Quantum Computation - Lecture 09 - Quantum Simulation

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February 20, 2013

A tree decomposition of a graph G:

- A tree $\mathcal{T}$
- A function that maps each vertex $w \in V(\mathcal{T})$ to a subset $B_w \subseteq V(G)$.
  - $\bigcup_{v \in V(\mathcal{T})} B_v = V(G)$
  - $\forall \{u, v\} \in E(G), \exists w \in V(\mathcal{T}), \{u, v\} \subseteq B_w$
  - $\forall u \in V(G)$ the set of vertices $w \in V(\mathcal{T})$ with $u \in B_w$ form a connected subtree.
- Width of a tree: $max_{w \in V(\mathcal{T})} |B_w| - 1$

- Equivalent definition: Elimination Width:
- Let π be an ordering of the vertices of G.
- Start with $\pi(1)$.
- Add an edge connecting any two pair of neighbors of $\pi(1)$ that were not previously adjacent.
- remove $\pi(1)$.
- Restart the process with $\pi(2)$ and so on...
- The induced width of a vertex is the number of neighbors of a vertex at the time it is being eliminated.
- The elimination width is equal to the tree-width.

- State Space of a Qubit: $\mathcal{H} = \mathbb{C}^2$
- Basis for $\mathcal{H}$: $|0\rangle, |1\rangle$
- Space of operators on a vector space V: $L(V)$
- Density Operator on n qubits: $\rho \in L(\mathcal{H}^{\otimes n})$, with $Tr(\rho) = 1$.
- If $x = x_1 x_2 ... x_n \in \{0,1\}^n$ then $\rho_x = \otimes_{i=1}^{n} |x_i\rangle\langle x_i|$ is a density operator of the state $|x\rangle = \otimes_{i=1}^{n} |x_i\rangle$.

- Graph of a quantum circuit C: G_C
  - A vertex v_U for each gate U.
  - A vertex v_i for each input i
  - A vertex v_o for each output o
  - An edge $(v_U, v_{U'})$ if an output of U is an input of U'.

- Rank-k tensor in an m-dimensional space $g = [g_{i_1,i_2,\ldots,i_k}]$ is a m^k-dimensional array
- rank-0 tensor: Complex number
- rank-1 tensor: vector
- rank-2 tensor: Usual Matrix

- Here we will be interested in dimension-4 tensors
- Each index of the tensor will run trough the set

$$\Pi = \{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\} \tag{1}$$

- Tensor associated to a density operator ρ:

$$\rho_{\sigma_1,\sigma_2,\ldots,\sigma_a} = tr(\rho \cdot (\otimes_{i=1}^{a} \sigma_i)^{\dagger}) \tag{2}$$

- Tensor associated to a superoperator Q acting on a input qubits and b output qubits.

$$Q_{\sigma_1,\ldots,\sigma_a,\tau_1,\ldots,\tau_b} = tr(Q(\otimes_{i=1}^{a} \sigma_i) \cdot (\otimes_{j=1}^{b} \tau_j)^{\dagger}) \tag{3}$$

- $g = [g_{i_1,\ldots,i_l,j_1,\ldots,j_k}]$
- $h = [h_{i_1,\ldots,i_l,j'_1,\ldots,j'_{k'}}]$
- $$f_{j_1,\ldots,j_k,j'_1,\ldots,j'_{k'}} = \sum_{i_1,\ldots,i_l} g_{i_1,\ldots,i_l,j_1,\ldots,j_k} \cdot h_{i_1,\ldots,i_l,j'_1,\ldots,j'_l} \quad (4)$$
- If all outputs of a density matrix ρ are connected to the inputs of Q then the contraction of ρ and Q gives the tensor corresponding to the density matrix $Q\rho Q^\dagger$.

- Measurement scenario on m qubits: $\tau : [m] \rightarrow L(\mathbb{C}^2)$ such that $\tau(i)$ is a single-qubit POVM measurement element.
- If a qubit is not to be measured we can set $\tau(i) = I$
- Contracting all the gates to the single output measurement gives the output probability.

- Contraction of an edge e: removes e and replaces its end vertices with a single vertex.
- Contraction ordering: Ordering of the edges of G

$$\pi : \pi(1), \pi(2), ..., \pi(|E(G)|)$$

- Complexity of π is the maximum degree of a merged vertex during the contraction process.
- Contraction complexity of G: $cc(G)$ is the minimum complexity of a contraction ordering.
- Obs: Only the degree of the merged vertex count: For instance, if G is a path $cc(G) = 1$ and $\Delta(G) = 2$

- Line Graph of G: G^* whose vertices are the edges of G and two vertices e_1, e_2 of G^* are connected if $e_1 \cap e_2 \neq \emptyset$.
- For any graph G, $cc(G) = tw(G^*)$
- Given a tree decomposition of G^* of width d, one can determine efficiently a contraction ordering $cc(\pi) \leq d$.
- Robertson-Seymour: There is a deterministic algorithm that given a graph G outputs a tree decomposition of G of width $O(tw(G))$ in time $|V(G)|^{O(1)} \exp[O(tw(G))]$.
- $(\frac{tw(G)-1)}{2} \leq tw(G*) \leq \Delta(G)(tw(G)+1)-1$
- Corollary: If the gates of a quantum circuit have at most d outputs, we have that

$$(tw(G)-1)/2 \leq cc(G) = tw(G^*) \leq d(tw(G_n)+1)-1$$