# Quantum Computation - Lecture 02 - Basics

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1. Elementary General Knowledge
2. Dirac's Notation
3. Quantum Circuits
4. Postulates
5. Universal Gates

Imaginary number:

$$i = \sqrt{-1}$$

Complex Number:

$$c = a + bi, \qquad a, b \in \mathbb{R}$$

Addition:

$$c + c' = (a + a') + (b + b')i$$

Multiplication:

$$cc' = (aa' - bb') + (ab' + a'b)i$$

Conjugation:

$$\overline{c} = a - b \cdot i$$

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Multiplication:

$$C = A \cdot B \text{ then } c_{ij} = \sum_j a_{ij} b_{ji}$$

Tensor Product:

$$A \otimes B = \begin{bmatrix} a_{11} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{bmatrix}$$

$$A^\dagger = \overline{A}^T = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} \\ \overline{a}_{12} & \overline{a}_{22} \end{bmatrix}$$

Unitary matrix:

$$UU^{\dagger} = I$$

In other words the lines of a unitary matrix in $\mathbb{C}^{n \times n}$ form a orthonormal basis for $\mathbb{C}^n$

Hermitian Matrix:

$$A = A^{\dagger}$$

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\langle\varphi| = \begin{bmatrix} \overline{b}_1 & \overline{b}_2 \end{bmatrix}$$

$$\langle\varphi|\psi\rangle = \begin{bmatrix} \overline{b}_1 & \overline{b}_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \overline{b}_1 a_1 + \overline{b}_2 a_2$$

$$|\psi\rangle\langle\varphi| = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} \overline{b}_1 & \overline{b}_2 \end{bmatrix} = \begin{bmatrix} a_1 \cdot \begin{bmatrix} \overline{b}_1 & \overline{b}_2 \end{bmatrix} \\ a_2 \cdot \begin{bmatrix} \overline{b}_1 & \overline{b}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1\overline{b}_1 & a_1\overline{b}_2 \\ a_2\overline{b}_1 & a_2\overline{b}_2 \end{bmatrix}$$

$$|\psi\rangle|\varphi\rangle = |\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ a_2 \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$$

Some 1-qubit unitaries:

- $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (Hadamard)
- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $Y = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$
- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Some 2-qubit unitaries:

- $C_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- $CR_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$

The 3-qubit gate.

- Toffoli Gate: $CCNOT$: $|a, b, c\rangle \rightarrow |a, b, c \oplus ab\rangle$
- $$C_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

## Theorem

*There is no unitary $U$ such that $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ for all states $|\psi\rangle$.*

- Suppose there exists unitary $U$ such that sends the state $|\psi\rangle|0\rangle$ to the state $|\psi\rangle|\psi\rangle$.
- Take any state $\varphi$: $U|\varphi\rangle|0\rangle = |\varphi\rangle|\varphi\rangle$
- Take any state $\psi$: $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$
- $\langle\varphi|\psi\rangle = \langle 0|0\rangle\langle\varphi|\psi\rangle = (\langle\varphi|\otimes\langle 0|)(|\psi\rangle\otimes|0\rangle) = (\langle\varphi|\otimes\langle 0|U^\dagger)(U|\psi\rangle\otimes|0\rangle)$
- $(\langle\varphi| \otimes \langle\varphi|)(|\psi\rangle \otimes |\psi\rangle) = \langle\varphi|\psi\rangle^2$

- States
- Dynamics
- Measurements
- Composite Systems

- To any physical system one can associate a complex vector space called the *state space* of the system.
- If the system is closed, then it can be completely described by its state vector.
- The state vector $v$ is a *unit vector* : $|v| = 1$
- If $|\psi_i\rangle$ is an orthonormal basis for the state space, then

$$|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle \tag{1}$$

is a superposition of the basis states $|\psi_i\rangle$ with amplitude $|\alpha_i|$ where $\sum_i |\alpha_i|^2 = 1$.

- The simplest quantum mechanical system, is the *qubit*.
- A qubit is a two-dimensional state space.
- Let $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then an arbitrary state in which the qubit may be encountered is

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

- where $\alpha_0$ and $\alpha_1$ are complex numbers satisfying $|\alpha_0|^2 + |\alpha_1|^2 = 1$. In other words $\langle\psi|\psi\rangle = 1$.

Some Physical Systems in which a qubit may be implemented (Outside the scope of this course). See Nielsen-Chuang (Ch. 7).

