Quantum Computation - Lecture 11 - The Local Hamiltonian Problem

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TCS-KTH

February 20, 2013

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- If every eigenvalue of H exceeds b then

$$p = 1 - r^{-1}\langle\eta|H|\eta\rangle \leq 1 - r^{-1}b$$

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- Then the probability of getting outcome 1 is $\sum_j \frac{1}{r}(1 - \langle\eta|H|\eta\rangle) = 1 - r^{-1}\langle\eta|H|\eta\rangle$

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- The term H_{in} adds a penalization of 1 to the function $\langle\eta|H|\eta\rangle$ whenever the qubit s is in state $|1\rangle$ while the counter is in state $|0\rangle$.

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- Each term H_j corresponds to a transition from $j-1$ to j.

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        - Thus $W^\dagger H_j W = I \otimes E_j$ where

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  - $W^\dagger H_{prop} W = I \otimes E$ where $E = \sum_j E_j$

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- Let $|\tilde{\eta}\rangle = |\chi, 0\rangle \otimes |\psi\rangle$ where
  - $|\psi\rangle = \frac{1}{\sqrt{L+1}} \sum_{j=0}^{1} |j\rangle$

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- Suppose the original circuit gives the answer YES with probability greater than $1 - \varepsilon$ on some input vector $|\chi\rangle$
- Then the Probability of measuring 0 is

$$P(0) = \langle \chi, 0 | U^\dagger \Pi_1^{(0)} U | \chi, 0 \rangle \leq \varepsilon$$

- We prove that $\tilde{H}$ has a small eigenvalue. Since the eigenvalues are preserved on change of basis H will also have such a small eigenvalue.
- Let $|\tilde{\eta}\rangle = |\chi, 0\rangle \otimes |\psi\rangle$ where
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- Finally

$$\langle \tilde{\eta} | \tilde{H}_{out} | \tilde{\eta} \rangle = \langle \tilde{\eta} | (U^\dagger \Pi_1^{(0)} U \otimes |L\rangle\langle L|) | \tilde{\eta} \rangle = P(0) \frac{1}{L+1} \leq \frac{\varepsilon}{L+1}$$