Quantum Computation - Lecture 06 - Quantum Search

Mateus de Oliveira Oliveira

TCS-KTH

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- Thus applying $U_f(|x\rangle H|1\rangle) = (-1)^{f(x)}|x\rangle H|1\rangle$
- The action of U_f on the first n qubits can be described by:

$$V|x\rangle = (-1)^{f(x)}|x\rangle = \begin{cases} |x\rangle & \text{if } x \neq a \\ -|a\rangle & \text{if } x = a \end{cases}$$

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- So V can be rewritten as $V = I - 2|a\rangle\langle a|$

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- Both V and W acting on $|\phi\rangle$ and $|a\rangle$ give rise to vectors in the plane spanned by $|\phi\rangle$ and $|a\rangle$.

$$\begin{aligned} V|a\rangle &= -|a\rangle & V|\phi\rangle &= |\phi\rangle - \frac{2}{2^{n/2}}|a\rangle \\ W|\phi\rangle &= |\phi\rangle & W|a\rangle &= \frac{2}{2^{n/2}}|\phi\rangle - |a\rangle \end{aligned} \tag{1}$$

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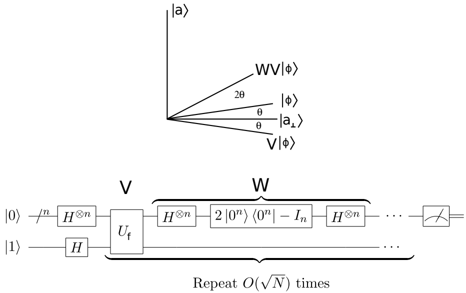
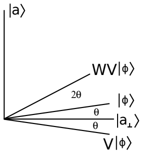
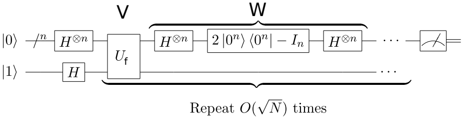
- But $\sin\theta = \cos\theta = 2^{-n/2} = \frac{1}{\sqrt{N}}$
- For small enough θ, $\sin\theta \simeq \theta$. Thus $\theta \simeq 2^{-n/2}$.



- W leaves $|\phi\rangle$ invariant, and reverses the direction of any vector orthogonal to $|\phi\rangle$. This is a reflection along $|\varphi\rangle$.





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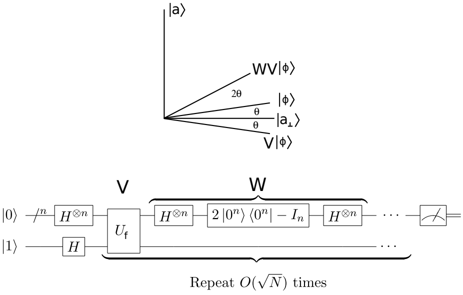
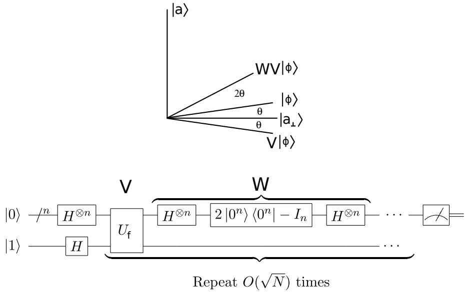
- V reverses the direction of $|a\rangle$ while leaving any vector orthogonal to $|a\rangle$ invariant. This is a reflection along $|a_\perp\rangle$.



- Each Application of WV rotates the vector 2θ in the direction of $|a\rangle$.



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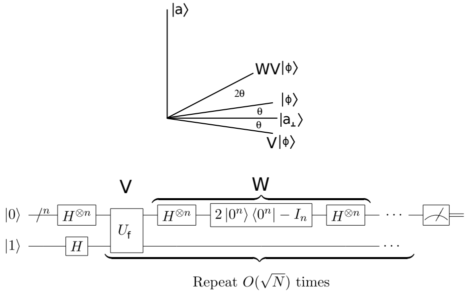

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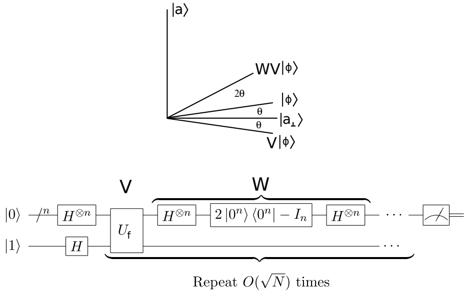

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- Measuring the final vector we will have the result with high probability.

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- Now replace the state $|a\rangle$ by

$$|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |a_i\rangle$$

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- Now $\sin \Theta = \cos(\pi/2 - \Theta) = \langle\psi|\phi\rangle = \sqrt{m/2^n}$
- If $m/2^n$ is much smaller than 1, we can get a state that is very close to $|\psi\rangle$ with $\frac{\pi}{4}\frac{2^{n/2}}{\sqrt{m}}$.

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  - $= \frac{1}{N} \sum_{y=0}^{N-1} (e^{2\pi i (\omega - \frac{z}{N})})^y = \frac{1}{N} \frac{1 - e^{2\pi i (\omega - \frac{z}{N}) N}}{1 - e^{2\pi i (\omega - \frac{z}{N})}}$

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  - Not proved here. Measuring $QFT^{-1}|\psi\rangle$ we have that with high probability we will get a z for which $\frac{z}{N}$ is close to ω.

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- $|y\rangle|\psi\rangle = \left(\otimes_{j=1}^{n}\frac{|0\rangle+e^{2\pi i 2^j\omega}|1\rangle}{\sqrt{2}}\right)|\psi\rangle = \frac{1}{\sqrt{N}}\sum_x e^{2\pi i\omega x}|x\rangle|\psi\rangle$