Quantum Computation - Lecture 07 - Quantum Error Correction I

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TCS-KTH

January 20, 2013

- No cloning theorem

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- Errrors are continuous

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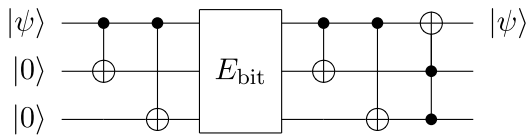
- Measurements destroys information



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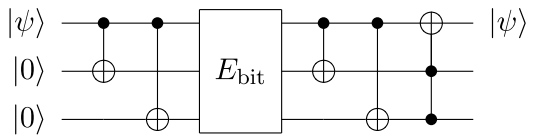

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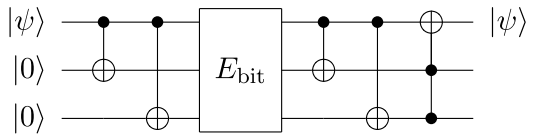
  - $|0\rangle \rightarrow |000\rangle$



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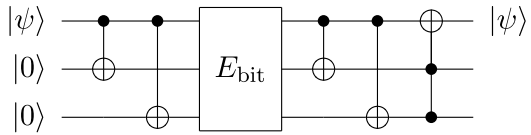

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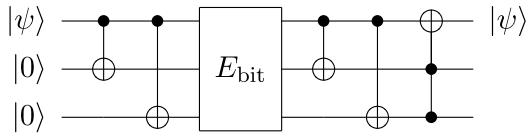

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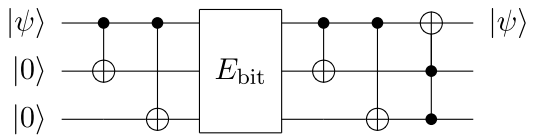

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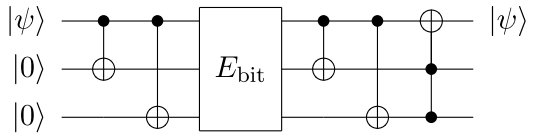

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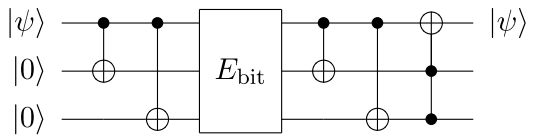

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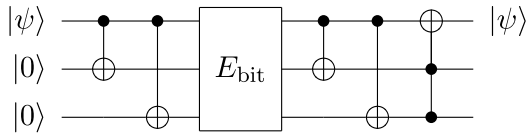

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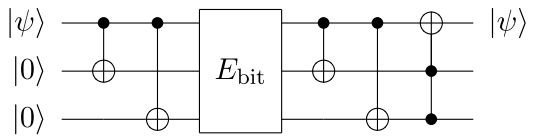

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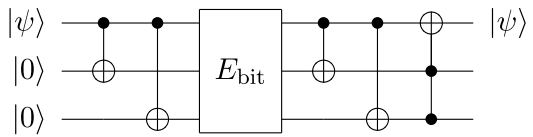

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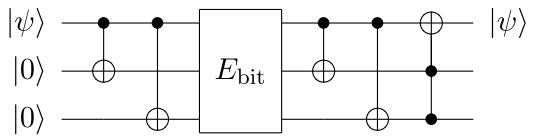

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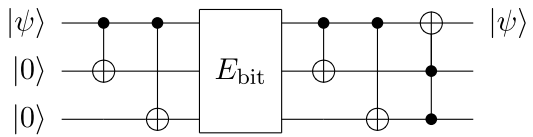

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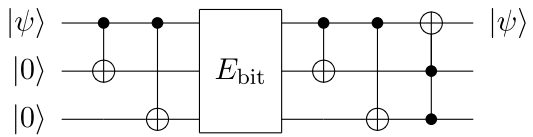

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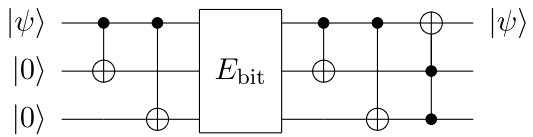

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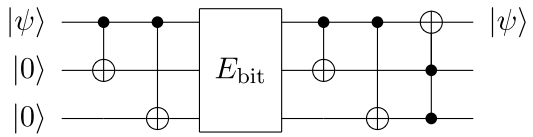
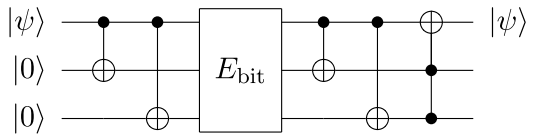
- After the measurement the state remains the same: $P_1|\psi\rangle = \Psi$.



- Recovery: Use the value of the error syndrome to apply the appropriate recovering procedure.