- Harmonic Oscilator
- Optical photon quantum computer
- Optical cavity quantum electrodynamics
- Ion traps
- Nuclear Magnetic Resonance ...

- The evolution of a closed quantum system is described by a unitary transformation.
- The state at time $t_1$ is related to the state at time $t_2$ by a unitary transformation $U$ which depends only on times $t_1$ and $t_2$.
- $|\psi'\rangle = U|\psi\rangle$

# Postulate II - Continuous version: Schrödinger equation.

$$ih\frac{d|\psi\rangle}{dt} = \mathcal{H}|\psi\rangle$$

- $\mathcal{H}$ is the Hamiltonian of the system, i.e. a Hermitian operator. $h$ is the plank constant.
- Knowing the Hamiltonian of a system and $h$, we have a complete understanding of the dynamics of the system.
- Since $\mathcal{H}$ is Hermitian, we have $\mathcal{H} = \sum_E E|E\rangle\langle E|$, where $E$ are eigenvalues, and $|E\rangle$ normalized eigenvectors.
- $|E\rangle$ are the *energy eigenstates* or *stationary states*.
- the eigenvalue $E$ is the energy of the state $|E\rangle$.
- Why stationary states: only the phase changes with time $|E\rangle \rightarrow exp(-iEt/h)|E\rangle$.
- The lowest energy is the ground state energy of the system. The corresponding eigenvalue is the ground state of the system.

- Ex: Suppose a qubit has Hamiltonian $H = h\omega X$.
- $\omega$ is a parameter that needs to be determined.
- The eigenstates are $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$.
- The corresponding energies are $h\omega$ and $-h\omega$.
- The ground state energy is $-h\omega$
- The solution to Schrödinger's equation is

$$|\psi(t_2)\rangle = exp\left[\frac{-i\mathcal{H}(t_2 - t_1)}{h}\right]|\psi(t_1)\rangle = U(t_1, t_2)|\psi(t_1)\rangle \qquad (2)$$

- where $U(t_1, t_2) = exp\left[\frac{-i\mathcal{H}(t_2 - t_1)}{h}\right]$

Matrix Exponential: Just for the sake of completeness.

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

- $e^0 = I$
- If $XY = YX$ then $e^X e^Y = e^{X+Y}$
- $\frac{d}{dX} e^{f(X)} = f'(X) e^{f(X)}$

- Quantum measurements are described by a collection $\{M_m\}$ of measurement operators.
- Each operator $M_m$ acts on the state of the system to be measured.
- The index $m$ refers to measurement outcomes that may occur in the experiment.
- If the state of the system is $|\psi\rangle$ before the measurement the probability that $m$ occurs as outcome is $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$
- The state after the measurement is $\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M|\psi\rangle}}$
- Completeness relation: $\sum_m M_m^\dagger M_m = I$
- By the completeness relation, probabilities sum to 1 : $\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M|\psi\rangle = 1$

Measurement in the computational basis: $M_0 = \langle 0||0\rangle$, $M_1 = \langle 1||1\rangle$.

- $M_0^\dagger M_0 = M_0^2 = M_0$
- $M_1^\dagger M_1 = M_1^2 = M_1$
- $I = M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1$
- Given $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ the probability of measuring 0 is

$$p(0) = \langle\psi|M_0|\psi\rangle = |\alpha_0|^2$$

and the probability of measuring 1 is

$$p(0) = \langle\psi|M_0|\psi\rangle = |\alpha_1|^2$$

The state space of a composite quantum system is the tensor product of the state space of the component physical systems. Moreover if we have systems numbered 1 through $n$, and $|\psi_i\rangle$ is the quantum state of the $i$-th system, then the joint state of the system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$.

$$|a\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad |b\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$|a\rangle \otimes |b\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

Checkpoint: What are the states $|0\rangle|0\rangle|0\rangle$, $|0\rangle|0\rangle|1\rangle$, $|0\rangle|1\rangle|0\rangle$, ..., $|1\rangle|1\rangle|1\rangle$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Preparing the Bell State:

- Start with $|0\rangle|0\rangle$
- Apply Hadamard in the first qubit:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{|0\rangle|0\rangle + |1\rangle|0\rangle}{\sqrt{2}}$$

- After C-NOT where the first bit is the control bit:

$$\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$

Prepare the other three 2-qubit Bell states.

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$\frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}}$$