
1	1	$(1,0,0,0)$	$X^2=0$	The plane $X=0$

Ruled vs. Unruled Quadrics

- Ruled quadrics contain straight lines (called generators)



Ruled Quadrics



Unruled Quadrics

Projective Transformation of 3-Space

- Are identified by their matrix form, Or
- Their invariants

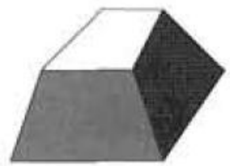
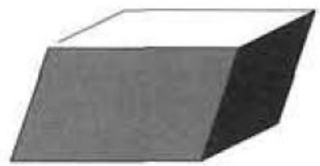
Just as the case with 2-Space Transforms

Projective Transformation of 3-Space

The 15 degrees of freedom are accounted for as:

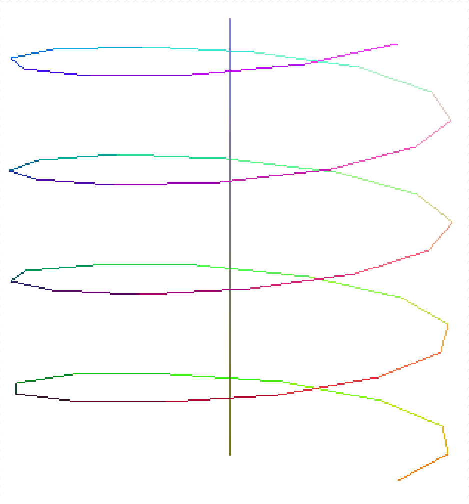
- Seven for similarity
  - 3 for rotations
  - 3 for translations
  - 1 for isotropic scaling
- Five for affine scaling
- Three for projective part

Projective Transformation of 3-Space

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Volume.

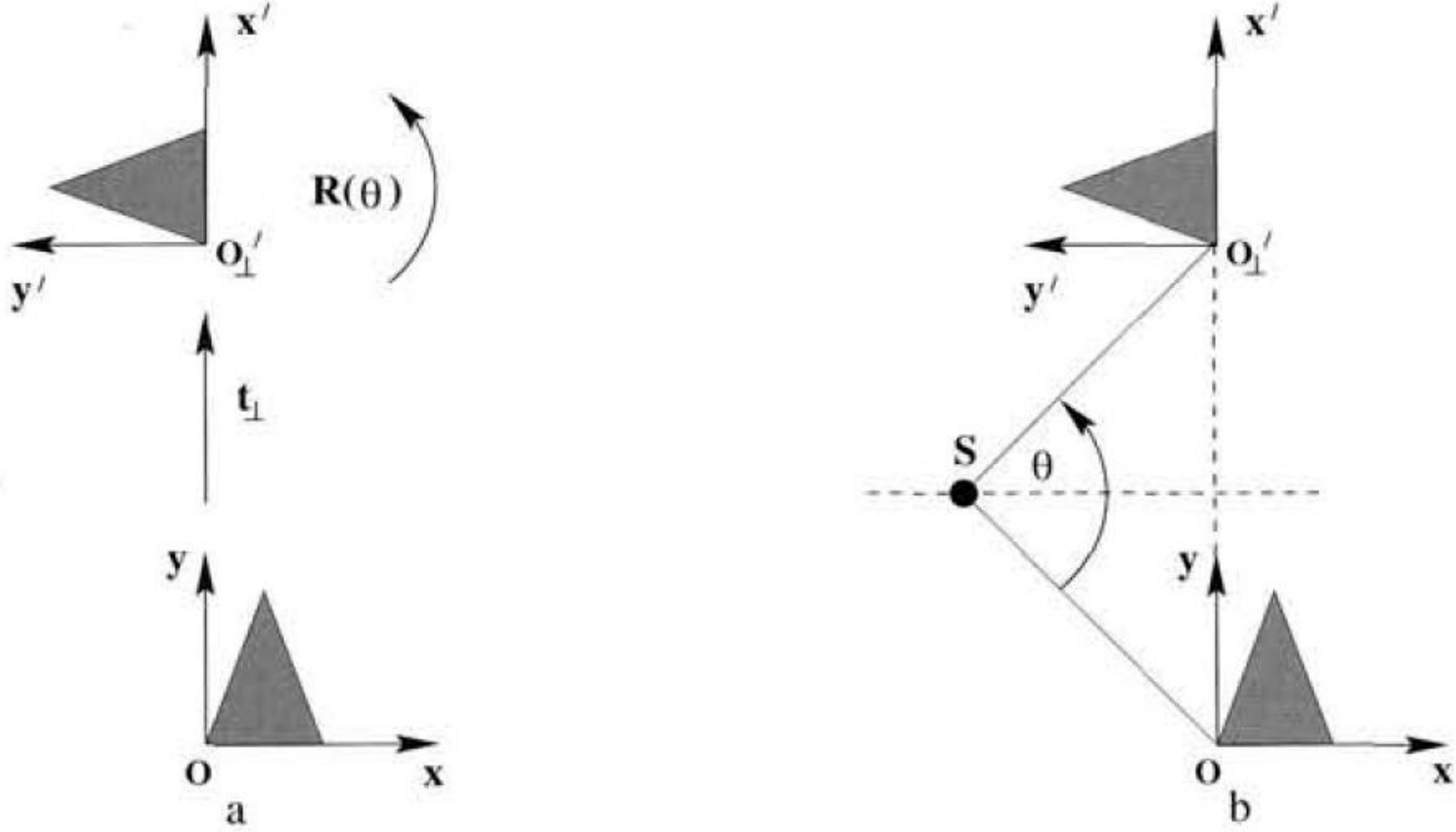
Screw Decomposition

Euclidean transformation on 3-Space is more general than Euclidean transformation on 2-Space



Any particular translation and rotation is equivalent to a rotation about a screw axis together with a translation along the screw axis. The screw axis is parallel to the rotation axis.

Screw Decomposition



Screw Decomposition