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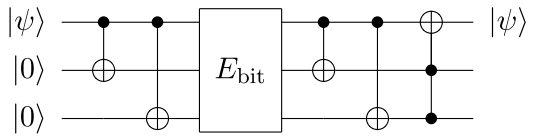

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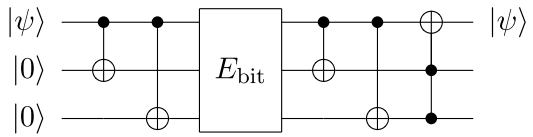

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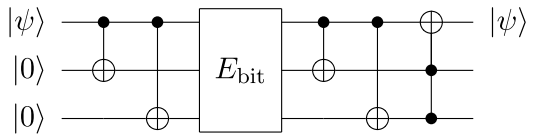

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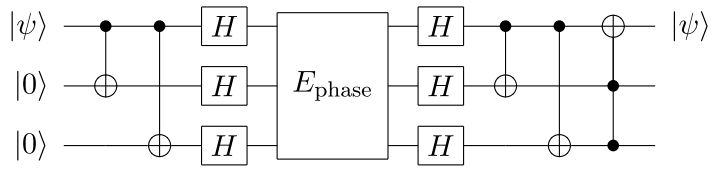

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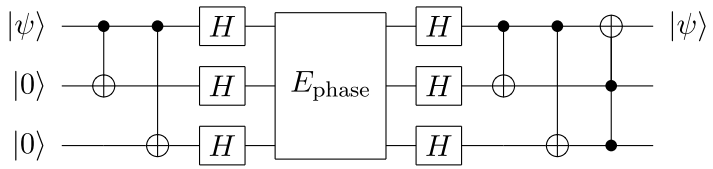

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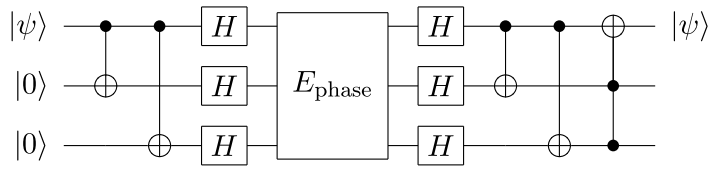

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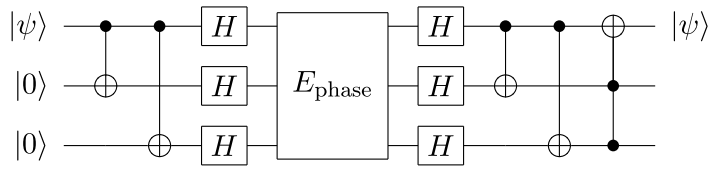

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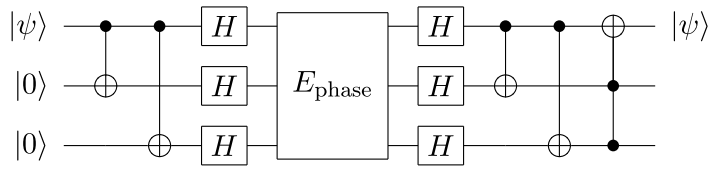

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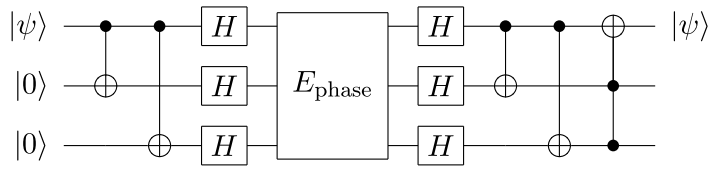

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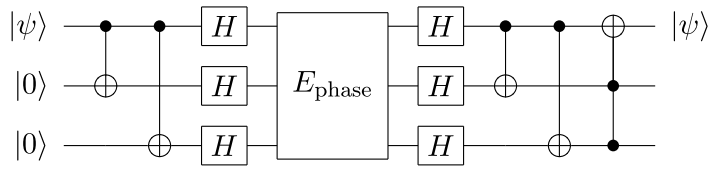

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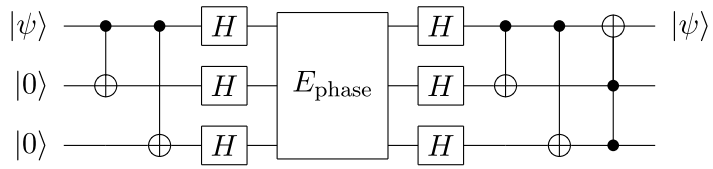
- First send $|0\rangle \rightarrow |000\rangle$ and $|1\rangle \rightarrow |111\rangle$. Then apply Hadamard $H^{\otimes 3}$.



