Quantum Computation - Lecture 10 - Approximation of the Jones Polynomial

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TCS-KTH

February 20, 2013

- Knots

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  - Kauffman bracket polynomial:

$$\langle L \rangle = \sum_{allstates \sigma} \sigma(L)$$

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- $\langle L\rangle$ is the Kauffman Bracket ignoring the orientation.

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- Obs: If an algebra is specified by a set of generators, then the representation may be specified by the images of the generators.
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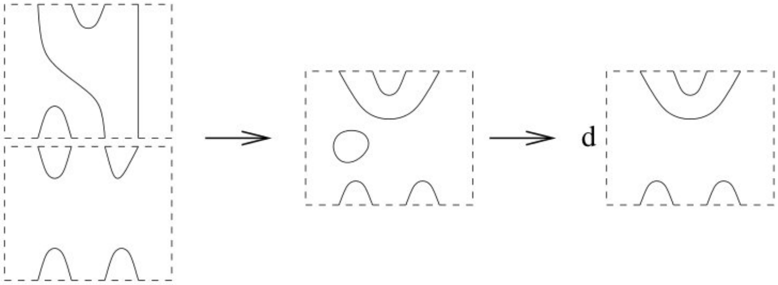
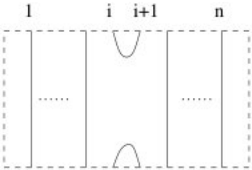
  - $E_i^2 = dE_i$



$\psi(E_i) =$ 

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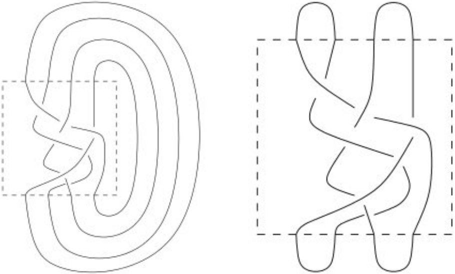
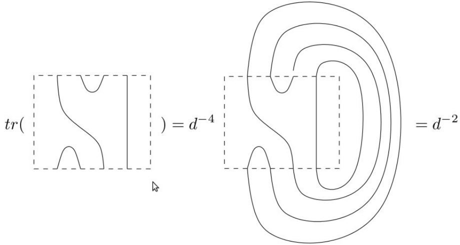
  - Exercise: Show that $\tau(\rho_A(\sigma_i))\tau(\rho_A(\sigma_i))^\dagger = I$

- A tangle is a braid in which some of the crossings have been replaced by a picture with the form $\overset{\bigcup}{\cap}$.

From braids to Knots.



- A linear function from an algebra to the complex numbers is called a trace if it satisfies the equation $tr(XY) = tr(YX)$ for every elements X, Y in the algebra.



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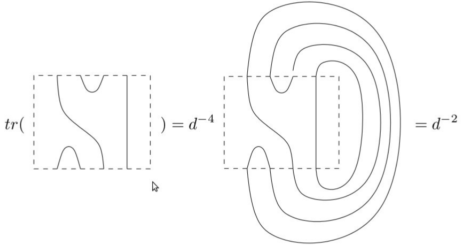

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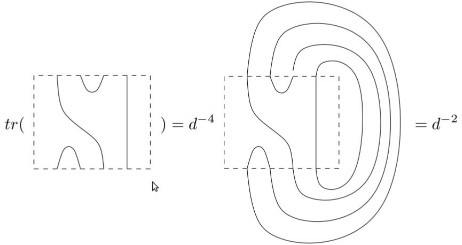

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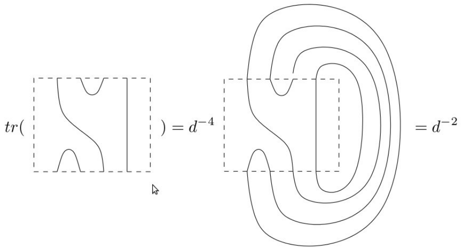

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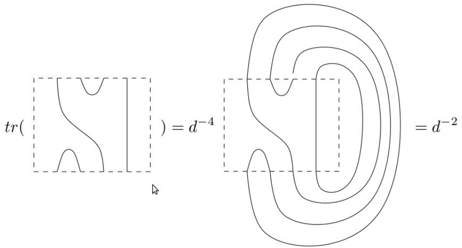
    If a is the number of circles resulting from this operation, then $tr(K) = d^{a-n}$.



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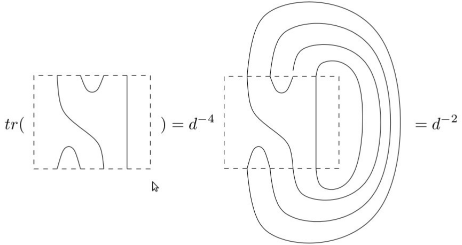
- By the isomorphism between $TL_n(d)$ and $gTL_n(d)$ this operation also induces a trace on $TL_n(d)$.



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- $tr(XY) = tr(YX)$ for any $X, Y \in TL_n(d)$
- If $X \in TL_{n-1}(d)$ then $tr(XE_{n-1}) = \frac{1}{d} tr(X)$

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Given a braid B, let B^{tr} denote its trace closure. Then

$$V_{B^{tr}}(A^{-4}) = (-A)^{3w(B^{tr})} d^{n-1} tr(\rho_A(B))$$

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  6. Exercise: What is the expectation of the output if we start with the state $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \otimes |\alpha\rangle$?

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- Do the same for the random variables y_j whose expectation value is $Im\langle\alpha|Q(B)|\alpha\rangle$ using an appropriated version of the Hadamard test.