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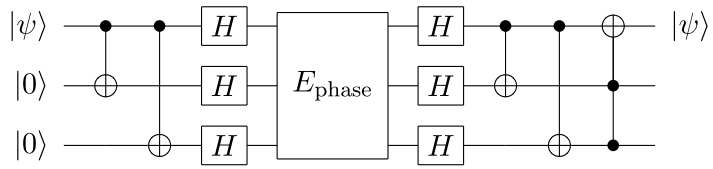

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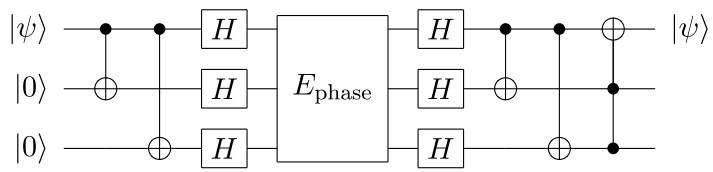

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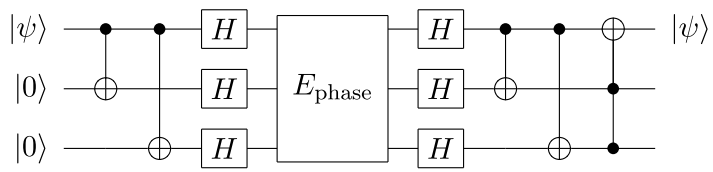

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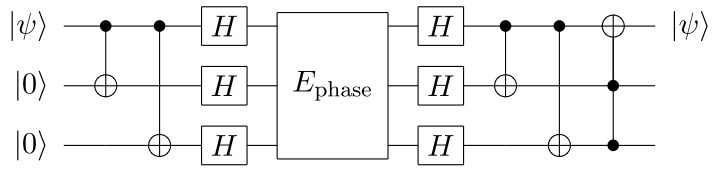

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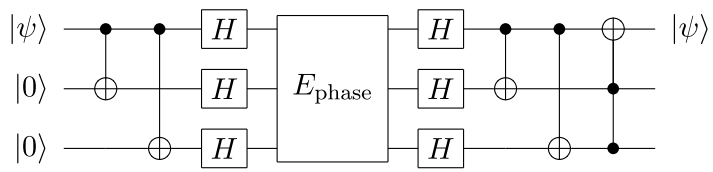

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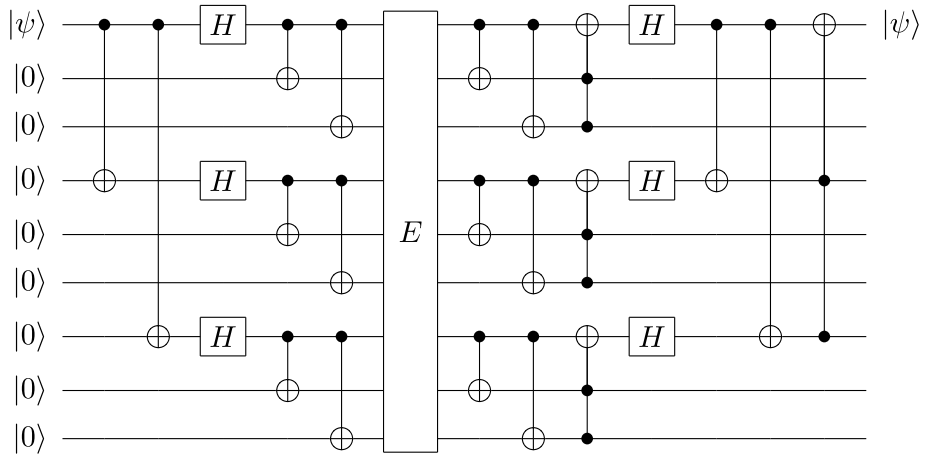
- First protect against phase flips: $|0\rangle \rightarrow |+++\rangle$, $|1\rangle \rightarrow |---\rangle$.
- Then protect each of the qubits against bit-flip. Encode

$$|+\rangle \rightarrow \frac{|000\rangle + |111\rangle}{\sqrt{2}} \text{ and } |-\rangle \rightarrow \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

.

$$|0\rangle \rightarrow \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$


$|\psi\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
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H
H
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E
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$|\psi\rangle$

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- $U = c_0 I + c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z$.
- where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

,

- A linear code C of length n over a field F is a subspace of F^n. Here we will consider that $F = \mathbb{Z}_2$.
- The words of the codespace F^n are vectors. Call them codewords.
- A linear code $C$is a $[n, k]$ code if C has dimension k.
- If C is a $[n, k]$ code then $|C| = |F|^k$. Thus we can consider that C encodes k bits of information into n qubits.
- The distance of the code is $d = min_{v,u \in C}\{i | v_i \neq u_i\}$
- Example: Repetition Code of length n over F.

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- Equivalently: $C^{\perp}$ consists of all codewords y such that y is orthogonal to all codewords in C.

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- Dimension of $CCS(C_1, C_2) = |C_1|/|C_2| = 2^{k_1 - k_2}$, and thus $CSS(C_1, C_2)$ is an $[n, k_1 - k_2]$ quantum code.

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- Thus the state becomes

$$\frac{1}{\sqrt{C_2}}\sum_{y\in C_2}(-1)^{(x+y)\cdot e_2}|x + y + e_1\rangle|H_1 e_1\rangle$$

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- Error Correction: Simply apply NOT gates to the qubits at whichever the positions a flip occurred.

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- Detecting Phase Flips: Apply H to each qubit.

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- Apply H to each qubit obtaining:

$$\frac{1}{|C_2|} \sum_{y \in C_2} |x \oplus y\rangle = |x \oplus C_2\rangle$$